Çankaya University Journal of Science and Engineering
Volume 9 (2012), No. 2, 139–147

Approximate Analytical Solutions of the Fractional Sharma-Tasso-Olver Equation Using Homotopy Analysis Method and a Comparison with Other Methods

Alaattin Esen[1,*], Orkun Taşbozan[1] and N. Murat Yağmurlu[1]

[1] *İnönü University, Faculty of Art and Sciences, Department of Mathematics, 44280 Malatya, Turkey*
[*] *Corresponding author: aesen@inonu.edu.tr*

Özet. Bu çalışmada, kesirli mertebeden Sharma-Tasso-Olver (STO) denkleminin yaklaşık analitik çözümlerini elde etmek için homotopi analiz metodu (HAM) başarılı bir şekilde uygulandı. Elde edilen sonuçlarla varyasyonel iterasyon metodu (VIM), Adomian ayrıştırma metodu (ADM) ve homotopi pertürbasyon metodu (HPM) ile elde edilen sonuçların karşılaştırılması; önemli ölçüde anlamlı sonuçlar elde ettiğimiz sonucuna varmamıza sebep oldu. HAM çözümü, çözüm serilerinin yakınsaklık bölgesini kontrol etmek ve ayarlamak için uygun bir yol sağlayan bir $\hbar$ yardımcı parametresini içerir.

Anahtar Kelimeler. Homotopi analiz metodu, yaklaşık analitik çözüm, kesirli mertebeden Sharma-Tasso-Olver denklemi, kesirli kalkülüs.

Abstract. In this paper, the homotopy analysis method (HAM) is successfully applied to the fractional Sharma-Tasso-Olver equation to obtain its approximate analytical solutions. Comparison of the obtained results with those of variational iteration method (VIM), Adomian's decomposition method (ADM) and homotopy perturbation method (HPM) has led us to conclude that the method gives significantly important consequences. The HAM solution includes an auxiliary parameter $\hbar$ which provides a convenient way of adjusting and controlling the convergence region of solution series.

Keywords. Homotopy analysis method, approximate analytical solution, fractional Sharma-Tasso-Olver equation, fractional calculus.

1. Introduction

Since many important phenomena in physics and engineering can only be well described by fractional differential equations, a considerable interest in them has been aroused recently due to their widespread applications in physics and engineering. However, in general, there exists no method that gives an exact solution for a fractional differential equation. Thus, their approximate analytical solutions have been

Received March 20, 2012; accepted February 6, 2013.

ISSN 1309 - 6788

sought and found by various methods such as VIM [1, 2], ADM [3, 4], HPM [5, 6] and HAM [7, 8, 9]. The HAM which was first proposed by Liao [10, 11] is a powerful tool for searching the approximate solutions of nonlinear evolution equations (NLEEs). Unlike perturbation techniques, the HAM is not limited to any small physical parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems [12, 13]. In this paper, we will apply the HAM to the fractional Sharma-Tasso-Olver equation. One of the fractional differential equations arising in science and engineering is Sharma-Tasso-Olver equation with time-fractional derivative of the form

$$D_t^\alpha u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1 \tag{1}$$

where a is an arbitary constant, α is a parameter describing the order of the fractional time-derivative.

Several definitions of fractional integration and derivation such as Riemann-Liouville's and Casputo's have been proposed. The Riemann-Liouville integral operator [14] having order $\alpha > 0$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \quad (x > 0)$$

and as

$$J^0 f(x) = f(x)$$

for $\alpha = 0$. Its fractional derivative of order $\alpha > 0$ is generally used

$$D^\alpha f(x) = \frac{d^n}{dx^n} J^{n-\alpha} f(x) \quad (n-1 < \alpha < n)$$

where n is an arbitrary integer. The Riemann-Liouville integral operator has an important role for the development of the theory of both fractional derivatives and integrals. In spite of this fact, it has certain disadvantages when it comes to modelling real-world phenomena with fractional differential equations. This problem has been solved by M. Caputo first in his article [15] and then in his book [16]. Caputo definition, which is a modification of Riemann-Liouville definition, can be given as

$$D^\alpha f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \ \alpha > 0, \ (n-1 < \alpha < n).$$

Note that Caputo derivative has the following two important properties

$$D^{\alpha} J^{\alpha} f(x) = f(x)$$

and

$$J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{x^k}{k!} \quad (n-1 < \alpha < n).$$

