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A Robust Quintic Hermite Collocation Method for One-Dimensional Heat Conduction Equation

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Abstract

In this work, a new robust numerical solution scheme constructed on Quintic Hermite Collocation Method (QHCM) utilizing the traditional Crank-Nicolson type approximation technique is developed for solving 1D heat conduction equation with certain initial and boundary conditions which is mostly handled as a prototype equation to support the reliability of many proposed new numerical methods. All temporal and spatial quantities in the equation are fully discretized using a usual Crank-Nicolson type finite difference approximation and a QHCM, respectively. In obtaining the present scheme, all the roots of the fourth degree Legendre and Chebyshev polynomials shifted to the unit interval are used as suitable inner collocation points. The obtained results from the developed scheme are found to be good enough and better than those from other schemes encountered in the literature. The scheme is also shown to be unconditionally stable by Fourier stability test.

1. Introduction

The considered problem in this study consists of one-dimensional heat conduction equation

$$\alpha^2 u_{xx} - u_t = 0, \quad t > 0, x \in [x_l, x_r] \tag{1.1}$$

given by the appropriate initial condition

$$u(x, t_{initial}) = f(x) \tag{1.2}$$

and homogenous Dirichlet boundary conditions

$$u(x_l, t) = u(x_r, t) = 0, \quad t > 0 \tag{1.3}$$

where $t_{initial} = 0$, $x_l = 0$, $x_r = l$, α and $f(x)$ respectively stand for the thermal diffusivity coefficient and a smooth function to be prescribed in the process of numerical calculations. The initial and boundary value (IBV) problem governed by Eqs.(1.1)-(1.3) is generally considered as one of the most outstanding Partial Differential Equations (PDEs) appearing especially in physics and engineering mathematics. The current problem is among the widely-known second order linear partial differential equation. This problem illustrates the fact that heat equation defines irreversible process and also at the same time presents a distance between previous and the next steps. Those equations generally arise in various areas of engineering and science to describe the variation of the temperature on a predefined solution domain over a given time period. For more information about the characteristics of various heat equations, the reader may refer to Refs. [1, 2] and references therein.

Since the classical heat conduction equation (1.1) with various types of IBVs is handled by many researchers as a pioneer prototype test problem in the first applications of many new numerical techniques, there exist many proposed numerical techniques in the literature to compute its approximate solutions. For example, some of their frequently used are finite difference method [3–8], finite element

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method [9–14], wavelet method [15], spectral method [16]. To our knowledge, all studies constructed on finite elements method available in the literature have used one of the usual polynomial, trigonometric and exponential functions as B-spline bases. But recently, Kutluay *et al*. [17] have successfully applied collocation finite element method with cubic Hermite basis functions to generate more precise approximate numerical solutions of the heat conduction initial and boundary problem. There are also recently published and related articles with the presented method, problem and the equations such as given in Refs. [18–20] and therein. In this study, in order to obtain much more accurate and precise approximate solutions for the IBV problem, a new effective numerical scheme basically constructed on collocation finite element method using quintic Hermite spline basis functions, not the usual basis mentioned above, have been developed. The rest of the article is structured as follows: In Section 2, the solution domain of the problem is firstly divided into uniform partition in spatial and temporal directions. Then, the handled problem is fully discretized using Crank-Nicolson like difference approximation for the time quantities and QHCM utilizing the shifted roots of Chebyshev and Legendre polynomials as inner collocation points for the space quantities. Additionally, utilizing the initial and boundary conditions, the initial vector to be needed for starting the recursive scheme is also constructed. Thus, the heat problem is converted into a solvable system consisting of algebraic equations. In Section 3, it is shown that the scheme is unconditionally stable by using von-Neumann stability test for the modest values of spatial and temporal step sizes. In Section 4, the developed numerical scheme is used to obtain the approximate solutions of the experimental problem, and the computed results are compared with those of other researchers. In Section 5, this paper with a brief conclusion and future works is summarized.

2. Quintic Hermite Collocation Method

Throughout the manuscript, the 1D heat conduction equation which is widely given by Eq. (1.1) by the appropriate initial (1.2) and boundary conditions (1.3) is going to be handled. For generating the approximate numerical solutions of the problem, the finite element collocation method with quintic Hermite base functions is selected as a useful and powerful tool. To be able to apply this method, the solution domain $[x_l,x_r]\times[0,t_f]$ of the problem should be discretized. For this purpose, the spatial interval $[x_l,x_r]$ is divided into M equal width finite subintervals by means of the mesh points $x_j,\ j=1(1)M+1$ such that $x_l=x_1<x_2\cdots<x_M<x_{M+1}=x_r$ and $\Delta x=x_{j+1}-x_j$, and similarly the temporal interval $[0,t_f]$ is divided into N equal width finite subintervals by means of the mesh points $t_n,\ n=0(1)N$ such that $t_{initial}=t_0<t_1<\cdots<t_{N-1}<t_N=t_f$ and $\Delta t=t_{n+1}-t_n$. In fact, a non-uniform partition of the region could also be selected. But, since the non-uniform choice would increase the workload, computer storage capacity and running time, the uniform one has been preferred.

