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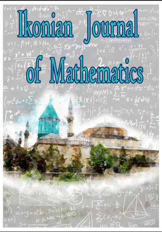

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Formulas for Bernoulli and Euler Numbers and Polynomials with the aid of Applications of Operators and Volkenborn Integral

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Abstract — We study on applications of operators and the (p-adic) Volkenborn integral in order to investigate fundamental properties of the special numbers and polynomials. The aim of this article is to derive new formulas for these numbers and polynomials and finite sums by using operators and the Volkenborn integral. These formulas are related to the Stirling numbers, array polynomials, the Fubini-type polynomials and numbers, and also the Bernoulli and Euler numbers and polynomials. Moreover, in the light of our new formulas, we set new special number families with their generating functions, and give very important footnotes about their definitions and properties.

Subject Classification (2020): 12D10, 11B68, 11S80, 26C05, 26C10.

1. Introduction

Recently, examining the properties of polynomials with operator theory and deriving special numbers with the help of operators are among the trendy topics in mathematics. Because special numbers and polynomials are among the basic tools that can be easily applied in

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mathematical modeling problems used in problem solving. Especially the special numbers and polynomials have also been used in almost all areas of mathematics, and in all applied sciences (*cf.* [1]-[40]). Investigating formulas and finite sums for certain family of polynomials and numbers using operators and Volkenborn integral methods also form the basis of the motivation of this article.

We use the following basic standard notations and definitions:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$\mathbb{C}$ denotes a set of complex numbers,

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}) \end{cases}$$

and

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{v} = \frac{\lambda(\lambda-1)\cdots(\lambda-v+1)}{v!} = \frac{(\lambda)^{(v)}}{v!},$$

where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$ (see [1]-[40]).

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.1}$$

where $|t| < 2\pi$ and when $x = 0$, we have $B_n := B_n(0)$ denotes the Bernoulli numbers (see [1]-[40]).

The Euler numbers are defined by

$$h(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where $|t| < \pi$ (see [1]-[40]).

The Euler polynomials are defined by

$$g(t,x) = h(t) e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.2}$$

which satisfies $E_n := E_n(0)$ (see [1]-[40]).

The Stirling numbers of the second kind $S_2(n,k)$ are defined by means of the following generating function:

$$F_s(t,k) = \frac{\left(e^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{1.3}$$

which satisfies

$$S_2(n,k) = 0$$

if $n < k$ or $k < 0$ and $k \in \mathbb{N}_0$ (see [1]-[40]).

By combining (1.2) and (1.3), assuming $\left|e^t - 1\right| < 1$, we reach the following functional equation:

$$g(t,x) = h(t)\sum_{m=0}^{\infty}\binom{x}{m} m! F_s(t,m) \tag{1.4}$$

and

$$e^{t(x+v)} = \sum_{m=0}^{\infty}\binom{x+v}{m} m! F_s(t,m). \tag{1.5}$$

By using Eq. (1.4), we obtain

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v} E_{n-v}\sum_{m=0}^{v}\binom{x}{m} m! S_2(v,m)\frac{t^n}{n!}.$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 1.1. Let $n \in \mathbb{N}_0$. Then we have

$$E_n(x) = \sum_{v=0}^{n}\binom{n}{v} E_{n-v}\sum_{m=0}^{v}\binom{x}{m} m! S_2(v,m).$$

By using Eq. (1.5), we obtain

$$\sum_{n=0}^{\infty}(x+v)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x+v}{m} m! S_2(n,m)\frac{t^n}{n!}.$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 1.2. Let $n, v \in \mathbb{N}_0$. Then we have

$$(x+v)^n = \sum_{m=0}^{n}\binom{x+v}{m} m! S_2(n,m). \tag{1.6}$$

When $v = 0$, Eq. (1.6) reduces to

$$x^n = \sum_{m=0}^{n} \binom{x}{m} m! S_2(n,m) \tag{1.7}$$

(*cf.* [6, 27, 28, 39]).

The λ-array polynomials $S_k^n(x;\lambda)$ are defined by means of the following generating function:

$$\frac{1}{k!} e^{tx} \left(\lambda e^t - 1\right)^k = \sum_{n=0}^{\infty} S_k^n(x;\lambda) \frac{t^n}{n!} \tag{1.8}$$

(see [1, 28]).

