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ON MAGIC TYPE LABELINGS OF ZERO-DIVISOR GRAPHS

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Abstract. In this article, we investigate magic type labelings of zero-divisor graphs. In particular, we turn our attention to semi-magic, magic, and super-magic labelings. We are able to construct infinitely many rings which admit these magic type labelings as well as infinitely many rings which do not have these magic type labeling. We further proceed to classify the magic type labeling properties for all of the rings which have zero-divisor graphs with up to 14 vertices. We then conclude with some conjectures about how these patterns may extend for larger zero-divisor graphs.

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1. Introduction

In this article, we investigate a property called magic labeling, as well as several related labeling types such as semi-magic and super-magic labelings on the zero-divisor graph of a commutative ring with unity, $1 \neq 0$. The zero-divisor graph of a commutative ring R was originally defined by I. Beck in [3] in 1988 as a graph with vertex set R and edges between distinct vertices x and y if and only if $xy = 0$. After observing that 0 is always adjacent to every other vertex in R, the more modern treatment, as in [2], is to use the vertex set $Z(R)^*$, where $Z(R) = \{a \in R \mid \exists b \in R,\ b \neq 0 \text{ such that } ab = 0\}$, the nonzero zero-divisors and the same edge relation for distinct $x, y \in Z(R)^*$, x is adjacent to y if and only if $xy = 0$. This new zero-divisor graph is denoted $\Gamma(R)$. Thus, $\Gamma(R)$ is an undirected and simple graph (no multi-edges or loops) and will be the graph we are interested in for this article. We will discuss further properties of the zero-divisor graph in Section 2.

Much of the initial interest in these zero-divisor graphs surrounded an early conjecture of Beck as to whether zero-divisor graphs were perfect (the coloring number is the same as the clique number). This was shown to not be true in general in [1]. This interest in coloring zero-divisor graphs motivates us to continue the investigation of labelings of zero-divisor graphs. Here, we turn our attention to a labeling introduced by J. Sedláček in [7] in 1976 called *magic labeling* and some variations of this labeling. We will define these magic type labelings more formally in the next section, but for now one can think of a magic labeling as an edge labeling such that we label edges so that for every vertex, the sum of the labels of the incident edges is the same fixed number, the magic constant for the graph. Recently, graceful and harmonious labeling of zero-divisor graphs were studied by the third author in [4] so we have continued the investigation of labeling zero-divisor graphs with perhaps the next most well-studied and popular labelings, the magic type labelings.

The article is organized as follows. In Section 2, we formally define the zero-divisor graph and the magic type labelings of interest throughout the article as well as provide the known results that will be of use to us. In Section 3, we are able to construct some infinite families of commutative rings which admit the various magic type labelings as well as infinitely many commutative rings which have no such magic type labelings. This leads us to Section 4 where we are able to prove several results which allow us to determine which commutative rings will have these various magic type zero-divisor graphs for many commutative rings. In Section 5, we begin the process of determining which rings satisfy these different magic type conditions for their zero-divisor graphs by utilizing the results in [5,6] where the author was able to find all possible zero-divisor graphs on up to 14 vertices. We are able to determine whether a magic type labeling exists on every zero-divisor graph which is possible on up to 14 vertices. We conclude the paper with several open questions about how these patterns may extend to larger zero-divisor graphs.

2. Preliminaries

We begin with some important definitions and results about the zero-divisor graph of a commutative ring R with unity $1 \neq 0$. As in [2], given a commutative ring R, we construct an associated graph $\Gamma(R) = (V, E)$ whose vertex set is $V = Z(R)^*$ and whose edge set, E, is defined by distinct vertices $x, y \in V$ which are adjacent if and only if $xy = 0$. This graph is simple as $\Gamma(R)$ is undirected (R is commutative), has no multi-edges, and has no loops. If R is an integral domain, then this graph is empty, so we will generally insist that there are non-trivial zero-divisors present in our rings R.

We pause briefly to provide an example of a zero-divisor graph for the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ below to illustrate how these graphs can be constructed for the reader.