2.1. Discretization of space variable

In this method, the approximate solution $u_M(x,t)$ corresponding to analytical solution $u(x,t)$ will be sought by means of quintic Hermite basis functions H_{ji} as [21]

$$u(x,t)\approx u_M(x,t)=\sum_{j=1}^{M+2}a_{j+4k-4}(t)H_{ji} \tag{2.1}$$

where a's stand for the time dependent coefficients which are going to be determined later, k stands for the element number in each subinterval, $H_{4j-R}(x)$ for $R=-2(1)3$ are defined over the interval $[x_{j-1},x_{j+1}]$ as given in Ref. [22]

$$\begin{aligned}
H_{4j-3}(x) &= 6\frac{(x-x_{j-1})^5}{h^5}-15\frac{(x-x_{j-1})^4}{h^4}+10\frac{(x-x_{j-1})^3}{h^3}, && x_{j-1}\le x\le x_j\\
H_{4j+1}(x) &= 6\frac{(x_{j-1}-x)^5}{h^5}-15\frac{(x_{j-1}-x)^4}{h^4}+10\frac{(x_{j-1}-x)^3}{h^3}, && x_j\le x\le x_{j+1}\\
H_{4j-2}(x) &= 3\frac{(x-x_{j-1})^5}{h^4}-7\frac{(x-x_{j-1})^4}{h^3}+4\frac{(x-x_{j-1})^3}{h^2}, && x_{j-1}\le x\le x_j\\
H_{4j+2}(x) &= 3\frac{(x_{j-1}-x)^5}{h^4}-7\frac{(x_{j-1}-x)^4}{h^3}+4\frac{(x_{j-1}-x)^3}{h^2}, && x_j\le x\le x_{j+1}\\
H_{4j-1}(x) &= \frac{1}{2}\frac{(x-x_{j-1})^5}{h^3}-\frac{(x-x_{j-1})^4}{h^2}+\frac{1}{2}\frac{(x-x_{j-1})^3}{h^1}, && x_{j-1}\le x\le x_j\\
H_{4j}(x) &= \frac{1}{2}\frac{(x_{j-1}-x)^5}{h^3}-\frac{(x_{j-1}-x)^4}{h^2}+\frac{1}{2}\frac{(x_{j-1}-x)^3}{h}, && x_j\le x\le x_{j+1}
\end{aligned} \tag{2.2}$$

and finally i $(i=1(1)4)$ stands for the inner-collocation points to be taken from the shifted roots of Legendre and Chebyshev polynomials. Throughout this study, the following shifted roots of the fourth-degree Legendre and Chebyshev polynomials [23] computed by the symbolic programming language Matlab are respectively used as inner-collocation points

$$\begin{aligned}
\eta_1^L &= 0.069431844202974,\ \eta_2^L=0.330009478207572\\
\eta_3^L &= 0.669990521792428,\ \eta_4^L=0.930568155797026.\\
\eta_1^C &= 0.038060233744357,\ \eta_2^C=0.308658283817455\\
\eta_3^C &= 0.691341716182545,\ \eta_4^C=0.961939766255643
\end{aligned}$$

When the following local coordinate system has been utilized on the j^{th} element

$$\eta=\frac{x-x_j}{h}$$

the interval $[x_j,x_{j+1}]$ is converted into an unit interval $[0,1]$ in which η represents both Chebyshev and Legendre shifted roots. Thus, from Eq. (2.2), quintic Hermite spline functions $H_j(\eta)$ $(j=1(1)6)$ in terms of the local coordinate η are written as

$$\begin{array}{lll} H_1(\eta) &=& 1-10\eta^3+15\eta^4-6\eta^5, \qquad H_2(\eta)=(\eta-6\eta^3+8\eta^4-3\eta^5)\Delta x, \\ H_3(\eta) &=& \left(0.5\eta^2-1.5\eta^3+1.5\eta^4-0.5\eta^5\right)(\Delta x)^2, \quad H_4(\eta)=\left(0.5\eta^3-\eta^4+0.5\eta^5\right)(\Delta x)^2, \\ H_5(\eta) &=& 10\eta^3-15\eta^4+6\eta^5, \qquad H_6(\eta)=(-4\eta^3+7\eta^4-3\eta^5)\Delta x. \end{array} \tag{2.3}$$