2. HAM Solutions of Fractional Sharma-Tasso-Olver Equation

The Eq. (1) is considered with the initial condition

$$u(x,0) = \frac{2k(w + \tanh(kx))}{1 + w \tanh(kx)} \tag{2}$$

where $k, w \in C$. To investigate the series solution of Eq. (1) with initial condition (2) and to make a comparison with VIM, ADM and HPM solutions in Ref. [17], we choose the linear operator

$$\mathcal{L}[\phi(x,t;q)] = D_t^{\alpha}[\phi(x,t;q)]$$

with the property

$$\mathcal{L}[c] = 0$$

where c is constant. From Eq. (1), we can now define a nonlinear operator as

$$\begin{aligned} \mathcal{N}[\phi(x,t;q)] &= \frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}} + 3a\left(\frac{\partial\phi(x,t;q)}{\partial x}\right)^2 \\ &\quad + 3a\left(\phi(x,t;q)\right)^2 \frac{\partial\phi(x,t;q)}{\partial x} + 3a\phi(x,t;q)\frac{\partial^2\phi(x,t;q)}{\partial x^2} + a\frac{\partial^3\phi(x,t;q)}{\partial x^3}. \end{aligned}$$

Therefore, we construct the zero-order deformation equation as follows

$$(1-q)\mathcal{L}[\phi(x,t;q) - u_0(x,t)] = q\hbar\mathcal{N}[\phi(x,t;q)]. \tag{3}$$

Obviously, if we choose $q = 0$ and $q = 1$, then we obtain

$$\begin{aligned} \phi(x,t;0) &= u_0(x,t) = u(x,0), \\ \phi(x,t;1) &= u(x,t) \end{aligned}$$

respectively. Thus, as the embedding parameter q increases from 0 to 1, the solution

$\phi(x,t;q)$ varies from the initial value $u_0(x,t)$ to the solution $u(x,t)$. By expanding $\phi(x,t;q)$ in Taylor series with respect to the embedding parameter q, we obtain

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, the above series converges at $q=1$, and one can have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [11, 18]. By differentiating Eq. (3) m times with respect to the embedding parameter q, we obtain the mth-order deformation equation

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\vec{u}_{m-1}\right) \tag{4}$$

where

$$\begin{aligned} R_m(\vec{u}_{m-1}) = {} & \frac{\partial^\alpha u_{m-1}(x,t)}{\partial t^\alpha} + 3a\sum_{n=0}^{m-1}\frac{\partial u_n(x,t)}{\partial x}\frac{\partial u_{m-1-n}(x,t)}{\partial x} \\ & + 3a\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x} \\ & + 3a\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial^2 u_{m-1-n}(x,t)}{\partial x^2} + a\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3} \end{aligned}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

The solution of the mth-order deformation Eq. (4) for $m \ge 1$ leads to

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar J_t^\alpha\left[R_m\left(\vec{u}_{m-1}\right)\right]. \tag{5}$$

By using Eq. (5) with the initial condition given by (2), we successively obtain

$$u_0(x,t) = u(x,0) = \frac{2k(w+\tanh(kx))}{1+w\tanh(kx)},$$

$$
\begin{aligned}
u_1(x,t) \;&=\; \frac{16ak^4(w^2-1)^2\hbar t^\alpha}{\Gamma(\alpha+1)(\cosh(kx)+w\sinh(kx))^4}+\frac{8ak^4(w^4-1)\hbar t^\alpha\cosh(2kx)}{\Gamma(\alpha+1)(\cosh(kx)+w\sinh(kx))^4}\\
&\quad+\frac{16ak^4w(w^2-1)\hbar t^\alpha\sinh(2kx)}{\Gamma(\alpha+1)(\cosh(kx)+w\sinh(kx))^4},\\
&\vdots
\end{aligned}
$$

etc. Therefore, the series solution expressed by the HAM can be written in the form