Example 2.1. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. The nonzero zero-divisors of the ring R are $Z(R)^* = \{(0,1),(0,2),(0,3),(1,0),(1,2)\}$. Then, by pairwise multiplying each nonzero zero-divisor together, we are able to determine the edge relationship for $\Gamma(R)$ shown below.

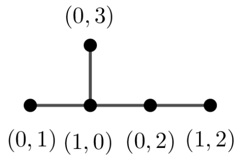


FIGURE 1. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$

Notice that we see why it is nice to not include the zero element of the ring since not much information is gained. This zero vertex would simply be always adjacent to all other vertices.

We begin with a proposition where we collect some well-known and useful results about the zero-divisor graph of a commutative ring.

Proposition 2.2. *Let R be a commutative ring with $1 \neq 0$. Then we have the following results about $\Gamma(R)$.*

(1) $\Gamma(R)$ *is connected.*
(2) $\Gamma(R)$ *has diam*$(\Gamma(R)) \leq 3$.
(3) $\Gamma(R)$ *is finite if and only if R is an integral domain or R is finite.*

Proof. (1)-(3) are proven in [2]. □

We now formally define the various magic type labelings of interest to us. For a graph, $G = (V, E)$, we say a function, $f : V \to S$ is a *vertex labeling* from S and $g : E \to S$ is an *edge labeling* from S. Then, let $m \in \mathbb{Z}$ be a constant, called the *magic constant.* For a graph G, an edge labeling $g : E \to \mathbb{Z}$ is a *semi-magic labeling* if for every $v \in V$, $\sum g(e_i) = m$ where the sum is taken over all edges e_i which are incident to v as introduced in [8]. We shall refer to a graph G as *trivially semi-magic* if there is only one semi-magic labeling, the trivial labeling of G, given by $g(e) = 0$ for all edges $e \in E$. Any graph G which admits a semi-magic labeling is sometimes called a semi-magic graph; however, because every graph always admits at least the trivially semi-magic labeling, we will consider a graph G to be a *semi-magic graph* when the graph admits at least one non-trivial semi-magic labeling. We will often use the phrase *non-trivially semi-magic* to stress that we mean a graph G admits at least one semi-magic labeling that is not just all edges receiving the edge label of 0, especially in situations where there may be some ambiguity.

Next, for a graph $G = (V, E)$, let $g : E \to \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of positive integers, and let $m \in \mathbb{N}^*$. Then, if g is an injective function and for all $v \in V$, $\sum g(e_i) = m$ where the sum is taken over all edges e_i which are incident to v, we call g a *magic labeling* of G [8]. If a graph G has a magic labeling, then G is said to be a *magic graph.*

Lastly, for a graph $G = (V, E)$, let $g : E \to \{n, n+1, \ldots, n+|E|-1\}$ for any $n \in \mathbb{N}^*$, and let $m \in \mathbb{N}$. Then, if g is a bijection and $\forall\ v \in V$, $\sum g(e_i) = m$ where the sum is taken over all edges e_i which are incident to v, we call g a *super-magic labeling* of G [8]. If a graph G has a super-magic labeling, G is a *super-magic graph.*

While the following relationships between these graph properties are essentially immediate from definitions, they are important enough for the reader that we state them in the form of the following proposition.

Proposition 2.3. *Let $G = (V, E)$ be a simple graph. Then we have the following relationship between these classes of graphs as indicated by the diagram below.*

super-magic $\Rightarrow$ *magic* $\Rightarrow$ *non-trivially semi-magic* $\Rightarrow$ *trivially semi-magic.*

Moreover, none of these implications can be reversed.

Proof. Let G be a simple, connected graph. Suppose G is a super-magic graph. Then there is an edge labeling $g : E \to \{n, n+1, \ldots, n+|E|-1\}$ such that g is a bijection and there exists a magic constant $m \in \mathbb{N}$ such that for all $v \in V$, $\sum g(e_i) = m$ where the sum is taken over all edges e_i which are incident to v. Because $\{n, n+1, \ldots, n+|E|-1\} \subset \mathbb{N}^*$ and g is an injection, g is also a magic labeling, so G is magic.