and therefore the approximate solution $u_M(x,t)$ given by Eq. (2.1) becomes

$$u_M(\eta,t)=\sum_{j=1}^{6} a_{j+4k-4}(t)H_j(\eta). \tag{2.4}$$

From (2.3), the second order derivatives of $H_j(\eta)$ are found as follows

$$\begin{array}{lll} H_1''(\eta) &=& -60\eta+180\eta^2-120\eta^3, \qquad H_2''(\eta)=\left(-36\eta+96\eta^2-60\eta^3\right)\Delta x, \\ H_3''(\eta) &=& \left(1-9\eta+18\eta^2-10\eta^3\right)(\Delta x)^2, \quad H_4''(\eta)=\left(3\eta-12\eta^2+10\eta^3\right)(\Delta x)^2, \\ H_5''(\eta) &=& 60\eta-180\eta^2+120\eta^3, \qquad H_6''(\eta)=\left(-24\eta+84\eta^2-60\eta^3\right)\Delta x. \end{array}$$

Thus, the point wise values of the approximation (2.4) with its second order derivative at inner collocation points η_i for $i=1(1)4$ are found as follows

$$\begin{array}{lll} u_i &=& u(\eta_i)=a_{4k-3}H_1(\eta_i)+a_{4k-2}H_2(\eta_i)+a_{4k-1}H_3(\eta_i)+a_{4k}H_4(\eta_i)+a_{4k+1}H_5(\eta_i)+a_{4k+2}H_6(\eta_i), \\ (\Delta x)^2u_i'' &=& (\Delta x)^2u''(\eta_i)=a_{4k-3}H_1''(\eta_i)+a_{4k-2}H_2''(\eta_i)+a_{4k-1}H_3''(\eta_i)+a_{4k}H_4''(\eta_i)+a_{4k+1}H_5''(\eta_i)+a_{4k+2}H_6''(\eta_i) \end{array} \tag{2.5}$$

where $H_1(\eta)=H_5(1-\eta)$, $H_2(\eta)=-H_6(1-\eta)$, $H_3(\eta)=-H_4(1-\eta)$ and $H_{ji}=H_j(\eta_i)$.
In the building of the numerical scheme, the finite element method constructed on quintic Hermite spline basis functions in the discretizaion of spatial quantities and the finite difference method constructed on Crank-Nicolson like approximation in the discretization of temporal quantities will be used. This choice has the advantages of several vital characteristics such as easy to handle algorithms produced by quintic Hermite spline basis functions and low level of storage requirement. Besides those advantages, both of the non-linear and linear systems resulted from the usage of splines are usually not ill-conditioned and thus permit the required coefficients to be found out in an easy manner. Furthermore, the newly obtained approximate solutions generally don't result in numerical instability.

2.2. Discretization of time variable

Now, we are ready to discretize the aforementioned 1D heat conduction equation given by (1.1). To do this, one can use Crank-Nicolson type formula. First of all, one discretizes Eq. (1.1) as

$$\frac{u^{n+1}-u^n}{\Delta t}-\alpha^2\left[\frac{(u_{xx})^n+(u_{xx})^{n+1}}{2}\right]=0.$$

Before proceeding more, let us separate the above equation such that the unknown values at $(n+1)$. time level are on the left-hand side and the known values n. time level are on the right-hand side as

$$\frac{u^{n+1}}{\Delta t}-\alpha^2\frac{(u_{xx})^{n+1}}{2}=\frac{u^n}{\Delta t}+\alpha^2\frac{(u_{xx})^n}{2}. \tag{2.6}$$

If one puts Eq. (2.5) in Eq. (2.6), the following fully discretized difference equation system with $4M$ difference equations and $4M+2$ coefficients in both time and space variables is obtained for the coefficients **a** to be calculated

$$\begin{array}{ll} & \frac{1}{\Delta t}\left[a_{4k-3}^{n+1}H_{1i}+a_{4k-2}^{n+1}H_{2i}+a_{4k-1}^{n+1}H_{3i}+a_{4k}^{n+1}H_{4i}+a_{4k+1}^{n+1}H_{5i}+a_{4k+2}^{n+1}H_{6i}\right] \\ - & \frac{\alpha^2}{2(\Delta x)^2}\left[a_{4k-3}^{n+1}B_{1i}+a_{4k-2}^{n+1}B_{2i}+a_{4k-1}^{n+1}B_{3i}+a_{4k}^{n+1}B_{4i}+a_{4k+1}^{n+1}B_{5i}+a_{4k+2}^{n+1}B_{6i}\right] \\ = & \frac{1}{\Delta t}\left[a_{4k-3}^{n}H_{1i}+a_{4k-2}^{n}H_{2i}+a_{4k-1}^{n}H_{3i}+a_{4k}^{n}H_{4i}+a_{4k+1}^{n}H_{5i}+a_{4k+2}^{n}H_{6i}\right] \\ + & \frac{\alpha^2}{2(\Delta x)^2}\left[a_{4k-3}^{n}B_{1i}+a_{4k-2}^{n}B_{2i}+a_{4k-1}^{n}B_{3i}+a_{4k}^{n}B_{4i}+a_{4k+1}^{n}B_{5i}+a_{4k+2}^{n}B_{6i}\right]. \end{array} \tag{2.7}$$

