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Generalized mixtures of Gaussian processes with an application to Bitcoin daily price analysis

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Abstract

In this paper, we apply Rice's formula, typically employed to calculate the mean number of upcrossings for stationary Gaussian processes, and extend it to the broader framework of generalized mixtures of Gaussian processes. The class of generalized mixtures of Gaussian distributions, recently introduced by [3], is highly comprehensive and includes significant subclasses such as mean mixtures of Gaussian, variance mixtures of Gaussian, mean-variance mixtures of Gaussian, and even scale mixtures of skew-Gaussian distributions. Consequently, our results hold substantial generality, enabling the extension of Rice's formula to address specific scenarios within these subclasses.

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1. Introduction

The expected number of upcrossings provides valuable information on the frequency with which a stochastic process exceeds a specified threshold. This is particularly relevant in fields such as finance, engineering, and insurance [13, 15, 23]. In structural engineering, upcrossings are frequently used to assess structural reliability, particularly in materials exposed to random loads over time. These crossings can indicate potential fatigue or risk of failure, helping engineers design structures with better resilience to varying stresses [24, 25]. In financial risk management, the frequency with which asset prices cross certain thresholds, such as strike prices in options, directly affects pricing, hedging strategies, and overall risk assessment [9]. Similarly, in insurance, analyzing upcrossings aids in estimating risk exposure and determining appropriate premiums by assessing the likelihood of extreme events, such as catastrophic losses or threshold-based claims. The applications of the upcrossing concept also extend to hydrology, as discussed by [8] and [20].

Upcrossings are closely related to the first passage time, defined as the time it takes for a stochastic process to reach a specified level for the first time [10]. When a process

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exhibits frequent upcrossings, it indicates that the process repeatedly hits and exceeds the threshold, implying shorter intervals between successive crossings. In this sense, the time between upcrossings can be interpreted as a "first passage time" for each cycle.

Consider $\{X_t = X(t);\ t > 0\}$ as a continuous stationary stochastic process, and for a fixed level u, let $N(u, X_t)$ be the number of points at which X_t up crosses u in the unit interval $(0, 1)$. Then, Rice demonstrated in [22] and [23] that the expected value of $N(u, X_t)$ can be obtained by

$$E(N(u, X_t)) = \int_0^{+\infty} x f_{X_t}(u) f_{X_t'|X_t=u}(x)\, dx = E\left(X_t^{'+} |X_t = u\right) f_{X_t}(u),$$

where X_t' represents the L^2-derivative of X_t and $X_t^{'+}$ is defined as the maximum of 0 and X_t'.

Consider the case of a stationary Gaussian process, i.e., when X_t is a standardized stationary Gaussian process with finite second spectral moment λ_2 , i.e., $Var\left(X_t'\right) = \lambda_2$. In such situations, from the result of [23], the above equation simplifies to

$$E(N(u, X_t)) = \frac{\sqrt{\lambda_2}}{2\pi} \exp\left(\frac{-u^2}{2}\right) = \frac{\sqrt{\lambda_2}}{\sqrt{2\pi}} \phi(u),$$

where $\phi(.)$ is the probability density function (PDF) of a standard Gaussian distribution. For a non-standard stationary Gaussian process, i.e., when for each t, $X_t{\sim}N(\mu, \sigma^2)$, the aforementioned equation becomes

$$E(N(u, X_t)) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} \phi\left(u; \mu, \sigma^2\right), \tag{1.1}$$

where $\phi(.; \mu, \sigma^2)$ denotes the PDF of $N(\mu, \sigma^2)$.

Notably, Rice's original derivation does not strictly require the process $X(t)$ to be fully stationary, which would mean that the process's statistical properties, such as mean, variance, and autocovariance, remain invariant under time shifts. Instead, Rice's formula can also be applied to processes with stationary increments, where the distribution of $X(t + s) - X(s)$ depends only on the time difference t, regardless of the starting point s. This weaker assumption of stationary increments is sufficient for many applications, as it captures the uniformity of changes over time. However, for the simplified and widely used version of the formula, the assumption of full stationarity is often adopted to facilitate analysis and interpretation.

One of the most noteworthy applications of Rice's formula is its association with the maximum tail distribution. If we define $M(t) = \max_{0 \le s \le t} X(s)$, then the upper Rice bound for the maximum tail can be expressed as follows (see [14]):

$$\Pr(M(t) > u) \le \Pr(X(0) \ge u) + tE(N(u, X_t)). \tag{1.2}$$

Specifically, in the case of a stationary Gaussian process, utilizing (1.1), this relationship can be written as

$$\Pr(M(t) > u) \le 1 - \Phi\left(\frac{u - \mu}{\sigma}\right) + t\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} \phi\left(u; \mu, \sigma^2\right).$$

Note that the preceding discussion has focused on upcrossings. However, similar results hold for downcrossings as well. Specifically, the mean number of downcrossings is also given by analogous formulae.

Most applications and generalizations of Rice's formula have traditionally focused on Gaussian processes. However, they have also been extended to encompass functions of Gaussian processes, such as lognormal processes [8] or χ^2 processes [12]. Furthermore, Klüppelberg and Rasmussen [11] have demonstrated analogous results for the class of multivariate generalized hyperbolic processes. Recently, Masoliver and Palassini [17] generalized Rice's formula for general Gaussian processes, removing restrictions on the independence of X_t and its derivative $X_t^{'}$, allowing them to follow from a bivariate Gaussian process. However, in this field, limited attention has been paid to analyzing datasets with asymmetrical distributions and data skewness. Due to the fact that in engineering, economics, and other applied fields, we encounter data with heavy tails and asymmetric distributions, addressing this topic will be of paramount importance.

Our primary objective in this paper is to extend Rice's formula to a highly versatile class of continuous mixtures of Gaussian processes, where for each t, X_t follows a generalized mixture of Gaussian (GMG) distribution. The GMG class, recently introduced by [3], is exceptionally comprehensive and encompasses many well-known and significant subclasses of continuous mixtures of Gaussian distributions, such as the mean mixture of Gaussian (MMG), variance mixture of Gaussian (VMG), mean-variance mixture of Gaussian (MVMG) and also scale mixtures of skew-Gaussian (SMSG) distributions. These classes of distributions encompass a wide range of essential distributions, including Student's t, skew-Gaussian (SG) and its extended versions, generalized hyperbolic (GH), Laplace, skew-*t* (ST), and skew-Gaussian generalized hyperbolic distributions, all as special cases. Hence, our results are highly general and can be applied to extend Rice's formula for specific cases involving MMG, VMG, MVMG, SMSG processes, and more.