Let G be magic. Then there is an edge labeling $g : E \to \mathbb{N}^*$ where g is injective and there exists a magic constant $m \in \mathbb{Z}$ such that $\forall\ v \in V$, $\sum g(e_i) = m$ where the sum is taken over all edges e_i which are incident to v. Since $\mathbb{N}^* \subset \mathbb{Z}$ and g is an injective function, g cannot be the trivial labeling (unless $|E| = 1$ where it is super-magic). This makes g a non-trivial, semi-magic labeling of G.

Lastly, every simple graph is trivially semi-magic by simply assigning $g(e) = 0$ for all $e \in E$.

We prove the final statement of the theorem through the following examples where we exhibit graphs to show none of these implications can be reversed. $\square$

Example 2.4. K_5 is a graph which is magic, but not super-magic.

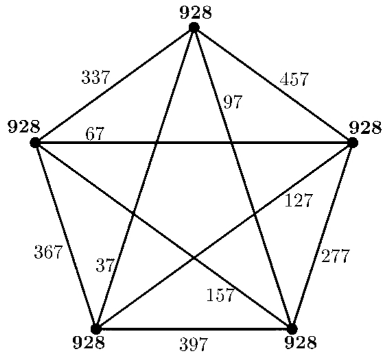


FIGURE 2. The graph K_5.

Clearly K_5 is magic, as it admits a magic labeling (as shown above). Thus, K_5 is also semi-magic and trivially semi-magic. However, as shown in Theorem 2 of Stewart's *Supermagic Complete Graphs* [9], K_5 is *not* super-magic.

Example 2.5. K_3 is a graph which is semi-magic, but not magic.

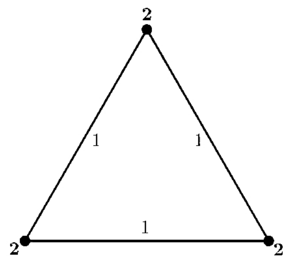


FIGURE 3. The graph K_3.

This graph admits a semi-magic labeling, and so it is also trivially semi-magic. However, this graph is *not* magic, and we prove this by contradiction. Suppose K_3 is magic. Then it admits a magic labeling, $g : E \to \mathbb{N}^+$. By definition, this function is injective, meaning our edge labels, $g(e_1) = x$, $g(e_2) = y$, $g(e_3) = z$, must all be distinct. Let m be the magic constant for this magic labeling of K_3. Consider the vertex v_1 in the figure incident to edges e_1 and e_2. Then it must be the case that $x + y = m$. Similarly, consider the vertex v_2 incident to edges e_2 and e_3. Then $y + z = m$. This implies that $x + y = y + z$ so that $x = z$. However, this is a contradiction that g was an injective mapping, so it must be the case that K_3 is not magic.

Example 2.6. P_3 is a graph which is trivially semi-magic, but not what we have called semi-magic, i.e., the only semi-magic labeling admitted is the trivial semi-magic labeling.

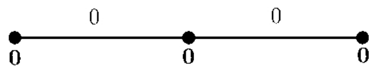


FIGURE 4. The graph P_3.

Suppose $g : E \to \mathbb{Z}$ is a semi-magic labeling. Without loss of generality, suppose one edge receives the label of x. Since each edge contains a leaf vertex (a vertex of degree 1), the magic constant must be x for the graph. This means the other edge must also receive the label of x since this edge also contains a leaf vertex. We now consider the non-leaf vertex whose sum of incident edge labels is $x+x$ and yet must also equal the magic constant, x. Thus $2x = x$ for integers implies $x = 0$ and we have shown the only semi-magic labeling must be the trivial semi-magic labeling.

3. Magic type properties of infinite families of rings

In this section, we are able to construct infinite families of rings which admit semi-magic, magic, and super magic zero-divisor graphs. We are also able to prove that there are also infinitely many rings which are not even semi-magic and hence not magic or super-magic. We are able to use complete and complete bipartite graphs to accomplish this goal since both these zero-divisor graphs and the magic type labeling properties of these types of graphs are well understood.