These newly obtained equations are clearly recursive in nature. Thus, the unknown vector $\mathbf{a}^n=(a_1^n,\ldots,a_{4M+1}^n,a_{4M+2}^n)$ can be recursively determined up to the requested final time t_f. If one utilizes the conditions given at the boundary of the solution domain by Eq. Eq. (1.3) and eliminates the coefficients a_1^n, a_{4M+1}^n in Eq. (2.7), the following statements are easily obtained. Using the boundary condition given at the left of the solution domain, the value of $u(x_l,t)$ is written as

$$u(x_l,t)=a_1^nH_{11}+a_2^nH_{21}+a_3^nH_{31}+a_4^nH_{41}+a_5^nH_{51}+a_6^nH_{61}=0.$$

Since $H_{21} = H_{31} = H_{41} = H_{51} = H_{61} = 0$ and $H_{11} \neq 0$, the value $a_1^n = 0$ is found. Similarly, by the boundary condition given at the right of the solution domain, the value of $u(x_r, t)$ is obtained as

$$u(x_r,t) = a_{4M-3}^n H_{14} + a_{4M-2}^n H_{24} + a_{4M-1}^n H_{34} + a_{4M}^n H_{44} + a_{4M+1}^n H_{54} + a_{4M+2}^n H_{64} = 0.$$

Since $H_{14} = H_{24} = H_{34} = H_{44} = H_{64} = 0$ and $H_{54} \neq 0$, the value $a_{4M+1}^n = 0$ is found.
After all of the above preparations and mathematical procedures, the iterative numerical scheme resulting in the following matrix algebraic system excluding a_1^n and a_{4M+1}^n is obtained as

$$\mathbf{L}\mathbf{a}^{n+1} = \mathbf{R}\mathbf{a}^n. \tag{2.8}$$

Here, both $\mathbf{L}$ and $\mathbf{R}$ are $4M \times 4M$ diagonal band matrices, and $\mathbf{a}^{n+1}$ and $\mathbf{a}^n$ are $4M$ column matrices.
The entries of $\mathbf{a}$ in Eq. (2.8) are obtained and then the numerical results of the 1D heat conduction equation at the next time level are calculated. This iterative process is done in a successive manner until the wanted time t_f. To be able to start this iterative calculation, the initial vector $\mathbf{a}^0$ should be found. The initial vector is determined by the usage of the initial condition given together with the governing equation. Utilizing the initial vector, Eq. (2.8) results in $4M$ unknowns. To solve the system Eq. (2.8), a script written in MatlabR2021a is used on a digital computer.

2.3. Determination of initial vector

It is obvious that there is a need for determining the initial vector to iterate the numerical scheme (2.8). This initial vector $\mathbf{a}^0$ is constructed using the conditions given at the initial time and boundaries of the solution domain. The approximate numerical solution given by Eq. (2.1) should be rewritten in terms of the initial vector as

$$u(x,t) \approx u_M(x,t) = \sum_{j=1}^{M+2} a_{j+4k-4}^0(t) H_{ji}.$$

Since the initial approximate solution $u_M(x,0)$ should satisfy the initial prescribed solution $u(x,0)$, it is clearly