2. Extending Rice's formula for GMG processes

In recent decades, various formulations have been discussed in the literature that involve continuous mixtures of Gaussian variables. These formulations allow a mixing variable to act on the mean [2,19], the variance, or both the mean and variance of a Gaussian variable [7,19]. This approach transforms these basic components from fixed constants to random quantities, introducing greater flexibility. As mentioned above, Arellano-Valle and Azzalini [3] proposed a highly general framework for continuous mixture distributions, known as the GMG distribution, which unifies all these formulations and can be defined as follows:

Definition 1. Let $X{\sim}N\left(0,1\right)$ be a standard Gaussian random variable independent of univariate random variables U and V with joint cumulative distribution function (CDF) $G_{U,V}$. Then, a random variable Y is said to have a GMG distribution if it has the stochastic representation as

$$Y{=}\xi+r\left(U,V\right)\gamma+s\left(U,V\right)\sigma X{=}\xi+R\gamma+S\sigma X, \tag{2.1}$$

where $r=r\left(u,v\right)$ is any real-valued function, $s=s\left(u,v\right)$ is a positive-valued function, and $\xi{\in}\mathbb{R}$, $\gamma{\in}\mathbb{R}$, and $\sigma{\in}\mathbb{R}^{+}$. For notational simplicity, we denote the random variables $r\left(U,V\right)$ and $s\left(U,V\right)$ by R and S, respectively. We denote the random vector with representation in (2.1) by $Y\sim GMG(\xi,\gamma,\sigma;G_{U,V})$.

From (2.1), if U and V have a joint PDF $g\left(u,v;\boldsymbol{\tau}\right)$, then an integral form of the PDF of $Y\sim GMG(\xi,\gamma,\sigma;G_{U,V})$ can be derived as

$$f_{GMG}\left(y;\xi,\gamma,\sigma;G_{U,V}\right)=\iint_{\mathbb{R}^2}\phi\left(y;\xi+r\gamma,s^2\sigma^2\right)g\left(u,v;\boldsymbol{\tau}\right)dudv.$$

Definition 2. (GMG process) Let $X=\{X\left(t\right);\ t>0\}$ be a standardized stationary Gaussian process in (2.1), and the random variables U and V be defined as in Definition 1. Then, the stochastic process

$$Y\left(t\right){=}\xi+R\gamma+S\sigma X\left(t\right), \tag{2.2}$$

is referred to as the GMG process with parameters (ξ,γ,σ) and mixing distribution $G_{U,V}$.

In this section, we aim to extend Rice's formula for the class of GMG processes. To achieve this, we introduce the concepts of univariate and bivariate weighted distributions, along with a key lemma, essential for subsequent derivations. The notion of univariate weighted distributions was first introduced by [21]. Suppose U be a univariate random variable with a PDF $f_U(u)$ and let $w(u)$ be a non-negative weight function such that $E\left(w(U)\right) < \infty$. Then, the PDF of the corresponding weighted random variable U_w is defined as

$$f_{U_w}(u) = \frac{w\left(u\right) f_U(u)}{E\left(w(U)\right)}.$$

Furthermore, the extension of weighted distributions to the bivariate case has been thoroughly explored by [16]. Let (U, V) be paired random variables with joint PDF $f_{U,V}(u, v)$ and a non-negative weight function $w(u, v)$ such that $E\left(w(U, V)\right)$ be finite. Then, the PDF of the corresponding weighted random variables $(U, V)_w$ is given by

$$f_{(U,V)_w}(u, v) = \frac{w\left(u, v\right) f_{U,V}(u, v)}{E\left(w(U, V)\right)}. \tag{2.3}$$

Now, let $(U, V)_s$ denote the special case where in (2.3), $w\left(u, v\right) = s\left(u, v\right)$ and $G_{(U,V)_s}$ denote the CDF of $(U, V)_s$. For this special case, we present the following lemma.

Lemma 1. Let U and V be bivariate random variables with a joint CDF $G_{U,V}$. Then, we have

$$E\left[S\phi\left(y;\ \xi + R\gamma, S^2\sigma^2\right)\right] = E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right),$$

where $r\left(U, V\right) = R$ and $s\left(U, V\right) = S$ and $f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right)$ is the PDF of $Y \sim$

$GMG(\xi, \gamma, \sigma; G_{(U,V)_s})$.

Proof. Let $Y \sim GMG(\xi, \gamma, \sigma; G_{(U,V)_s})$, then from the stochastic representation of the GMG distribution given by (2.1), we have

$$Y\left|((U, V)_s = (u, v)\right) \sim N\left(\xi + r\gamma, s^2\sigma^2\right),$$

and so from Bayes' rule the PDF of weighted random variables $(U, V)_s$ given $Y = y$ can be written as

$$f_{(U,V)_s|Y=y}(u, v) = \frac{f_{Y\left|(U,V)_s=(u,v)\right.}(y) f_{(U,V)_s}(u, v)}{f_Y\left(y\right)} = \frac{\phi\left(y;\ \xi + r\gamma, s^2\sigma^2\right) s f_{U,V}(u, v)}{E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right)}. \tag{2.4}$$

Therefore, we have

$$E\left[S\phi\left(y;\ \xi + R\gamma, S^2\sigma^2\right)\right] = \iint_{\mathbb{R}^2} s\phi\left(y;\ \xi + r\gamma, s^2\sigma^2\right) f_{U,V}(u, v)dudv$$

Substituting from (2.4), we get

$$\iint_{\mathbb{R}^2} s\phi\left(y;\ \xi + r\gamma, s^2\sigma^2\right) f_{U,V}(u, v)dudv = E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right) \times \iint_{\mathbb{R}^2} f_{(U,V)_s|Y=y}(u, v)dudv.$$

Since $f_{(U,V)_s|Y=y}(u, v)$ is the conditional PDF of $(U, V)_s$ given $Y = y$, we have

$$\iint_{\mathbb{R}^2} f_{(U,V)_s|Y=y}(u, v)dudv = 1,$$

and so we arrive at the desired result.

Theorem 1. Let Y_t be a GMG process with parameters (ξ, γ, σ) and mixing distribution $G_{U,V}$. Then, the mean number of upcrossings per unit time by the process Y_t is given by

$$E\left(N\left(y, Y_t\right)\right) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right)$$

Proof: From (2.2), $Y_t\left|(U = u, V = v)\right.$ is Gaussian process with mean $\xi + r\gamma$ and variance $s^2\sigma^2$. So, from (1.1), we have

$$E\left(N\left(y, Y_t\right)\left|(U = u, V = v)\right.\right) = \frac{\sqrt{\lambda_2}\sigma s}{\sqrt{2\pi}} \phi\left(y; \xi + r\gamma, s^2\sigma^2\right).$$

Using the result from Lemma 1, we have

$$\begin{aligned} E\left(N\left(y, Y_t\right)\right) &= E\left[E\left(N\left(y, Y_t\right)\left|(U = u, V = v)\right.\right)\right] = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(S\phi\left(y; \xi + R\gamma, S^2\sigma^2\right)\right) \\ &= \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right). \end{aligned}$$

which completes the proof.