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\ldots. \tag{6}$$

To demonstrate the efficiency of the method, we compare the HAM solutions of fractional Sharma-Tasso-Olver equation given by Eq. (6) for $\alpha=1$ with exact solutions [17]

$$u(x,t)=\frac{2k(w+\tanh(k(x-4ak^2t)))}{1+w\tanh(k(x-4ak^2t))}. \tag{7}$$

Note that our HAM solution series contains the auxiliary parameter $\hbar$ which provides us with a simply way to adjust and control the convergence of the solution series. To obtain an appropriate range for $\hbar$, we consider the so-called $\hbar$-curve to choose a proper value of $\hbar$ which ensures that the solution series is convergent, as pointed by Liao [11], by discovering the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis. In Fig.1, we demonstrate the $\hbar$-curves of $u(2,0.01)$ given by 4th-order HAM solution (6) for $\alpha=1$, $\alpha=0.9$ and $\alpha=0.8$. It can be seen from the figure that the valid range of $\hbar$ is approximately $-1.4\le\hbar\le-0.7$.

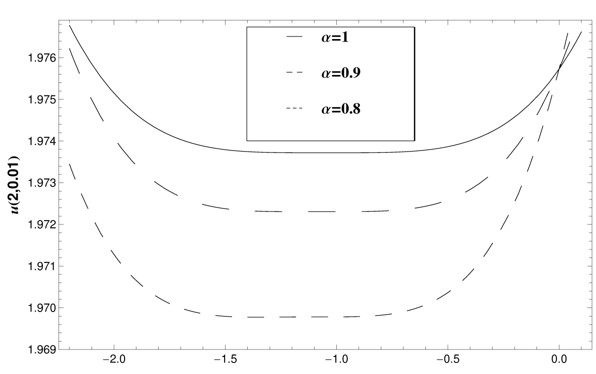

FIGURE 1. The $\hbar$-curves of 4th-order approximate solutions obtained by the HAM.

The comparison of the results of the HAM, VIM [17], ADM [17], HPM [17] and exact solution for $\alpha = 1$ is given in Table 1. It shows that 4th-order approximate solution obtained by the HAM for $\hbar = -1$ is in good agrement at almost all points (x, t).

TABLE 1. The results obtained by the HAM for $\hbar = -1$ by 4th-order approximate solution in comparison with the VIM, ADM, HPM in Ref. [17] and exact solution at $t = 0.01$ for $\alpha = k = a = 1$, and $w = \frac{1}{2}$.

x	u_{VIM} [17]	u_{ADM} [17]	u_{HPM} [17]	$u_{\mathrm{HAM}}(x,t)$	Exact Solution
0	0.938798380	0.938800000	0.938800000	0.938808800	0.938808808
1	1.813642383	1.813642415	1.813642415	1.813631681	1.813631681
2	1.973721044	1.973721044	1.973721044	1.973719022	1.973719022
3	1.996423221	1.996423221	1.996423221	1.996422935	1.996422935
4	1.999515561	1.999515561	1.999515561	1.999515522	1.999515522
5	1.999934431	1.999934431	1.999934431	1.999934426	1.999934426
6	1.999991127	1.999991127	1.999991127	1.999991125	1.999991125
7	1.999998799	1.999998799	1.999998799	1.999998799	1.999998799
8	1.999999839	1.999999839	1.999999839	1.999999837	1.999999837
9	1.999999978	1.999999978	1.999999978	1.999999978	1.999999978
10	1.999999997	1.999999997	1.999999997	1.999999997	1.999999997

Fig. 2 shows the absolute error between numerical solution of $u(x, t)$ during $0 \leq t \leq 0.1$ for $-100 \leq x \leq 100$, $a = k = \alpha = 1$, $w = \frac{1}{2}$ and $\hbar = -1$ obtained by 4th-order HAM and analytical solutions. It can be seen from this figure and Table 1 that the choice of $\hbar = -1$ is a suitable one.

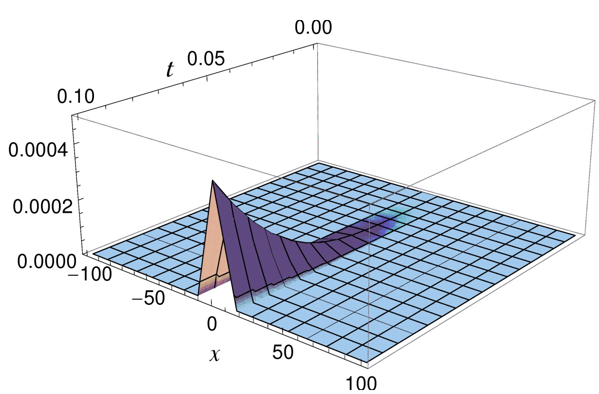

FIGURE 2. The absolute error between the exact solution and the 4th-order approximate solution obtained by the HAM for $a = k = \alpha = 1$, $w = \frac{1}{2}$ and $\hbar = -1$.