Substituting $\lambda = 1$ into (1.8), we have

$$S_k^n(x) := S_k^n(x;1) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(x+j\right)^n \tag{1.9}$$

with

$$S_0^0(x) = S_n^n(x) = 1, S_0^n(x) = x^n.$$

If $k > n$, then

$$S_k^n(x) = 0$$

(see [1, 3, 28]; and also the references cited therein).

The Fubini-type numbers and polynomials of order k are defined, respectively, by

$$\left(\frac{2}{2-e^t}\right)^k = \sum_{n=0}^{\infty} a_n^{(k)} \frac{t^n}{n!} \tag{1.10}$$

and

$$\left(\frac{2}{2-e^t}\right)^k e^{xt} = \sum_{n=0}^{\infty} a_n^{(k)}(x) \frac{t^n}{n!} \tag{1.11}$$

which satisfies $a_n^{(k)} := a_n^{(k)}(0)$ (see [9]; and also [8, 10, 12, 13, 36]).

When $k = 1$ in (1.10), we have

$$a_n := a_n^{(1)}$$

and

$$a_n = 2 \sum_{j=0}^{n} \binom{n}{j} w_g(j) w_g(n-j),$$

where $w_g(n)$ denote the Fubini numbers which are defined by

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!} \tag{1.12}$$

(see [4]; and also [8–10, 12, 13, 36]).

Using (1.3) and (1.12), we have the following well-known relation [4]:

$$w_g(n)=\sum_{j=0}^{n} j!S_2(n,j).$$

From (1.11) and (1.3), Kilar and Simsek [9] gave the following formula:

$$x^n=2^{-k}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x). \tag{1.13}$$

1.1. The operators $\mathcal{O}_\lambda\left[f;a,b\right]$ and $T_\lambda\left[f;a,b\right]$

Let

$$E^a\left[f\right](x)=f(x+a),$$

(see [1, 18, 23, 37]). We [30] gave the following operator $\mathcal{O}_\lambda\left[f;a,b\right]$ for real parameters λ, a and b:

$$\mathcal{O}_\lambda\left[f;a,b\right](x)=\lambda E^a\left[f\right](x)+E^b\left[f\right](x), \tag{1.14}$$

where $x\in\mathbb{R}$ and

$$T_\lambda\left[f;a,b\right](x)=\frac{\mathcal{O}_\lambda\left[f;a,b\right](x)}{a+b+1}. \tag{1.15}$$

We [30] showed that

$$\begin{aligned}
\frac{1}{2}T_1\left[f;0,0\right](x) &= I\left[f\right](x), \text{ (Identity Operator)}\\
-2T_{-1}\left[f;1,0\right](x) &= \Delta\left[f\right](x), \text{ (Forward Difference Operator)}\\
I\left[f\right](x)+\frac{1}{2}T_1\left[f;-1,-1\right](x) &= \nabla\left[f\right](x), \text{ (Backward Difference Operator)}\\
T_1\left[f;1,0\right](x) &= M\left[f\right](x), \text{ (Means Operator)}\\
-T_{-1}\left[f;\frac{1}{2},-\frac{1}{2}\right](x) &= \delta\left[f\right](x), \text{ (Central Difference Operator)}\\
\frac{1}{2}T_1\left[f;\frac{1}{2},-\frac{1}{2}\right](x) &= \mu\left[f\right](x), \text{ (Averaging Difference Operator)}
\end{aligned}$$

and also

$$\begin{aligned} -(2a+b+1)T_{-1}[f;a+b,a](x) &= \Delta_b E^a[f](x), (a\neq b, \text{Gould Operator}) \\ -2T_{-\lambda}[f;1,0](x) &= \Delta_\lambda[f](x). \end{aligned}$$

For details about the above operators and their applications, see [30] and also [38].

We [32] modified the operators $\mathscr{O}_\lambda[f;a,b]$ and $T_\lambda[f;a,b]$ as follows:

$$\mathbb{Y}_{\lambda,\beta}[f;a,b](x) = \lambda E^a[f](x) + \beta E^b[f](x) \tag{1.16}$$

and

$$\mathbb{Y}_{\lambda,\beta}[f;a,b](x) = \beta\mathscr{O}_{\frac{\lambda}{\beta}}[f;a,b](x) = \beta(a+b+1)\,T_{\frac{\lambda}{\beta}}[f;a,b](x),$$

where λ and β are complex or real parameters, a and b are real parameters.