We begin by collecting some useful results about the semi-magic, magic, and super-magic properties of complete and complete bipartite graphs in the form of the next three propositions. These results are known and we have tried to give credit to the first person to prove these results where possible; however, many of these terms of magic type labelings have been used to mean different things in the literature over the years so it is possible we have proved things that are well known by a different name or have cited someone who was not actually the first to prove the result.

Proposition 3.1. *We have the following known results about which graphs are non-trivially semi-magic and which ones are not.*

(1) *A complete graph, K_n is always non-trivially semi-magic for any $n \geq 2$.*
(2) *A complete bipartite graph, $K_{m,n}$ is non-trivially semi-magic with nonzero magic constant if and only if $m = n$.*

(3) *A complete bipartite graph, $K_{m,n}$ with $m \neq n$ and $m, n \geq 2$, is non-trivially semi-magic, but the magic constant must always be 0 in every semi-magic labeling.*

(4) *A star graph, $K_{1,n}$ is non-trivially semi-magic if and only if $n = 1$.*

Proof. (1) and (2) We note that all regular graphs are non-trivially semi-magic since one could simply label every edge c for any constant $c \in \mathbb{Z}$. Then if the graph is k-regular, then each vertex has sum kc which would be the magic constant for the k-regular graph. Since K_n (resp. $K_{n,n}$) are $n-1$ (resp. n)-regular graphs, they are non-trivially semi-magic by simply picking c to be nonzero.

To see why $m = n$ is necessary for a complete bipartite graph to admit a non-trivial semi-magic labeling with magic constant $k \neq 0$, we suppose $K_{m,n}$ for $m \neq n$ is semi-magic. Then for each vertex sum of the labels of incident edges must add to k. We consider the sum of the edge labels for all of the edges in the entire graph in two different ways.

We first consider the sum of all the labels of the edges incident to the vertices in the part with m vertices and see that the sum of all of these edges must be km since each of the m vertices has sum k. Because the graph is a complete bipartite graph, we have added up the labels of every edge in the entire graph. On the other hand, the same argument by adding up the labels of the edges incident to the vertices in the part with n vertices must sum to kn. This must be the same number since it is a complete bipartite graph and we have added up the labels of all of the edges in the graph both ways. But this means $kn = km$ and $k \neq 0$, hence $m = n$.

(3) Let G be a complete bipartite graph, $K_{m,n}$ with $m \neq n$ and $m, n \geq 2$. When there are at least two vertices in each of the parts of the complete bipartite, graph, then there is a subgraph which is $K_{2,2}$ which is isomorphic to a 4-cycle, C_4. We may then label this subgraph as below and then assign 0 to every other edge in the graph and we will get a non-trivial labeling, but as in (2), we see the magic constant is indeed 0 for the graph.

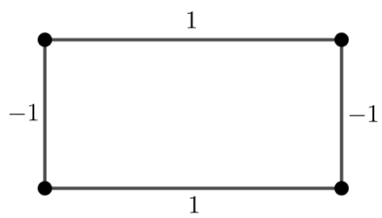


FIGURE 5. The $K_{2,2}$ subgraph

(4) If we have a star graph, $K_{1,n}$ with $n \geq 2$, then we see that this is a complete bipartite graph, with unbalanced parts so the magic constant must be 0. Every non-central vertex is a leaf vertex, so the edge must receive 0 as the label. This is all of the edges in $K_{1,n}$, thus every semi-magic labeling must be trivial. Conversely, if $n = 1$, then we may simply assign the only edge in the graph any nonzero label and we get a non-trivial semi-magic labeling. □

We present a similar proposition that gives some important results about which graphs are magic and which are not.

Proposition 3.2. *We have the following known results about which graphs are magic and which ones are not.*

(1) *A complete graph, K_n is magic if and only if $n = 2$ or $n \geq 5$.*
(2) *A complete bipartite graph, $K_{m,n}$ is magic if and only if $m = n \geq 3$.*

Proof. The proofs of (1) and (2) are given as Examples 2 and 1, respectively, in Section 7 of [8]. □

We present a similar proposition that gives some important results about which graphs are super-magic and which are not.