Fig. 3 shows the numerical solutions of $u(x,t)$ during $0 \le t \le 0.1$ for $-100 \le x \le 100$, $a = k = \alpha = 1$, $w = \frac{1}{2}$ and $\hbar = -1$ obtained by 4th-order HAM for $\alpha = 0.9$ and $\alpha = 0.8$, respectively.

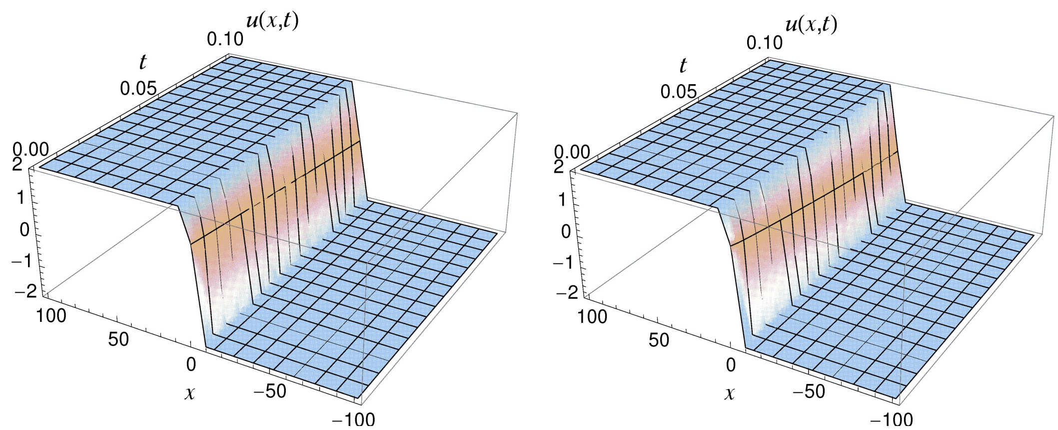

FIGURE 3. The results obtained by the HAM for $\alpha = 0.9$, $\alpha = 0.8$, respectively, and $\hbar = -1$ by 4th-order approximate solution when $a = k = 1$ and $w = \frac{1}{2}$.

In order to investigate the state of the parameter $\hbar$ for smaller values of α, in Fig. 4, we illustrate the $\hbar$-curve of $u(2, 0.01)$ given by the 4th-order HAM solution (6) for, the same parameters as used in Ref. [17], $\alpha = 0.5$ when $a = \ln 10, k = -\frac{\pi}{2}$ and $w = -\frac{1}{5}$. It can clearly be seen from the figure that the valid range of $\hbar$ lies approximately in $-0.1 \le \hbar \le 0.5$

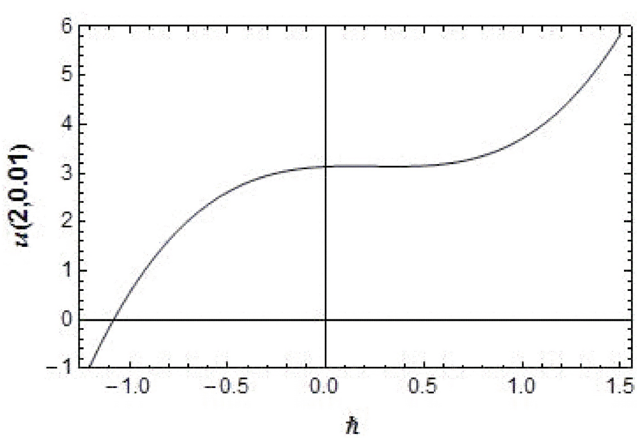

FIGURE 4. The $\hbar$-curve of 4th-order approximate solutions obtained by the HAM for $\alpha = 0.5$, $a = \ln 10, k = -\frac{\pi}{2}$ and $w = -\frac{1}{5}$.

Fig. 5 shows the numerical solution of $u(x,t)$ during $0 \le t \le 1$ for $-100 \le x \le 100$, $a = \ln 10$, $k = -\frac{\pi}{2}$, $w = -\frac{1}{5}$ and $\hbar = 0.2$ obtained by 4th-order HAM for $\alpha = 0.5$.