We [32] showed that

$$\mathbb{Y}_{-\lambda,1}[f;1,0](x) = -\Delta_\lambda[f](x)$$

(see also [1]),

$$E^a[f](x) = \mathbb{Y}_{1,0}[f;a,0](x)$$

and

$$\Delta_a[f](x) = \mathbb{Y}_{1,-1}[f;a,0](x),$$

where Δ_a denotes the forward difference operator,

$$\nabla_{-b}[f] = \mathbb{Y}_{1,-1}[f;0,-b],$$

which yields

$$\mathbb{Y}_{1,-1}[f;b,0]\,\mathbb{Y}_{1,0}[f;-b,0] = \mathbb{Y}_{1,-1}[f;b,0]\,\mathbb{Y}_{1,0}[f;0,-b],$$

where ∇_{-b} denotes the backward difference operator.

$$\delta_a[f] = \mathbb{Y}_{1,-1}\left[f;\frac{a}{2},-\frac{a}{2}\right],$$

where δ_a denotes the central difference operator. The Gould operator

$$G_{a,b}[f] = \mathbb{Y}_{1,0}[f;a+b,0] - \mathbb{Y}_{1,0}[f;a,0],$$

where $a \neq b$. Let $k \in \mathbb{N}$. With the aid of (1.14), we [32] also showed that

$$\mathbb{Y}_{\lambda,\beta}^{k}\left[f;a,b\right](x)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}\beta^{j}f(x+jb+(k-j)a). \tag{1.17}$$

Putting $b=0$ and $\beta=-1$ in (1.17), we have

$$\mathbb{Y}_{\lambda,-1}^{k}\left[f;1,0\right](x)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}(-1)^{j}f(x+(k-j)a)=\Delta_{\lambda}^{k}\left[f\right](x)$$

(see [1, p. 155, Eq. (29)], [32]).

Putting $b=0$ and $\beta=-1$ in the above equation, we have

$$\begin{aligned}\mathbb{Y}_{\lambda,-1}^{k}\left[x^{n};1,0\right](x) &= \Delta_{\lambda}^{k}\left[x^{n}\right](x)\\ &= S_{k}^{n}(x,\lambda)\end{aligned}$$

(*cf.* [1, p. 155], [32]).

Therefore,

$$S_{k}^{n}(x)=\frac{1}{k!}\Delta^{k}\left[x^{n}\right]$$

(*cf.* [1, p. 155], [3], [32]).

The results of this article are briefly summarized for the reader as follows, section by section.

In Section 2, some basic properties of the Euler polynomials are given with the help of operators. We also give formulas for the Fubini-type polynomials, the Stirling numbers of the second kind and the Euler polynomials.

In Section 3, we derive some formulas, identities and finite sums for the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the array polynomials, and the Stirling numbers of the second kind with the aid of operators and Volkenborn integrals.

In Section 4, we give a conclusion section.

2. Formulas for Euler Polynomials in terms of Operators

The purpose of this section is to study the Euler polynomials with the help of operators and to provide an introductory discussion of some of their properties and applications. Here, we note that the operators $T_{\lambda}\left[f;a,b\right]$ and derivative operator D action the variable x (see [22, p.

406]). Using the averaging operator

$$M[f] = T_1[f;1,0] = \frac{E+I}{2}[f],$$

we have

$$T_1[E_n(x);1,0](x) = x^n, \tag{2.1}$$

which satisfies

$$\frac{E_n(x+1)+E_n(x)}{2} = x^n$$

and

$$E_n(x) = T_1^{-1}[x^n;1,0](x).$$

Thus, we see that

$$E_n(x) = \sum_{j=0}^{\infty}\left(T_{-1}[x^n;1,0](x)\right)^j.$$

For $j > n$,

$$\Delta^j\{x^n\} = 0,$$

the Euler polynomials are given by

$$E_n(x) = \sum_{j=0}^{n}\frac{(-1)^j}{2^j}\Delta^j\{x^n\} \tag{2.2}$$

(see [22, p. 406]).