As a direct result of Theorem 1 and (1.2), we can derive an upper bound for the maximum tail distribution in the case of a stationary GMG process as

$$\Pr\left(M\left(t\right) > y\right) \leq 1 - F_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right) + t\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(S\right) f_{GMG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right). \tag{2.5}$$

3. Some special cases

In this section, we employ the general results established in the previous section to address several specific sub-classes of GMG processes.

3.1. Mean mixture of Gaussian (MMG) process

The class of MMG process is a simplified form of (2.2) where $r\left(u, v\right) = u$ and $s\left(u, v\right) = 1$. Consequently, we can express it as

$$Y\left(t\right) = \xi + U\gamma + \sigma X\left(t\right). \tag{3.1}$$

Several distributional properties of the MMG class of distributions have been obtained by [19]. If U has a PDF, denoted by $f\left(u, \boldsymbol{\nu}\right)$, then the integration form for the PDF of the MMG distribution can be derived as

$$f_{MMG}\left(y; \xi, \gamma, \sigma; G_U\right) = \int_{\mathbb{R}} \phi\left(y;\ \xi + u\gamma, \sigma^2\right) f_U\left(u, \boldsymbol{\nu}\right) du. \tag{3.2}$$

Using the results in Theorem 1 by letting $r\left(u, v\right) = u$ and $s\left(u, v\right) = 1$, we can derive the mean number of upcrossings per unit time by the MMG process as presented in the following corollary.

Corollary 1. Let Y_t be an MMG process with parameters (ξ, γ, σ) and mixing distribution G_U. Then, we have

$$E\left(N\left(y, Y_t\right)\right) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} f_{MMG}\left(y; \xi, \gamma, \sigma; G_U\right). \tag{3.3}$$

Simply from (2.5), for a stationary MMG process, we can formulate an upper bound for the maximum tail distribution as follows:

$$\Pr\left(M\left(t\right)>y\right)\leq 1-F_{MMG}\left(y;\xi,\gamma,\sigma;G_U\right)+t\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}f_{MMG}\left(y;\xi,\gamma,\sigma;G_U\right). \tag{3.4}$$

3.1.1. Some noteworthy special cases. In this subsection, we present three notable special cases of MMG processes and proceed to calculate the mean number of upcrossings per unit time for these particular instances.

Rayleigh mean mixture of Gaussian (RMMG) process: Let U in (3.1) have a standard Rayleigh distribution with density $g_U\left(u\right)=ue^{-\frac{1}{2}u^2}I_{[0,+\infty]}\left(u\right)$. Then, we say that the process $Y\left(t\right)$ with stochastic representation in (3.1) follows a RMMG process. Using (3.3), we have

$$E\left(N\left(y,Y_t\right)\right)=\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}f_{RMMG}\left(y;\xi,\gamma,\sigma\right),$$

where $f_{RMMG}\left(.;\xi,\gamma,\sigma\right)$ can be derived from (3.2) by letting U be a standard Rayleigh random variable as follows

$$f_{RMMG}\left(y;\xi,\gamma,\sigma\right)=\sqrt{\frac{2\pi}{1+\eta^2}}\frac{1}{\sigma^*}\phi\left(\frac{y-\xi}{\sigma^*}\right)\Phi\left(\eta\left(\frac{y-\xi}{\sigma^*}\right)\right)\left[\frac{\phi\left(\eta\left(\frac{y-\xi}{\sigma^*}\right)\right)}{\Phi\left(\eta\left(\frac{y-\xi}{\sigma^*}\right)\right)}+\eta\left(\frac{y-\xi}{\sigma^*}\right)\right],$$

where

$$\sigma^{*2}=\sigma^2+\gamma^2\text{and }\eta=\frac{\gamma}{\sigma}. \tag{3.5}$$

Skew-Gaussian (SG) process: The SG process is a significant special subclass of MMG processes when U in (3.1) follows a standard positive half-Gaussian distribution. If Y_t represents an SG process with parameters $\left(\xi,\gamma,\sigma\right)$, the mean number of upcrossings per unit time can be directly obtained from (3.3) as

$$E\left(N\left(y,Y_t\right)\right)=\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}f_{SG}\left(y;\xi,\gamma,\sigma\right), \tag{3.6}$$

where $f_{SG}\left(y;\xi,\gamma,\sigma\right)$ is the PDF of the SG distribution introduced by [4], given by

$$f_{SG}\left(y;\xi,\gamma,\sigma\right)=\frac{2}{\sigma^*}\phi\left(\frac{y-\xi}{\sigma^*}\right)\Phi\left(\eta\left(\frac{y-\xi}{\sigma^*}\right)\right),$$

with σ^{*2} and η defined as in (3.5).

Gamma mean mixture of Gaussian (GMMG) process: Suppose U in (3.1) follows a Gamma distribution with a shape parameter α and a rate parameter equal to 1. In this scenario, we refer to the stochastic process Y_t as a GMMG process with parameter $(\xi,\gamma,\sigma,\alpha)$. Employing the outcome from (3.3), we arrive at the following expression

$$E\left(N\left(y,Y_t\right)\right)=\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}f_{GMMG}\left(y;\xi,\gamma,\sigma,\alpha\right),$$

where $f_{GMMG}\left(.;\xi,\gamma,\sigma,\alpha\right)$ can be obtained from (3.2) after certain calculations as

$$\begin{aligned} f_{GMMG}\left(y;\xi,\gamma,\sigma,\alpha\right) &= \frac{\frac{1}{\sigma}\phi\left(\frac{y-\xi}{\sigma}\right)}{\Gamma\left(\alpha\right)\sqrt{1+\eta^2}}\zeta^{-1}\left(\left(\frac{y-\xi}{\sigma}\right)-\frac{1}{\eta}\right)\times \\ & MLTN\left(\frac{1}{\eta}\left(\frac{y-\xi}{\sigma}\right)-\frac{1}{\eta^2},\frac{1}{\eta^2},\alpha-1\right), \end{aligned}$$

where $\zeta(x) = \frac{\phi(\mathbf{x})}{\Phi(x)}$ and $MLTN(\mu, \sigma^2, n)$ represents the n-th moment of a truncated Gaussian distribution with location μ, scale σ and truncation set $(0, +\infty)$.

It should be noted that when we set $\alpha = 1$, the gamma distribution simplifies to the standard exponential distribution. In this case, we can refer to the associated random process Y_t as the exponential mean mixture Gaussian process. The distributional characteristics of this class have been thoroughly investigated by [19].