$$u_M(x_j, 0) = u(x_j, 0), \qquad j = 1(1)M+1.$$

From this equality, the following matrix equation is obtained

$$\mathbf{W}\mathbf{a}^0 = \mathbf{b}$$

where

$$\mathbf{W} = \left[\begin{array}{cccccccccccccc}
H_{21} & H_{31} & H_{41} & H_{51} & H_{61} & & & & & & & & & \\
H_{22} & H_{32} & H_{42} & H_{52} & H_{62} & & & & & & & & & \\
H_{23} & H_{33} & H_{43} & H_{53} & H_{63} & & & & & & & & & \\
H_{24} & H_{34} & H_{44} & H_{54} & H_{64} & & & & & & & & & \\
 & & & H_{11} & H_{21} & H_{31} & H_{41} & H_{51} & H_{61} & & & & & \\
 & & & H_{12} & H_{22} & H_{32} & H_{42} & H_{52} & H_{62} & & & & & \\
 & & & H_{13} & H_{23} & H_{33} & H_{43} & H_{53} & H_{63} & & & & & \\
 & & & H_{14} & H_{24} & H_{34} & H_{44} & H_{54} & H_{64} & & & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & & H_{11} & H_{21} & H_{31} & H_{41} & H_{51} & H_{61} & & & \\
 & & & & & H_{12} & H_{22} & H_{32} & H_{42} & H_{52} & H_{62} & & & \\
 & & & & & H_{13} & H_{23} & H_{33} & H_{43} & H_{53} & H_{63} & & & \\
 & & & & & H_{14} & H_{24} & H_{34} & H_{44} & H_{54} & H_{64} & & & \\
 & & & & & & & & & H_{11} & H_{21} & H_{31} & H_{41} & H_{61} \\
 & & & & & & & & & H_{12} & H_{22} & H_{32} & H_{42} & H_{62} \\
 & & & & & & & & & H_{13} & H_{23} & H_{33} & H_{43} & H_{63} \\
 & & & & & & & & & H_{14} & H_{24} & H_{34} & H_{44} & H_{64}
\end{array}\right],$$

$$\mathbf{a}^0 = (a_2, a_3, a_4, \ldots, a_{4M-1}, a_{4M}, a_{4M+2})^T$$

and

$$\mathbf{b} = (u(x_{11},0), u(x_{12},0), u(x_{13},0), u(x_{14},0), \ldots, u(x_{M3},0), u(x_{M4},0))^T.$$

Therefore, from the iterative scheme (2.8), the approximate numerical solutions of the heat conduction equation given by (1.1) at the desired time are calculated.

3. Stability Analysis of the Scheme

In order to examine the stability of the linear iterative scheme in Eq. (2.7), von-Neumann method will be used. For this aim, substituting the Fourier mode

$$a_j^n = \xi^n e^{ij\varphi}$$

into Eq. (2.7), one gets

$$\begin{aligned}
& \xi^{n+1} e^{i(4j-3)\varphi}(\alpha_1) + \xi^{n+1} e^{i(4j-2)\varphi}(\alpha_2) + \xi^{n+1} e^{i(4j-1)\varphi}(\alpha_3) + \xi^{n+1} e^{i(4j)\varphi}(\alpha_4) + \xi^{n+1} e^{i(4j+1)\varphi}(\alpha_5) + \xi^{n+1} e^{i(4j+2)\varphi}(\alpha_6) \\
= \; & \xi^{n} e^{i(4j-3)\varphi}(\beta_1) + \xi^{n} e^{i(4j-2)\varphi}(\beta_2) + \xi^{n} e^{i(4j-1)\varphi}(\beta_3) + \xi^{n} e^{i(4j)\varphi}(\beta_4) + \xi^{n} e^{i(4j+1)\varphi}(\beta_5) + \xi^{n} e^{i(4j+2)\varphi}(\beta_6)
\end{aligned}$$

where $\varphi=\beta h$, β is the mode number, h is the spatial step size, $i=\sqrt{-1}$ and

$$\begin{aligned} \alpha_1 &= H_{1i}-\tfrac{\alpha^2}{2h^2}kB_{1i} & \beta_1 &= H_{1i}+\tfrac{\alpha^2}{2h^2}kB_{1i} \\ \alpha_2 &= H_{2i}-\tfrac{\alpha^2}{2h^2}kB_{2i} & \beta_2 &= H_{2i}+\tfrac{\alpha^2}{2h^2}kB_{2i} \\ \alpha_3 &= H_{3i}-\tfrac{\alpha^2}{2h^2}kB_{3i} & \beta_3 &= H_{3i}+\tfrac{\alpha^2}{2h^2}kB_{3i} \\ \alpha_4 &= H_{4i}-\tfrac{\alpha^2}{2h^2}kB_{4i} & \beta_4 &= H_{4i}+\tfrac{\alpha^2}{2h^2}kB_{4i} \\ \alpha_5 &= H_{5i}-\tfrac{\alpha^2}{2h^2}kB_{5i} & \beta_5 &= H_{5i}+\tfrac{\alpha^2}{2h^2}kB_{5i} \\ \alpha_6 &= H_{6i}-\tfrac{\alpha^2}{2h^2}kB_{6i} & \beta_6 &= H_{6i}+\tfrac{\alpha^2}{2h^2}kB_{6i} \end{aligned}$$