Proposition 3.3. *We have the following known results about which graphs are super-magic and which ones are not.*

(1) *A complete graph, K_n is super-magic if and only if $n = 2$ or $n > 5$ and $n \not\equiv 0 \pmod 4$.*
(2) *A complete bipartite graph, $K_{m,n}$ is super-magic if and only if $m = n$ and $n \neq 2$.*

Proof. (1)-(2) are proven in [9]. □

Lemma 3.4. ([2, Theorem 2.10]) *Let R be a finite commutative ring with unity and $\Gamma(R)$ be the zero-divisor graph of R. Then the following are equivalent.*

(1) *$\Gamma(R)$ is complete.*
(2) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*
(3) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with maximal ideal M with $M^2 = 0$.*

Lemma 3.5. *Let $R \cong A \times B$ where A and B are integral domains. Then $\Gamma(R)$ is a complete bipartite graph, $K_{|A|-1,|B|-1}$.*

Proof. Let $M = \{(a, 0) \mid a \in A,\ a \neq 0\}$ and $N = \{(0, b) \mid b \in B,\ b \neq 0\}$. Then we see that every fixed vertex in M is adjacent to every vertex in N since

$(a,0)\cdot(0,b) = (0,0)$ but since A is an integral domain, $(a,0)\cdot(a',0) \neq (0,0)$. Similarly for the vertices in N. Thus $\Gamma(R)$ is a complete bipartite graph, and specifically, $\Gamma(R) = K_{|A|-1,|B|-1}$. □

Using Lemmas 3.4 and 3.5 along with the magic type results on complete and complete bipartite graphs, we are able to construct many rings which satisfy the various magic type labeling properties.

Proposition 3.6. *Let p be a prime. Then the following finite commutative rings with unity have zero-divisor graphs which are non-trivially semi-magic.*

(1) *$R = \mathbb{Z}_{p^2}$ with $p \geq 3$. In this case, $\Gamma(R) = K_{p-1}$.*
(2) *$R = Z_p[X]/(X^2)$ with $p \geq 3$. In this case, $\Gamma(R) = K_{p-1}$.*
(3) *$R = Z_p[X_1, X_2, \ldots, X_n]/I$ where $I = (\{X_iX_j \mid 1 \leq i, j \leq n\})$. In this case, $\Gamma(R) = K_{p^n-1}$.*
(4) *$R = F_1 \times F_2$ where F_i are finite fields of the same size, say $|F_1| = |F_2| = k$. In this case, $\Gamma(R) = K_{k-1,k-1}$.*

Proof. Each of these rings are either complete or complete bipartite graphs as shown in [2]. Since all complete graphs on at least two vertices and complete bipartite graphs where the parts have the same size are nontrivially semi-magic, the results follow. □

It is worth pointing out that the construction in Proposition 3.6 (3) demonstrates that we can find a semi-magic zero-divisor graph of size m as long as $m = p^n - 1$ for some prime p and some integer $n \geq 1$. Thus there are infinitely many finite commutative rings which admit non-trivially semi-magic zero-divisor graphs.

We follow with the natural analogue to construct families of rings which have magic zero-divisor graphs and collect some in the next proposition.

Proposition 3.7. *Let p be a prime. Then the following finite commutative rings have zero-divisor graphs which are magic.*

(1) *Let $R = \mathbb{Z}_9$ or $R = \mathbb{Z}_3[X]/(X^2)$. Then $\Gamma(R) = K_2$.*
(2) *Let $p \geq 7$ and let $R = \mathbb{Z}_{p^2}$ or $R = \mathbb{Z}_p[X]/(X^2)$. Then $\Gamma(R) = K_{p-1}$.*
(3) *$R = Z_p[X_1, X_2, \ldots, X_n]/I$ where $I = (\{X_iX_j \mid 1 \leq i, j \leq n\})$ chosen such that $p^n - 1 = 2$ or $p^n - 1 \geq 5$. In this case, $\Gamma(R) = K_{p^n-1}$.*
(4) *Let $R = F_1 \times F_2$ where F_i are finite fields and $|F_1| = |F_2| = k \geq 4$. Then $\Gamma(R) = K_{k-1,k-1}$.*

Proof. We have chosen families of rings so that their graphs will either be complete graphs on either two or at least 5 vertices or else complete bipartite graphs with parts having size at least 3. This is precisely the conditions outlined in Proposition 3.2 which have magic graphs. $\square$

Again, this means for any integer $m = p^n - 1$ for some prime p and integer $n \geq 1$ so that $m = 2$ or $m \geq 5$, we can build a finite commutative ring whose zero-divisor graph is magic.