Applying derivative operator D to the equation (2.1) yields

$$D[M[E_n(x)]] = D\{x^n\}.$$

Therefore

$$D\left\{\frac{E_n(x+1)+E_n(x)}{2}\right\} = nx^{n-1}.$$

Combining the above equation with the following derivative formula for the Euler polynomials, which are members of Appell polynomials,

$$D\{E_n(x)\} = nE_{n-1}(x),$$

we get

$$\frac{E_{n-1}(x+1)+E_{n-1}(x)}{2}=x^{n-1}.$$

Thus we get

$$D\{M[E_n(x)]\}=M[E_{n-1}(x)].$$

From the above equation, we get

$$M^{-1}[D\{M[E_n(x)]\}]=E_{n-1}(x).$$

Hence

$$D\{E_n(x)\}=\frac{M^{-1}[D\{M[E_n(x)]\}]}{n},$$

and

$$D^k\{E_n(x)\}=\begin{cases}(n)^{(k)}E_{n-k}(x), & 1\leq k<n\\ k!, & n=k\\ 0, & n<k\end{cases}$$

(see [22, p. 406]).

Combining (1.13) with (2.2), we have the following result:

Corollary 2.1. Let $k,n\in\mathbb{N}_0$. Then we have

$$T_1[E_n(x);1,0](x)=2^{-k}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x).$$

or, equivalently,

$$E_n(x+1)+E_n(x)=2^{-k+1}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x).$$

3. Formulas for the Bernoulli and Euler Numbers and Polynomials with the aid of Operators and Volkenborn Integrals

The purpose of this section is to derive formulas, finite sums and relations involving the Bernoulli and Euler numbers and polynomials, and the Stirling numbers using operators and applications of the Volkenborn integral.

Before giving the essential formulas of this section, the following some properties of the Volkenborn integral are given with a very brief introduction.

Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $f : \mathbb{Z}_p \to \mathbb{C}_p$, where $\mathbb{C}_p$ is a field of p-adic completion of algebraic closure of set of p-adic rational numbers. f is called a uniformly differential function at a point $a \in \mathbb{Z}_p$ if f satisfies the following conditions:

If the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(z)$ as $(x,y) \to (0,0)$ (with $x \neq y$). A set of uniformly differential functions is briefly indicated by $f \in UD(\mathbb{Z}_p)$ or $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$.

The Volkenborn integral of the uniformly differential function f is given as follows:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{3.1}$$

where $\mu_1(x)$ denote the Haar distribution, given by

$$\mu_1(x) = \frac{1}{p^N}$$

(see [7, 15, 17, 21, 25, 31, 34, 40]).

Let $n \in \mathbb{N}_0$. Some examples for p-adic integrals are given as follows:

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x) \tag{3.2}$$

and

$$B_n(y) = \int_{\mathbb{Z}_p} \left(x + y\right)^n d\mu_1(x), \tag{3.3}$$

where B_n and $B_n(y)$ denote the Bernoulli numbers and the Bernoulli polynomials, respectively (see [7, 15, 16, 21, 25, 31, 34]).

By applying the Volkenborn integral to the Eq. (2.2), we obtain

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \int_{\mathbb{Z}_p} \Delta^j \left\{x^n\right\} d\mu_1(x).$$

Combining the above equation with the following well-known formulas

$$\Delta = E - I$$

and

$$\Delta^{j} = \sum_{v=0}^{j} (-1)^{v} \binom{j}{v} E^{v},$$

we get

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^{v} \binom{j}{v} \int_{\mathbb{Z}_p} (x+v)^n \, d\mu_1(x).$$

Combining the above equation with (3.3) yields the following theorem:

Theorem 3.1. Let $n \in \mathbb{N}_0$. Then we have

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^{v} \binom{j}{v} B_n(v). \tag{3.4}$$

By combining (3.4) with the following known formula:

$$E_n(x) = \sum_{v=0}^{n} \binom{n}{v} x^{n-v} E_v,$$

we also get

$$\sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^{v} \binom{j}{v} B_n(v) = \sum_{v=0}^{n} \binom{n}{v} E_v \int_{\mathbb{Z}_p} x^{n-v} d\mu_1(x).$$

Combining the above equation with (3.2), we arrive at the following theorem:

Theorem 3.2. Let $n \in \mathbb{N}_0$. Then we have

$$(B+E)^n = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^{v} \binom{j}{v} B_n(v),$$

where

$$(B+E)^n = \sum_{v=0}^{n} \binom{n}{v} E_v B_{n-v}$$

and after applying binomial expansion, each index of B^n and E^n are to be replaced by the corresponding suffix: B_n and E_n, respectively.