Again, if one makes the required simplification and mathematical operations, one obtains

$$\xi=\frac{M_1-iM_2}{M_3-iM_4} \tag{3.1}$$

where

$$\begin{aligned} M_1 &= (\beta_3+\beta_5)\cos\varphi+(\beta_2+\beta_6)\cos 2\varphi+\beta_1\cos 3\varphi+\beta_4 \\ M_2 &= (\beta_3-\beta_5)\sin\varphi+(\beta_2-\beta_6)\sin 2\varphi+\beta_1\sin 3\varphi+\beta_4 \\ M_3 &= (\alpha_3+\alpha_5)\cos\varphi+(\alpha_2+\alpha_6)\cos 2\varphi+\alpha_1\cos 3\varphi+\alpha_4 \\ M_4 &= (\alpha_3-\alpha_5)\sin\varphi+(\alpha_2-\alpha_6)\sin 2\varphi+\alpha_1\sin 3\varphi+\alpha_4. \end{aligned}$$

When the modulus of Eq. (3.1) is taken, it is seen that the condition $|\xi|\le 1$ is satisfied. Thus, it is concluded that the numerical scheme is unconditionally stable.

4. Numerical Example and Results

In this section, the obtained scheme (2.8) will be applied to the 1D heat conduction problem given by Eqs. (1.1)-(1.3) for $x\in[0,1]$, $t\in[0,t_f]$, $f(x)=\sin(\pi x)$ and $\alpha=1$, as a test problem. The exact solution of the test problem is [7, 8]

$$u(x,t)=\sin(\pi x)e^{-\alpha^2\pi^2 t}.$$

Since the test problem has an exact solution, the error norms L_2 and L_∞ given as follows, respectively, are going to be used to test the accuracy and validity of the scheme

$$L_2=\sqrt{\left(h\sum_{i=1}^{M}|u_i-(u_M)_i|^2\right)}, \qquad L_\infty=\max_{1\le i\le M}|u_i-(u_M)_i|.$$

This manuscript carries out all of the numerical computations using both Quintic Hermite Collocation Method with Legendre roots (QHCM-L) and Quintic Hermite Collocation Method with Chebyshev roots (QHCM-C). All calculations have been made with MATLAB R2021a on 13th Gen Intel(R) Core(TM) i9-13900HX 2.20 GHz computer having 32,0 GB of RAM.
Some numerical results are presented for different spatial and temporal step sizes with final desired values of time to check the efficiency and accuracy of the scheme. The newly obtained results are also compared with some of the existing ones using the same parameter values.

	L_2			
Δt	QHCM-L	QHCM-C	[7]	[17]CHCM-L
0.01	5.8454×10^{-7}	5.8454×10^{-7}	4.2273×10^{-4}	4.1333×10^{-7}
0.005	1.4641×10^{-7}	1.4641×10^{-7}	1.0560×10^{-4}	1.0353×10^{-7}
0.0025	3.6621×10^{-8}	3.6621×10^{-8}	2.6395×10^{-5}	2.5895×10^{-8}

Table 4.1: A comparison of the computed error norms L_2 of the present scheme with those in Refs. [7, 17], for $M=1000$ and $\Delta t=0.01, 0.005, 0.0025$ $t_f=1$).

The values of the error norm L_2 are calculated from the scheme (2.8) for the parameters $\Delta t=1/100, 5/1000, 25/10000$ and $\Delta x=1/1000$ at $t_f=1$, and are listed in Table 4.1 with a comparison of those given in Refs. [7, 17]. The newly obtained error norm L_2 is remarkably small enough for using both QHCM-C and QHCM-L schemes. They are also in very good agreement with those in Ref. [17] and much better than those in Ref. [7].
In Table 4.2, a clear comparison of the newly computed error norm values L_2 of the scheme with those in Refs. [8, 17] for several values of $\Delta x=1/5, 1/10, 1/20$ and $\Delta t=1/10^6$ at $t_f=1$ is displayed. It can be easily seen from the table that the obtained results are extremely small

	L_2				
Δx	QHCM-L	QHCM-C	[8](CN-I)	[8](CN-II)	[17]CHCM-L
$1/5$	3.1385×10^{-12}	2.2952×10^{-9}	4.7696×10^{-6}	8.5859×10^{-3}	3.5716×10^{-8}
$1/10$	4.7257×10^{-14}	1.5029×10^{-10}	7.7143×10^{-9}	2.1412×10^{-3}	2.2848×10^{-9}
$1/20$	1.1057×10^{-14}	9.4983×10^{-12}	1.8820×10^{-11}	5.3498×10^{-4}	1.4361×10^{-10}
$1/40$	1.3947×10^{-15}	5.9155×10^{-13}			
$1/80$	5.1467×10^{-16}	3.3364×10^{-14}			

Table 4.2: A comparison of the computed error norms L_2 of the present scheme with those in Refs. [8, 17], for various values of $M = 5,10,20,40,80$ and $\Delta t = 1/10^6$ at $t_f = 1$.