It is worth noting that in order to derive an upper bound for the maximum tail distribution in these three cases, it is sufficient to substitute the MMG distribution in (3.4) with the appropriate distribution.

3.2. Mean-variance mixture of Gaussian (MVMG) process

The class of MVMG distributions provides a meaningful alternative to the Gaussian distribution, particularly for datasets exhibiting high skewness and kurtosis, often seen in economic, engineering, and financial data. A notable subclass within MVMG is the generalized hyperbolic (GH) distribution, which includes widely recognized distributions such as Student's t, Laplace, and variance gamma. For further information on MVMG distributions and their applications (see [1,6]).

In this subsection, we intend to define MVMG processes as special cases of GMG processes. To achieve this, it is sufficient to set $r(U,V) = V$, $s(U,V) = V^{\frac{1}{2}}$ in (2.2). Hence, we arrive at the formulation

$$Y(t) = \xi + V\gamma + V^{\frac{1}{2}}\sigma X(t). \tag{3.7}$$

Using the results in Theorems 1 by letting $r(u,v) = v$ and $s(u,v) = v^{\frac{1}{2}}$, we can derive the mean number of upcrossings per unit time for a MVMG process, as presented in the following corollary.

Corollary 2. Let Y_t be an MVMG process with parameters (ξ, γ, σ) and mixing distribution G_V. Then, we have

$$E(N(y, Y_t)) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(V^{\frac{1}{2}}\right) f_{MVMG}(y; \xi, \gamma, \sigma; G_{V_s}),$$

where $s = s(v) = v^{\frac{1}{2}}$.

3.2.1. Special case of GH process. A particularly significant formulation arises when V in (3.7) follows a Generalized Inverse Gaussian (GIG) distribution with density

$$g(v; \kappa, \chi, \psi) = \left(\frac{\psi}{\chi}\right)^{\frac{\kappa}{2}} \frac{v^{\kappa-1}}{2K_\kappa\left(\sqrt{\chi\psi}\right)} \exp\left(-\frac{1}{2}\left(v^{-1}\chi + v\psi\right)\right), \qquad v > 0,$$

where $K_\lambda(\cdot)$ represents the modified Bessel function of third kind with index $\kappa \in \mathbb{R}$. Parameters χ and ψ must satisfy certain conditions: $\chi \geq 0$, $\psi > 0$ if $\kappa > 0$, $\psi \geq 0$, $\chi > 0$ if $\kappa < 0$, and $\chi > 0$, $\psi > 0$ if $\kappa = 0$. Going forward, we'll denote $V \sim GIG(\kappa, \chi, \psi)$ for the random variable V following the aforementioned GIG distribution. In this context, the moments of V can be expressed as

$$E(V^m) = \left(\frac{\chi}{\psi}\right)^{\frac{m}{2}} \frac{K_{\kappa+m}\left(\sqrt{\chi\psi}\right)}{K_\kappa\left(\sqrt{\chi\psi}\right)}, \quad m \in \mathbb{R}. \tag{3.8}$$

In this scenario, we refer to the stochastic process Y_t as a GH process with parameter $(\xi, \gamma, \sigma, \kappa, \chi, \psi)$. The PDF of GH distribution is given for $y \in \mathbb{R}$ by

$$f_{GH}\left(y;\xi,\gamma,\sigma,\kappa,\chi,\psi\right) = c \times \frac{K_{\kappa-\frac{1}{2}}\left(\sqrt{\left(\chi+\left(\frac{y-\xi}{\sigma}\right)^2\right)\left(\psi+\frac{\gamma^2}{\sigma^2}\right)}\right)}{\left(\sqrt{\left(\chi+\left(\frac{y-\xi}{\sigma}\right)^2\right)\left(\psi+\frac{\gamma^2}{\sigma^2}\right)}\right)^{\frac{1}{2}-\kappa}} \exp\left(\gamma\frac{y-\xi}{\sigma^2}\right),$$

where

$$c = \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\kappa}{2}}\left(\psi+\frac{\gamma^2}{\sigma^2}\right)^{\frac{1}{2}-\kappa}}{\sqrt{2\pi}\sigma K_\kappa\left(\sqrt{\chi\psi}\right)}.$$

Klüppelberg and Rasmussen [11] originally calculated the mean upcrossings per unit time for the GH process in a multivariate scenario. Here, we utilize our previously outlined methodology to once again extract their findings, focusing this time on the univariate case. To achieve this, consider $V \sim GIG\left(\kappa,\chi,\psi\right)$. Consequently, we have $V_s \sim GIG\left(\kappa+\frac{1}{2},\chi,\psi\right)$. By utilizing this relationship alongside the insights from Corollary 2 and (3.8), we are able to deduce the mean number of upcrossings per unit time for the Y_t process as

$$E\left(N\left(y,Y_t\right)\right) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}\left(\frac{\chi}{\psi}\right)^{\frac{1}{4}}\frac{K_{\kappa+\frac{1}{2}}\left(\sqrt{\chi\psi}\right)}{K_\kappa\left(\sqrt{\chi\psi}\right)} f_{GH}\left(y;\xi,\gamma,\sigma,\kappa+\frac{1}{2},\chi,\psi\right).$$

The conclusions drawn from this subsection can be utilized to deduce an upper bound for the maximum tail distribution of the MVMG processes. For instance, in the case of a stationary GH process, we can formulate an upper bound for the maximum tail distribution as follows:

$$\begin{aligned}\Pr\left(M\left(t\right)>y\right) \quad \leq \quad & 1 - F_{GH}\left(y;\xi,\gamma,\sigma,\kappa,\chi,\psi\right) + t\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}\left(\frac{\chi}{\psi}\right)^{\frac{1}{4}} \times \\ & \frac{K_{\kappa+\frac{1}{2}}\left(\sqrt{\chi\psi}\right)}{K_\kappa\left(\sqrt{\chi\psi}\right)} f_{GH}\left(y;\xi,\gamma,\sigma,\kappa+\frac{1}{2},\chi,\psi\right).\end{aligned}$$

Remark. Variance mixtures of Gaussian processes can be obtained from the VMMG process simply by setting $\gamma = 0$, and so we do not present the corresponding expressions here for the sake of brevity.