We continue here in the same fashion to provide infinite families of rings whose zero-divisor graphs are super-magic.

Proposition 3.8. *Let p be a prime. Then the following finite commutative rings have zero-divisor graphs which are super-magic.*

(1) *Let $R = \mathbb{Z}_9$ or $R = \mathbb{Z}_3[X]/(X^2)$. Then $\Gamma(R) = K_2$.*
(2) *Let $p \geq 7$ and $p \equiv 3 \pmod 4$. Let $R = \mathbb{Z}_{p^2}$ or $R = \mathbb{Z}_p[X]/(X^2)$. Then $\Gamma(R) = K_{p-1}$.*
(3) *$R = Z_p[X_1, X_2, \ldots, X_n]/I$ where $I = (\{X_iX_j \mid 1 \leq i, j \leq n\})$ chosen such that $p^n - 1 = 2$ or $p^n - 1 \geq 6$ and $p^n - 1 \not\equiv 0 \pmod 4$. In this case, $\Gamma(R) = K_{p^n-1}$.*
(4) *Let $R = F_1 \times F_2$ where F_i are finite fields and $|F_1| = |F_2| = k \geq 4$. Then $\Gamma(R) = K_{k-1,k-1}$.*

Proof. Again, we have chosen rings which precisely meet the conditions to be super-magic as in Proposition 3.3. We note in (2) we have framed it in the more constructive way. Because the complete graph will be on $p-1$ vertices and we need $p-1 \not\equiv 0 \pmod 4$. Since primes larger than two are all either 1 or 3 $\pmod 4$, this is equivalent. $\square$

We again note that this is infinitely many finite commutative rings which are super-magic. There are an infinite number of primes which are congruent to 3 modulo 4 (a fun exercise to generalize Euclid's proof), so this construction in (2) alone generates infinitely many super-magic zero-divisor graphs.

We now turn our attention to generating infinitely many finite commutative rings which do not admit semi-magic, magic, or super-magic zero-divisor graphs.

Proposition 3.9. *We can find an infinite number of commutative rings with unity which are neither non-trivially semi-magic, magic, nor super-magic.*

Proof. Let $R = \mathbb{Z}_2 \times A$ where A is a finite integral domain (finite field) with $|A| \geq 3$. Then $\Gamma(R) = K_{1,|A|-1}$ by Lemma 3.5 and has only the trivial semi-magic labeling of $f(e) = 0$ for all $e \in E(\Gamma(R))$ by Proposition 3.1(4). This means $\Gamma(R)$ is not non-trivially semi-magic and hence not magic or super-magic from Proposition 2.3. □

We can slightly extend the classes of graphs which are not magic or super-magic through the following theorem.

Proposition 3.10. *There exists an infinite family of rings whose zero-divisor graphs are neither magic nor super-magic. Moreover, this family of rings is non-trivially semi-magic; however, the magic constant must be* 0.

Proof. Let $R = A \times B$ for finite integral domains A and B (finite fields) such that $|A| \neq |B|$ and $|A|, |B| > 2$. Then $\Gamma(R) = K_{|A|-1,|B|-1}$ by Lemma 3.5 with the parts of different sizes. This graph will not be magic from Proposition 3.2, so it will also not be super-magic. For the last sentence, we see that the graph is an unbalanced complete bipartite graph which forces the magic constant to be 0 as in Proposition 3.1 (3). When $|A|, |B| > 2$, we will have at least two vertices in each part of the bipartite graph which allows us to use the $K_{2,2}$ subgraph labeling in Figure 5 for a subgraph of $\Gamma(R)$ and label the rest of the edges 0 and get a non-trivial semi-magic labeling. □

4. Classification of small zero-divisor graphs

These results above indicate that magic type labelings of zero-divisor graphs are worth investigating since there are infinitely many which both do and do not satisfy these three magic type properties. This leads us to naturally study the classification question related to these magic type properties. Given a commutative ring, can we determine whether it has a semi-magic, magic, or super-magic zero-divisor graph?