	L_2			
Δx	QHCM-L	[7]		[17]CHCM-L
		$\theta = 0.1$	$\theta = 0.2$	
$1/10$	3.5183×10^{-10}	1.6534×10^{-4}	2.4968×10^{-4}	1.6957×10^{-6}
$1/20$	8.5012×10^{-12}	4.3906×10^{-7}	7.8152×10^{-7}	1.0426×10^{-7}
$1/40$	1.0432×10^{-12}	8.6154×10^{-10}	8.0111×10^{-10}	6.4916×10^{-9}
		$\theta = 0.3$	$\theta = 0.4$	
$1/10$	3.5183×10^{-10}	3.2357×10^{-4}	3.6276×10^{-4}	1.6957×10^{-6}
$1/20$	8.5012×10^{-12}	1.0791×10^{-6}	1.2556×10^{-6}	1.0426×10^{-7}
$1/40$	1.0432×10^{-12}	9.2171×10^{-10}	1.1166×10^{-9}	6.4916×10^{-9}
		$\theta = 0.5$	$\theta = 0.6$	
$1/10$	3.5183×10^{-10}	3.5225×10^{-4}	2.8316×10^{-4}	1.6957×10^{-6}
$1/20$	8.5012×10^{-12}	1.2494×10^{-6}	1.0109×10^{-6}	1.0426×10^{-7}
$1/40$	1.0432×10^{-12}	1.2167×10^{-9}	1.1107×10^{-9}	6.4916×10^{-9}
		$\theta = 0.7$	$\theta = 0.8$	
$1/10$	3.5183×10^{-10}	1.5954×10^{-4}	1.9640×10^{-4}	1.6957×10^{-6}
$1/20$	8.5012×10^{-12}	5.4419×10^{-7}	9.1503×10^{-7}	1.0426×10^{-7}
$1/40$	1.0432×10^{-12}	7.4804×10^{-10}	8.6751×10^{-10}	6.4916×10^{-9}
		$\theta = 0.9$	$\theta = 0.95$	
$1/10$	3.5183×10^{-10}	4.5197×10^{-4}	6.1645×10^{-4}	1.6957×10^{-6}
$1/20$	8.5012×10^{-12}	2.0820×10^{-6}	2.8818×10^{-6}	1.0426×10^{-7}
$1/40$	1.0432×10^{-12}	2.4668×10^{-9}	3.6858×10^{-9}	6.4916×10^{-9}

Table 4.3: A comparison of the calculated error norms L_2 of the present scheme with those in Refs. [7, 17] for $M = 10,20,40$ and $\Delta t = 1/10^6$ at $t_f = 0.1$.

and also when the element number is increased, the error norm L_2 decreases. Again, from the table it is obvious that the newly obtained results are much more better than those in Refs. [8, 17].

Table 4.3 shows a clear comparison of the L_2 error norms computed from the new scheme using QHCM-L with those in Refs. [7, 17] for values of $\Delta x = 1/10, 1/20, 1/40$ and $\Delta t = 1/10^6$ at $t_f = 0.1$. One can obviously seen that the scheme produces good enough results and clearly, the obtained L_2 error norms are relatively smaller than those in Refs. [7, 17].

	L_2			L_∞		
$\Delta x = \Delta t$	QHCM-L	QHCM-C	[17]CHCM-L	[11]	[6](CN)	[6](CBVM)
0.2	5.1455×10^{-5}	5.1439×10^{-5}	5.8498×10^{-5}	1.4145×10^{-1}	1.1×10^{-1}	2.8×10^{-2}
0.1	3.1581×10^{-5}	3.1589×10^{-5}	3.1584×10^{-5}	3.7195×10^{-2}	3.0×10^{-2}	3.8×10^{-3}
0.05	9.7102×10^{-6}	9.7106×10^{-6}	9.7065×10^{-6}	8.4588×10^{-3}	6.9×10^{-3}	2.7×10^{-4}
0.025	2.5488×10^{-6}	2.5489×10^{-6}	2.5485×10^{-6}	2.0698×10^{-3}	1.7×10^{-3}	1.3×10^{-5}
0.0125	6.4490×10^{-7}	6.4490×10^{-7}	6.4488×10^{-7}	5.1473×10^{-4}	4.2×10^{-4}	5.1×10^{-7}
0.00625	1.6171×10^{-7}	1.6171×10^{-7}	1.6171×10^{-7}		1.1×10^{-4}	3.6×10^{-8}
0.01	4.1332×10^{-7}	4.1333×10^{-7}	4.1332×10^{-7}			
0.005	1.0353×10^{-7}	1.0353×10^{-7}	1.0353×10^{-7}			
0.0025	2.5895×10^{-8}	2.5895×10^{-8}	2.5895×10^{-8}			
0.002	1.6574×10^{-8}	1.6574×10^{-8}	1.6574×10^{-8}			
0.001	4.1437×10^{-9}	4.1437×10^{-9}	4.1437×10^{-9}			