3.3. Scale mixtures of skew-Gaussian (SMSG) process

Branco and Dey [5] were the first to introduce a concept that closely resembles SMSG distributions. This class, being a scale mixture of skewed versions of Gaussian distributions, offers remarkable flexibility for modeling data with significant skewness and heavy tails. In this subsection, our aim is to introduce the class of SMSG processes as a special case of GMG processes. For this purpose, let $Y\left(t\right)$ be defined as

$$Y\left(t\right) = \xi + V^{\frac{1}{2}}Z\left(t\right),$$

where $Z\left(t\right)$ represents an SG process with parameter γ, σ and $\xi = 0$. By combining this representation with the additive form of SG processes, expressed as $Z\left(t\right) = U\gamma + \sigma X\left(t\right)$, where U is a standard half-Gaussian variable, we can write

$$Y\left(t\right) = \xi + UV^{\frac{1}{2}}\gamma + V^{\frac{1}{2}}\sigma X\left(t\right) \tag{3.9}$$

which falls into the category given by (2.2) with $r(u,v) = uv^{\frac{1}{2}}$ and $s(u,v) = v^{\frac{1}{2}}$. By employing the results from Theorems 1 and selecting $r(u,v) = uv^{\frac{1}{2}}$ and $s(u,v) = v^{\frac{1}{2}}$, we can calculate the mean number of upcrossings per unit time for an SMSG process.

Corollary 3. Let Y_t be an SMSG process with parameters (ξ, γ, σ) and mixing distribution $G_{U,V}$. Then, we have

$$E(N(y, Y_t)) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} E\left(V^{\frac{1}{2}}\right) f_{SMSG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right),$$

where $s = s(v) = v^{\frac{1}{2}}$, and $f_{SMSG}\left(y; \xi, \gamma, \sigma; G_{(U,V)_s}\right)$ is the PDF of the SMSG distribution with parameters (ξ, γ, σ) and the mixing distribution $G_{(U,V)_s}$.

Skew-Gaussian generalized hyperbolic (SGGH) process: The concept of SGGH distribution was initially introduced by [26] and represents a flexible family of distributions that subsumes several well-known parametric distributions as special cases, including the SG and ST distributions. Using the results of [26], we obtain the PDF of the univariate SGGH distribution as

$$\begin{aligned} f_{SGGH}(y; \xi, \gamma, \sigma, \kappa, \chi, \psi) \quad &= \quad \frac{2}{\sigma^*} f_{SYGH}\left(\left(\frac{y-\xi}{\sigma^*}\right); \kappa, \chi, \psi\right) \times \\ & F_{SYGH}\left(\eta\left(\frac{y-\xi}{\sigma^*}\right); \kappa - \frac{1}{2}, \sqrt{\left(\chi + \left(\frac{y-\xi}{\sigma^*}\right)^2\right)}, \psi\right), \end{aligned}$$

where σ^{*2} and η are defined as in (3.5) and $f_{SYGH}(.; \xi, \sigma, \kappa, \chi, \psi)$ and $F_{SYGH}(.; \xi, \sigma, \kappa, \chi, \psi)$ denote the PDF and CDF of the symmetric generalized hyperbolic distribution with parameter $(\xi, \sigma, \kappa, \chi, \psi)$, respectively, such that

$$f_{SYGH}(z; \kappa, \chi, \psi) = \frac{\psi^{\frac{1}{4}}}{\chi^{\frac{\kappa}{2}}\sqrt{2\pi}K_\kappa\left(\sqrt{\chi\psi}\right)} \frac{K_{\kappa-\frac{1}{2}}\left(\sqrt{\psi\left(\chi + z^2\right)}\right)}{\left(\sqrt{\left(\chi + z^2\right)}\right)^{\frac{1}{2}-\kappa}}.$$

To introduce the SGGH process, let $V \sim GIG(\kappa, \chi, \psi)$ in (3.9). Under this assumption, the stochastic process $Y(t)$, as represented in (3.9), conforms to an SGGH process. Since in this scenario, U and V are independent, we have $(U,V)_s = (U, V_s)$ where $V_s \sim GIG\left(\kappa + \frac{1}{2}, \chi, \psi\right)$. Using this relationship, along with insights from Corollary 3 and (3.8), the mean number of upcrossings per unit time for the Y_t process can be deduced as

$$E(N(y, Y_t)) = \frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}} \left(\frac{\chi}{\psi}\right)^{\frac{1}{4}} \frac{K_{\kappa+\frac{1}{2}}\left(\sqrt{\chi\psi}\right)}{K_\kappa\left(\sqrt{\chi\psi}\right)} f_{SGGH}\left(y; \xi, \gamma, \sigma, \kappa + \frac{1}{2}, \chi, \psi\right).$$

Skew-t (ST) process: A notable member of this class, which has garnered significant attention since 2001 both in theoretical and applied contexts, is the ST process. This process arises when V follows an inverse Gamma distribution with shape and scale parameters $\left(\frac{\nu}{2}\right)$, denoted as $V \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$. In this scenario, once again, U and V are independent. Thus, we have $(U,V)_s = (U, V_s)$, and additionally, we can demonstrate that $\frac{\nu-1}{\nu}V_s$ follows an inverse-gamma distribution, specifically $\frac{\nu-1}{\nu}V_s \sim IG\left(\frac{\nu-1}{2}, \frac{\nu-1}{2}\right)$. Utilizing this relationship and the insights from Corollary 3, along with the expression

$$E\left(V^{\frac{1}{2}}\right) = \frac{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)},$$

we can derive the mean number of upcrossings per unit time for the Y_t process as

$$E\left(N\left(y,Y_t\right)\right)=\frac{\sigma\sqrt{\lambda_2}}{\sqrt{2\pi}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}f_{ST}\left(y;\xi,\sqrt{\frac{\nu}{\nu-1}}\gamma,\sqrt{\frac{\nu}{\nu-1}}\sigma,\nu-1\right), \tag{3.10}$$

where $f_{ST}\left(.;\xi,\gamma,\sigma,\nu\right)$ is PDF of the ST distribution with parameters (ξ,γ,σ,ν). It can be expressed as

$$f_{ST}\left(y;\xi,\gamma,\sigma,\nu\right)=\frac{2}{\sigma^*}t\left(\left(\frac{y-\xi}{\sigma^*}\right);\nu\right)T\left(\eta\left(\frac{y-\xi}{\sigma^*}\right)\sqrt{\frac{\nu+1}{\nu+\left(\frac{y-\xi}{\sigma^*}\right)^2}};\nu+1\right),$$

where σ^* and η are as defined before and $t\left(.;\nu\right)$ and $T\left(.;\nu\right)$ represent the PDF and CDF of the standardized Student's t-distribution with ν degrees of freedom, respectively.

As previously mentioned in the previous sections, the findings of this subsection can also be used to derive an upper bound for the maximum tail distribution of SMSG processes, particularly in specific instances of these processes.