By applying the Volkenborn integral to the Eq. (1.6), we get

$$\int_{\mathbb{Z}_p} (x+v)^n \, d\mu_1(x) = \sum_{m=0}^{n} m! S_2(n,m) \int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x).$$

Combining the left-hand side of the above equation with (3.3), we obtain

$$B_n(v) = \sum_{m=0}^{n} m! S_2(n,m) \int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x).$$

Combining the right-hand side of the above equation with the following formula

$$\int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x) = \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{1}{k+1}$$

(see [34, p. 21]), we arrive at the following theorem:

Theorem 3.3. Let $n, v \in \mathbb{N}_0$. Then we have

$$B_n(v) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{m! S_2(n,m)}{k+1}. \tag{3.5}$$

Combining (3.4) with (3.5), we also arrive at the following theorem:

Theorem 3.4. Let $n \in \mathbb{N}_0$. Then we have

$$\begin{aligned}\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) &= \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} \\ &\quad \times \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{m! S_2(n,m)}{k+1}.\end{aligned}$$

By using (1.9) and (1.8), we have (*cf.* [28]):

$$\begin{aligned} S_k^n(x) &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) x^{n-j}. \end{aligned}$$

By applying the Volkenborn integral to the above equation, we get

$$\begin{aligned} & \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) \int_{\mathbb{Z}_p} x^{n-j} d\mu_1(x) \\ = & \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \int_{\mathbb{Z}_p} (x+j)^n d\mu_1(x). \end{aligned}$$

Combining the above equation with (3.3), we obtain the following theorem:

Theorem 3.5. Let $n, k \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^{n} \binom{n}{j} S_2(j,k) B_{n-j} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n(j). \tag{3.6}$$

Here we note that using (3.6), we set the following sequences of numbers:

$$Y_{10}(n,k) = \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) B_{n-j} \tag{3.7}$$

and

$$Y_{11}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n(j). \tag{3.8}$$

Thus, generating function for the numbers $Y_{10}(n,k)$ is defined by

$$F(t) = \sum_{n=0}^{\infty} Y_{10}(n,k) \frac{t^n}{n!} \tag{3.9}$$

and generating function for the numbers $Y_{11}(n,k)$ is defined by

$$G(t) = \sum_{n=0}^{\infty} Y_{11}(n,k) \frac{t^n}{n!}. \tag{3.10}$$

Examination of the fundamental properties of the functions $F(t)$ and $G(t)$ is left to the reader. With the help of these functions, interesting and applicable results can be derived by examining the fundamental properties of the numbers $Y_{10}(n,k)$ and $Y_{11}(n,k)$.

Let us end our article with guiding tips by giving the reader a brief introduction about the functions $F(t)$ and $G(t)$.

Observe that

$$F(t) = \frac{t(e^t-1)^{k-1}}{k!}, \tag{3.11}$$

where k is a positive integer. By using the above function, we get

$$\sum_{n=0}^{\infty} Y_{10}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty} S_2(n,k-1)\frac{t^{n+1}}{kn!} \tag{3.12}$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 3.6. Let $n, k \in \mathbb{N}$. Then we have

$$Y_{10}(n,k) = \frac{n}{k} S_2(n-1,k-1). \tag{3.13}$$

Thus, by combining (3.6) and (3.7) with (3.13), we also have the following result:

Theorem 3.7. Let $n, k \in \mathbb{N}$. Then we have

$$S_2(n-1,k-1) = \frac{k}{n}\sum_{j=0}^{n}\binom{n}{j} S_2(j,k)B_{n-j}. \tag{3.14}$$

With the help of similar operations and methods above, new and applicable formulas can be achieved by performing the function $G(t)$ and the numbers $Y_{11}(n,k)$.

4. Conclusions

We gave generating functions for the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Fubini-type polynomials, and the Stirling numbers. We also gave some properties of the operator. Some properties of the Euler polynomials were examined with the aid of operators.

By using operators and the Volkenborn integrals, we derived some formulas, identities and finite sums involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Fubini numbers and polynomials, the array polynomials, and Stirling numbers.

With the help of Theorem 3.5, we set new special number families with their generating functions, and gave very important footnotes about their definitions and properties.

We think that these formulas will have the potential to be used in mathematics, mathematical physics, and engineering.

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