With this in mind, we begin with small zero-divisor graphs and we turn our attention to the work of Redmond in [5,6]. The author was able to find all possible graphs that may arise as the zero-divisor graph of a commutative ring with unity on up to 14 vertices and start to determine which of these satisfy the various magic type conditions.

We begin with two useful propositions which can be used to quickly rule out certain graphs for being magic and hence super-magic.

Proposition 4.1. *If a zero-divisor graph has more than two vertices and has a leaf vertex (a vertex of degree* 1*), then the zero-divisor graph is not magic.*

Proof. Let G be a graph with at least three vertices and let $l \in V(G)$ be a leaf vertex of G. Suppose G is magic. Then there is an injection, $f : E(G) \mapsto \mathbb{N}^*$ where f is a magic labeling. Let m be the magic constant of G. Because l is a leaf vertex, it has only one adjacent vertex, call it $n \in V(G)$. Because zero-divisor graphs are connected [2] and $|V(G)| > 2$, n is adjacent to at least one other vertex, call these adjacent vertices (besides l), $n_1, \ldots, n_i \in V(G)$. Let $f(n-l) = k \in \mathbb{N}^*$. We then are able to consider the sum of the labels of the incident edges to l and n. Because f is a magic labeling, these sums must be equal; however, the sum of the labels of incident edges to l are simply $f(n-l) = k$ since it is a leaf vertex and hence $k = m$ since we supposed this labeling was magic. On the other hand, the sum of the labels of the edges incident to n must be strictly larger than this since each edge label is a positive integer in a magic labeling. This is a contradiction and G cannot be magic. □

Proposition 4.2. *If there are four vertices, $v_1, v_2, v_3, v_4 \in V(G)$ where G is a zero-divisor graph such that v_1 is adjacent to v_2, v_2 is adjacent to v_3, v_3 is adjacent to v_4 and $deg(v_2) = deg(v_3) = 2$ where $deg(v)$ denotes the degree of v, then G is not magic. Note that v_1 and v_4 can be the same vertex and the result still applies.*

Proof. Let G be a zero-divisor graph and let $v_1, v_2, v_3, v_4 \in V(G)$ such that v_1 is adjacent to v_2 by an edge e_1, v_2 is adjacent to v_3 by an edge e_2, v_3 is adjacent to v_4 by an edge e_3 and $deg(v_2) = deg(v_3) = 2$. Suppose also that G is magic. Then there is an injective function, $f : E(G) \mapsto \mathbb{Z}^+$ which is a magic labeling. Let m be the magic constant of G. We consider the sum of the labels of the incident edges to v_2 and v_3. Notice each of these has degree 2 by assumption and they share one of the edges, e_2. Thus we compare the two sum and see that $f(e_1)+f(e_2) = f(e_2)+f(e_3)$. But this means that $f(e_1) = f(e_3)$. This contradicts that f is an injection which means such a graph cannot be magic. □

An important note for these two propositions is that the converse is not true. A graph can fail to have a leaf vertex and still not be magic, and a graph can fail to meet the condition of Proposition 4.2 and still not be magic. For example, as in Proposition 3.2 (1) with $n = 4$.

Another result we can use to determine if a graph is semi-magic is to see if it contains a special type of semi-magic subgraph. If G is a graph, then we call G' a *spanning subgraph* of G if $V(G) = V(G')$.