4. Simulation study

To provide a more complete validation of the theoretical results, we conducted an extensive simulation study. We considered a multiple set of parameters and increasing sample sizes n to investigate the limiting behavior of the average number of cross sections and the precision of the theoretical formula. The processes studied include the SG and ST processes. For the simulation study, we considered the following parameter sets and steps:

- **Parameter sets:**
  - **SG process:** All combinations of $\xi=0.5$, 1.0; $\gamma=0.75$, 1.00, 1.50; $\sigma=1.0$, 2.5, 4.0; resulting in 18 different cases.
  - **ST process:** All combinations of $\xi=0.5,1.0$; $\gamma=0.75$, 1.50; $\sigma=2.5$, 4.0; $\nu=4$, 10, 15; resulting in 24 different cases.
- For each parameter set, $N=100,000$ sample paths each with $n=1000$ points were simulated over the interval $[0,1]$.
- Theoretical upcrossings were computed for each case using (3.6) and (3.10).

Figure 1 shows 25 simulated paths for SG and ST processes with parameters $\xi=1$, $\gamma=1.5$, $\sigma=2.5$, $\nu=10$, and an upcrossing level $b_0=5$, respectively. Tables 1 and 2 show the comparison of the simulated and theoretical average number of upcrossings for selected parameter sets. To determine the effect of the number of points in each simulated path, several values of $n=100$, 200, 500, 1000 were examined for one parameter set of each distribution. This analysis ensures that $n=1000$ points are sufficient for simulating the sample paths over $[0,1]$. Table 3 summarizes the results.

5. Real data application: Bitcoin daily closing prices

Bitcoin prices, known for their high volatility and dynamic nature, serve as an excellent data set to validate advanced statistical models. Their distinctive features, such as heavy tails, skewness, and volatility clustering, present unique challenges that require robust and sophisticated modeling frameworks. Given these characteristics, we analyze the daily closing prices of Bitcoin using the SG process, a flexible model capable of capturing the complex statistical properties of financial time series.

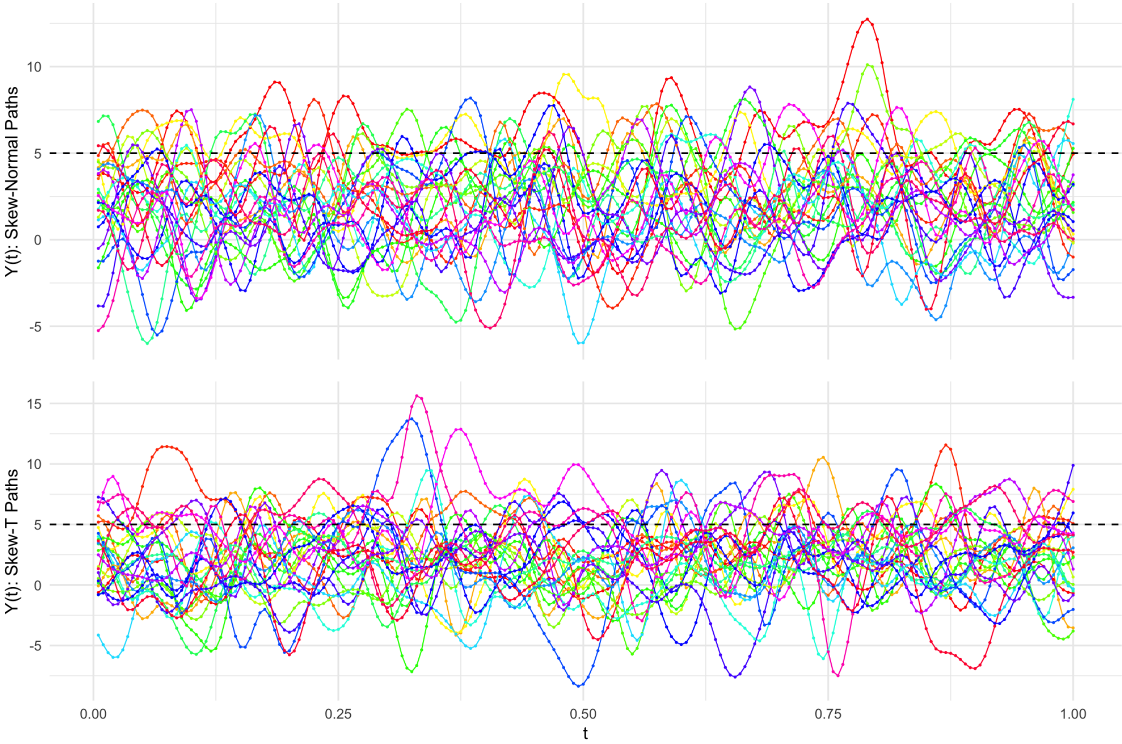

Figure 1. Time series plots of 25 simulated paths with $n = 200$ sample points. Top: SG process. Bottom: ST process. A horizontal line at the upcrossing level $b_0 = 5$ is included to illustrate upcrossings.

Table 1. Comparison of simulated and theoretical average number of level 2.5 upcrossings of SG process for 18 different parameter sets

ξ	γ	σ	$E(N)$	$\hat{E}(N)$	ξ	γ	σ	$E(N)$	$\hat{E}(N)$
0.5	0.75	1.0	3.133	3.135	1.0	0.75	1.0	5.057	5.057
		2.5	6.716	6.716			2.5	7.346	7.347
		4.0	7.441	7.434			4.0	7.710	7.707
	1.00	1.0	3.814	3.810		1.00	1.0	5.486	5.476
		2.5	6.917	6.926			2.5	7.442	7.440
		4.0	7.526	7.523			4.0	7.749	7.755
	1.50	1.0	4.542	4.545		1.50	1.0	5.583	5.585
		2.5	7.115	7.113			2.5	7.427	7.418
		4.0	7.608	7.598			4.0	7.740	7.737

5.1. Descriptive statistics and trend analysis

The data, obtained from Yahoo Finance, cover the period from March 1, 2024, to October 31, 2024, comprising 245 observations. To facilitate analysis, the data set was divided into five groups of 49 consecutive samples, each group being treated as an observed sample path of the SG process. Descriptive statistics, skewness, and kurtosis values for the entire series and its five groups are presented in Tables 4, 5, and 6. These results provide insights into the distributional properties and variability of Bitcoin prices. Figure 2 illustrates the time series for the entire dataset, highlighting overall trends and fluctuations. Figure 4 shows the five divided groups along with their mean series. To evaluate stationarity and trends, we conducted a Mann-Kendall test on the mean series. This approach aligns

Table 2. Comparison of simulated and theoretical average number of level 2.5 upcrossings of ST process for 24 different parameter sets

ξ	γ	σ	ν	$E(N)$	$\hat{E}(N)$	ξ	γ	σ	ν	$E(N)$	$\hat{E}(N)$
0.5	0.75	2.5	4	6.720	6.718	1	0.75	2.5	4	7.307	7.306
			10	6.718	6.723				10	7.330	7.323
			15	6.717	6.712				15	7.335	7.335
		4.0	4	7.422	7.416			4.0	4	7.689	7.685
			10	7.433	7.431				10	7.701	7.689
			15	7.435	7.426				15	7.704	7.697
	1.50	2.5	4	7.009	7.023		1.50	2.5	4	7.324	7.324
			10	7.071	7.072				10	7.384	7.385
			15	7.085	7.078				15	7.398	7.404
		4.0	4	7.550	7.550			4.0	4	7.692	7.689
			10	7.584	7.581				10	7.720	7.714
			15	7.592	7.593				15	7.726	7.724

Table 3. Limiting behaviour of simulated average of number of level $b = 2.5$ upcrossings for SG and ST with parameters $\xi = 1$, $\gamma = 1.5$, $\sigma = 2.5$, $\nu = 10$ with multiple n values.