Table 4.4: A comparison of the calculated error norms L_∞ of the present scheme with those in Refs. [6, 11, 17], for various values of $\Delta x = \Delta t$ at $t_f = 1$.

Table 4.4 displays a clear comparison of the L_∞ error norms obtained from the presented scheme with those given in Refs. [6, 11, 17] for some decreasing values of $\Delta x = \Delta t$ at $t_f = 1$. The table indicates that the computed L_∞ error norms are sufficiently small and in good harmony with those in Refs. [6, 17], but much more better than those in Ref. [11].

Finally, Table 4.5 compares the discrete values of both L_2 and L_∞ error norms for $M = 16$ and $\Delta t = 0.01$ at various t_f with those in Refs. [15, 17]. The table shows that the newly computed error norms are again small enough and better than both of those in Refs. [15, 17].

	L_2			L_∞		
t_f	QHCM-L	[17]CHCM-L	[15]	QHCM-L	[17]CHCM-L	[15]
0.1	4.2273×10^{-4}	2.9917×10^{-4}	4.86×10^{-3}	2.9889×10^{-4}	2.9891×10^{-4}	6.79×10^{-3}
0.3	1.7602×10^{-4}	1.2457×10^{-4}	8.87×10^{-5}	1.2446×10^{-4}	1.2447×10^{-4}	3.76×10^{-4}
0.5	4.0720×10^{-5}	2.8818×10^{-5}	1.73×10^{-3}	2.8791×10^{-5}	2.8793×10^{-5}	2.44×10^{-4}
0.7	7.9127×10^{-6}	5.5999×10^{-6}	2.04×10^{-4}	5.5446×10^{-6}	5.5953×10^{-6}	3.17×10^{-4}
0.9	1.4121×10^{-6}	9.9934×10^{-7}	2.14×10^{-3}	9.9840×10^{-7}	9.9856×10^{-7}	3.14×10^{-3}
1.0	5.8454×10^{-7}	4.1368×10^{-7}	2.15×10^{-3}	4.1329×10^{-7}	4.1338×10^{-7}	3.32×10^{-3}

Table 4.5: A comparison of the calculated error norms L_2 and L_∞ of the present scheme with those in Ref. [15, 17], for $M = 16$, $\Delta t = 0.01$ at various t_f.

From the presented tables one can clearly see that the newly calculated numerical results are in good harmony with both the analytical and other previously published numerical results. In fact, the present scheme in general produces better results with a smaller number of elements compared to other studies based on classical and trigonometric B-splines.
From the results given in the above tables, it is clearly seen that the approximate numerical solutions are getting closer and closer to exact ones as the mesh step sizes are refined. It should not be forgotten that the mesh step sizes must be chosen modestly small enough to avoid increase in the CPU simulation time and storage requirement capacity.

5. Conclusion

In this work, the developed fully discretized numerical scheme constructed using the conventional Crank-Nicolson-like finite difference technique for the integration of temporal quantities and a quintic Hermite B-spline finite element collocation technique for the integration of spatial quantities has been successfully applied to obtain approximate numerical solutions of the 1-D heat conduction equation for certain initial and boundary conditions. In order to confirm the accuracy and reliability of the suggested scheme, the error norms L_2 and L_∞ calculated from the exact and numerical solutions are compared with the results given by the previous ones existing in the literature. It is seen that the results obtained by applying the developed scheme to the example problem are mostly better than the previous results and at least in some cases in good agreement with the previous ones. Generally speaking, it is found a good quality harmony between the existing and the above aforesaid results. For the stability test of the current scheme, von-Neumann (Fourier) technique is performed which demonstrates that the scheme is unconditionally stable. The main contribution of this paper is the ability of the newly presented scheme to produce much better results by using less computer storage capacity and requiring less CPU time. In conclusion, the suggested scheme is producing stable, efficient and accurate results, and can be successfully used to find approximate numerical solutions of initial and boundary value problems consisting of many linear and especially nonlinear PDEs which have an important role in describing natural phenomena encountered in most branches of applied and engineering sciences.

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