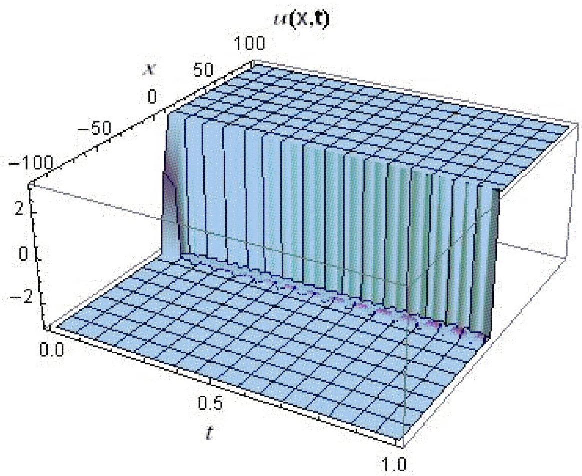

FIGURE 5. The results obtained by the HAM for $\alpha = 0.5$, and $\hbar = 0.2$ by 4th-order approximate solution when $a = \ln 10$, $k = -\frac{\pi}{2}$ and $w = -\frac{1}{5}$.

3. Conclusion

In this paper, the HAM has been successfully applied to obtain approximate analytical solution of fractional Sharmo-Tasso-Olver equation. It has also been seen that the HAM solution of the problem converges very rapidly to the exact one by choosing an appropriate auxiliary parameter $\hbar$. In conclusion, this study shows that the HAM is a powerful and efficient technique with respect to VIM, ADM and HPM in finding the approximate analytical solution of fractional Sharma-Tasso-Olver equation. Moreover, it can also be used to solve many other nonlinear evolution equations arising in science and engineering.

References

[1] J.-H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Computer Methods in Applied Mechanics and Engineering* **167** (1998), 57–68.

[2] R. Y. Molliq, M. S. M. Noorani and I. Hashim, Variational iteration method for fractional heat- and wave-like equations, *Nonlinear Analysis: Real World Applications* **10** (2009), 1854–1869.

[3] N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, *Applied Mathematics and Computation* **131** (2002), 517–529.

[4] S. Momani and Z. Obibat, Numerical approach to differential equations of fractional order, *Journal of Computational and Applied Mathematics* **207** (2007), 96–110.

[5] S. Momani and Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, *Physics Letters A* **365** (2007), 345–350.

[6] Z. Odibat, Exact solitary solutions for variants of the KdV equations with fractional time derivatives, *Chaos, Solitons & Fractals* **40** (2009), 1264–1270.

[7] H. Xu and J. Cang, Analysis of a time fractional wave-like equation with the homotopy analysis method, *Physics Letters A* **372** (2008), 1250–1255.

[8] L. Song and H. Q. Zhang, Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation, *Physics Letters A* **367** (2007), 88–94.

[9] H. Jafari and S. Seifi, Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation, *Communications in Nonlinear Science and Numerical Simulation* **14** (2009), 2006–2012.

[10] S. J. Liao, *The Proposed Homotopy Analysis Tecnique for the Solution of Nonlinear Problems*, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai 1992.

[11] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall/CRC Press, Boca Raton 2003.

[12] S. J. Liao, Homotopy analysis method: A new analytical technique for nonlinear problems, *Communications in Nonlinear Science and Numerical Simulation* **2** (1997), 95–100.

[13] S. J. Liao, On the homotopy analysis method for nonlinear problems, *Applied Mathematics and Computation* **147** (2004), 499–513.

[14] L. Podlubny, *Fractional Differantial Equations*, Academic Press, London 1999.

[15] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, *Geophysical Journal International* **13** (1967), 529–539.

[16] M. Caputo, *Elasticità e Dissipazione*, Zanichelli Publisher, Bologna, 1969.

[17] L. Song, Q. Wang and H. Zhang, Rational approximation solution of the fractional Sharma-Tasso-Olever equation, *Journal of Computational and Applied Mathematics* **224** (2009), 210–218.

[18] S. J. Liao, Notes on the homotopy analysis method: Some definitions and theorems, *Communications in Nonlinear Science and Numerical Simulation* **14** (2009), 983–997.