	$E(N)$	$\hat{E}(N)$				
n	-	50	100	200	500	1000
SG	7.427	6.771	7.255	7.383	7.413	7.425
ST	7.384	6.723	7.213	7.335	7.372	7.385

Table 4. Descriptive statistics of Bitcoin prices (in USD).

Statistic	Part 1	Part 2	Part 3	Part 4	Part 5	All
Min.	61276.69	58254.01	55849.11	53948.75	58192.51	53948.75
1st Qu.	65315.12	63113.23	60320.14	57648.71	62089.95	60811.28
Median	68300.09	65231.58	64118.79	59119.48	63394.84	64118.79
Mean	67443.13	65519.61	63380.25	60005.06	64542.47	64178.10
3rd Qu.	69702.15	68296.22	66490.30	61415.07	67361.41	67612.72
Max.	73083.50	71448.20	69647.99	68255.87	72720.49	73083.50
Std Dev	3080.85	3337.66	3939.58	3683.16	3474.62	4278.94

Table 5. Skewness analysis of Bitcoin prices.

Statistic	Part 1	Part 2	Part 3	Part 4	Part 5	All
Skewness	-0.3042	-0.1271	-0.2799	0.6016	0.3695	-0.1962
P-value	0.3387	0.6857	0.3777	0.0688	0.2485	0.2012

Table 6. Kurtosis analysis of Bitcoin prices.

Statistic	Part 1	Part 2	Part 3	Part 4	Part 5	All
Kurtosis	2.0391	2.1309	1.9753	2.7924	2.4059	2.2595
P-value	0.0363	0.0857	0.0170	0.9544	0.4076	0.0003

with the continuity assumptions of the underlying stochastic model, ensuring that trend analysis avoids bias from finite discontinuities between segments. The Mann-Kendall test result ($\tau = 0.163$, $p = 0.0997$) provides weak evidence against stationarity, supporting the suitability of the SG process for these data.

Figure 2. Time series of Bitcoin prices from March 1, 2024, to October 31, 2024.

5.2. Fitting the SG process

To validate the applicability of the SG process, we performed maximum likelihood estimation (MLE) on the five observed sample paths. The estimated parameters of the model were $\hat{\xi} = 5.9671$, $\hat{\gamma} = 0.5119$, and $\hat{\sigma}^2 = 0.1235$. These parameters were used to compute the theoretical expected number of upcrossings of a threshold value $b_0 = 6.8$. Figure 3 shows the time series of these groups with a horizontal line at $b_0 = 6.8$, representing the upcrossing threshold.
The empirical estimate of the expected number of upcrossings was $\hat{E}(N) = 3.211$. Using the fitted parameters, the theoretical expected number of upcrossings was calculated as $\hat{E}(N) = 3.2$, closely matching the empirical result. This agreement between theoretical and empirical results highlights the suitability of the model to capture key statistical features of the Bitcoin price dynamics.

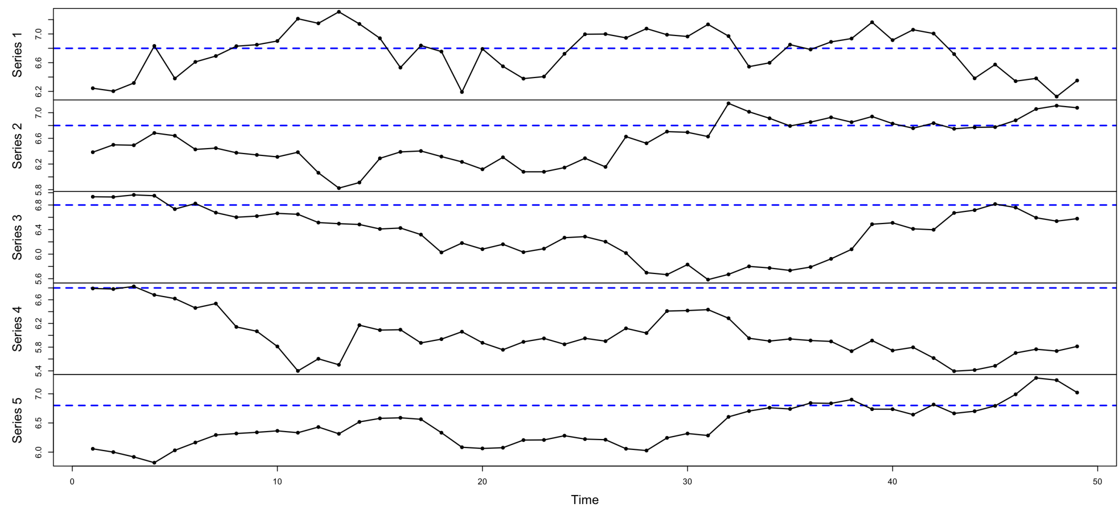

Figure 3. Time series plots of the 5 groups of bitcoin daily closing prices with $b_0 = 6.8$ as the upcrossing threshold.

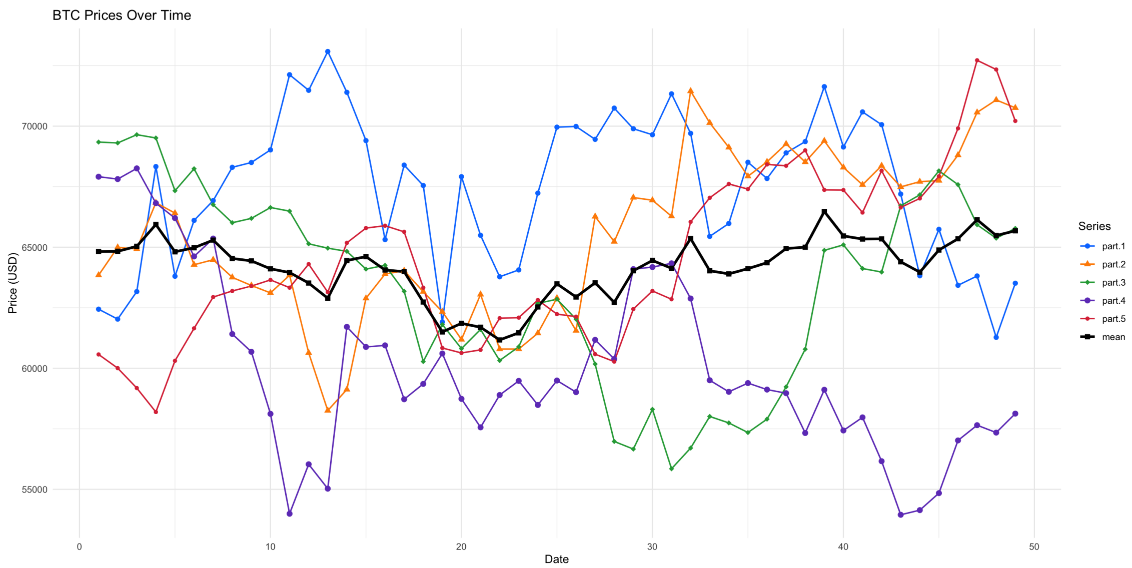

Figure 4. Time series of five parts of bitcoin prices and their mean series.

6. Concluding remarks

In this paper, we have extended the well-known Rice formula to obtain precise expressions for the mean number of upcrossings per unit time within the rich family of GMG processes, which in particular includes several significant subclasses of continuous mixtures of Gaussian processes, such as MMG, MVMG, and SMSG processes. Future directions for this work include extending the upcrossing framework to multivariate processes, enabling the study of simultaneous threshold crossings in multiple correlated processes, which is particularly relevant in portfolio risk analysis and multisystem reliability. Furthermore, investigating the relationship between upcrossings and first passage time for non-stationary or time-dependent processes could provide deeper insights into the dynamics of extreme events, thereby broadening the applicability of the proposed framework across diverse fields.

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References

[1] K. Aas and I.H. Haff, *The generalized hyperbolic skew Student's t-distribution*, J. Financ. Econ. **4**, 275-309, 2006.

[2] M. Abdi, M. Madadi, N. Balakrishnan and A. Jamalizadeh, *Family of mean-mixtures of multivariate normal distributions: Properties, inference and assessment of multivariate skewness*, J. Multivar. Anal. **181**, 104679, 2021.

[3] R.B. Arellano-Valle and A. Azzalini, *A formulation for continuous mixtures of multivariate Gaussian distributions*, J. Multivar. Anal. **185**, 104780, 2021.

[4] A. Azzalini, *A class of distributions which includes the Gaussian ones*, Scand. J. Stat. **12**, 171–178, 1985.

[5] M.D. Branco and D.K. Dey, *A general class of multivariate skew-elliptical distributions*, J. Multivar. Anal. **79(1)**, 99–113, 2001.

[6] O. Barndorff-Nielsen, J. Kent and M. Sorensen, *Normal variance-mean mixtures and Z distributions*, Int. Stat. Rev. **50**, 145–159, 1982.

[7] M.L. Bianchi and G.L. Tassinari, *Estimation for multivariate normal rapidly decreasing tempered stable distributions*, J. Stat. Comput. Simul. **94(1)**, 103–125, 2024.

[8] A.F. Desmond and B.T. Guy, *Crossing theory for non-Gaussian stochastic processes with an application to hydrology*, Water Resour. Res. **27(10)**, 2791–2797, 1991.

[9] P. Embrechts, C. Kluppelberg and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, New York, 1997.

[10] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences*, 4th Edition, Springer, New York, 2009.

[11] C. Klüppelberg and M.G. Rasmussen, *Outcrossings of safe regions by generalized hyperbolic processes*, Stat. Probab. Lett. **83**, 2197–2204, 2013.

[12] G. Lindgren, *Values and crossings for the χ^2-process and other functions of multidimensional Gaussian processes, with reliability applications*, Adv. Appl. Probab. **12(3)**, 746–774, 1980.

[13] G. Lindgren, *Stationary Stochastic Processes: Theory and Applications, In: Texts in Statistical Science*, Chapman & Hall/CRC, 2012.

[14] G. Lindgren, *Gaussian Integrals and Rice Series in Crossing Distributions–to Compute the Distribution of Maxima and Other Features of Gaussian Processes*, Stat. Sci. **34**, 100–128, 2019.

[15] G. Lindgren and I. Rychlik, *Slepian Models and Regression Approximations in Crossing and Extreme Value Theory*, Int. Stat. Rev. **59**, 195-225, 1991.

[16] M. Mahfoud and G.P. Patil, *On weighted distributions, In G. Kallianpur et al. (eds.) Statistics and Probability Essays in Honor of C. R. Rao*, Amsterdam: North Holland, 479-492, 1982.

[17] J. Masoliver and M. Palassini, *Counting of level crossings for inertial random processes: Generalization of the Rice formula*, Phys. Rev. E. **107**, 024-111, 2023.

[18] M. Naderi, W.L. Hung, T.I. Lin and A. Jamalizadeh, *A novel mixture model using the multivariate normal mean–variance mixture of Birnbaum Saunders distributions and its application to extrasolar planets*, J. Multivar. Anal. **171**, 126-138, 2019.

[19] H. Negarestani, A. Jamalizadeh, S. Shafiei and N. Balakrishnan,*Mean mixtures of Gaussian distributions: properties, inference and application*, Metrika. **82**, 501-528, 2019.

[20] C. F. Nordin and D. M. Rosbjerg, *Applicationsof crossingtheory in hydrology*, Bull. Int. Assoc. Sci. Hydrol. **15**, 27-43, 1970.

[21] C. R. Rao, *On discrete distributions arising out of methods of ascertainment*, Sankhya A. **27**, 311-324, 1965.

[22] S.O. Rice, *Mathematical analysis of random noise*, Bell Syst. Tech. J. **23**, 282–332, 1944.

[23] S.O. Rice, *Mathematical analysis of random noise*, Bell Syst. Tech. J. **24**, 46-156, 1945.

[24] I. Rychlik, *Note on Cycle Counts in Irregular Loads*, Fatigue Fract. Eng. Mater. Struct. **16**, 377-390, 1993.

[25] I. Rychlik, *On Some Reliability Applications of Rice Formula for Intensity of Level Crossings*, Extremes. **3**, 331-348, 2000.

[26] F. Vilca, N. Balakrishnan and C. Zeller, *Multivariate Skew-Normal Generalized Hyperbolic distribution and its properties*, J. Multivar. Anal. **128**, 73-85, 2014.