Proposition 4.3. *Let G be a graph. If G' is a semi-magic spanning subgraph of G, G is semi-magic.*

Proof. Let G be a graph and let G' be a semi-magic spanning subgraph of G. Then, $\exists f : E(G') \to \mathbb{Z}$ such that $\forall v \in V(G)$, $\sum f(e) = m$ where the sum is taken over each edge incident to v. Define a new function $g : E(G) \to \mathbb{Z}$ by

$$g(e) = \begin{cases} f(e) & e \in E(G') \\ 0 & otherwise. \end{cases}$$

Then, for each vertex $v \in E(G)$, the sum of $g(e)$ for each edge incident to v is given by $\sum g(e) = \sum f(e) + \sum 0 = m$. Thus g is a semi-magic labeling of G. □

Corollary 4.4. *If G is a graph that contains a Hamiltonian cycle, then G is semi-magic.*

Proof. Let G be a graph that contains a Hamiltonian cycle. Then, C_n where $|V(G)| = n$ is a spanning subgraph of G. Then, $f : E(C_n) \to \mathbb{Z}$ where $f(e) = 1$ is a semi-magic labeling on C_n. Thus, by Proposition 4.3, G is semi-magic. □

To find semi-magic labelings for many of these zero-divisor graphs, we will find the following corollary useful.

Corollary 4.5. *If G is a connected graph which contains a non-trivial subgraph G' which is non-trivially semi-magic with magic constant $m = 0$, then G itself non-trivially semi-magic.*

Proof. Let $f : E(G') \to \mathbb{Z}$ be the non-trivial semi-magic labeling of the subgraph with magic constant $m = 0$. Then we simply assign 0 to every edge in $E(G) \setminus E(G')$ and see that this is a non-trivial semi-magic labeling of G also with magic constant 0. □

5. Tables

In this section, we compile all of the information about which zero-divisor graphs satisfy the various magic type labeling properties for small zero-divisor graphs. We use the work of Redmond in [5,6] where every possible zero-divisor graph which can arise on up to 14 vertices was determined. Many of the graphs are complete or complete bipartite which have been discussed thoroughly above, so we do not provide the actual labelings of these graphs since they are not terribly interesting. We use 'Yes*' for semi-magic labelings of $K_{m,n}$ to indicate, as in Proposition 3.1, that yes, some of these are non-trivially semi-magic, but every labeling must have magic constant $k = 0$. We use semi-magic in the table to mean non-trivially semi-magic to save space.

Zero-divisor graphs with 1 vertex.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_4$	4	K_1	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_2[X]/(X^2)$	4	K_1	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**

Zero-divisor graphs with 2 vertices.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_9$	9	K_2	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	K_2	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_3[X]/(X^2)$	9	K_2	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**

Zero-divisor graphs with 3 vertices.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_6$	6	$K_{1,2}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_8$	8	$K_{1,2}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_2[X]}{(X^3)}$	8	$K_{1,2}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_4[X]}{(2X,X^2-2)}$	8	$K_{1,2}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2}$	8	K_3	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_4[X]}{(2,X)^2}$	8	K_3	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{F}_4[X]}{(X^2)}$	16	K_3	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_4[X]}{(X^2+X+1)}$	16	K_3	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**

Zero-divisor graphs with 4 vertices.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{F}_4$	8	$K_{1,3}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	$K_{2,2}$	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_{25}$	25	K_4	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_5[X]}{(X^2)}$	25	K_4	Yes - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**

Zero-divisor graphs with 5 vertices.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{Z}_5$	10	$K_{1,4}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_3 \times \mathbb{F}_4$	12	$K_{2,3}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	Fig. 6	No - **Prop. 5.1**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	8	Fig. 6	No - **Prop. 5.1**	No - **Prop. 4.1**	No - **Prop. 2.3**

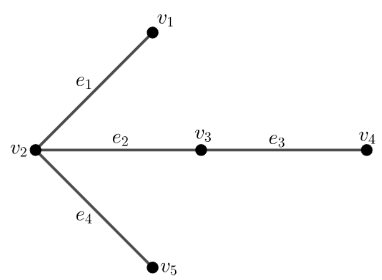


FIGURE 6

Proposition 5.1. *Figure 6 is not non-trivially semi-magic, and thus not magic or super-magic either.*

Proof. Consider the graph G with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_2v_5\}$ (Figure 6 is the graph G). Let e_1 be the edge-labeling for the edge v_1v_2, e_2 for the edge v_2v_3, e_3 for the edge v_3v_4 and e_4 for the edge v_2v_5. Then suppose for the sake of contradiction that G is semi-magic, i.e., G admits a semi-magic labeling.

Let $m \in \mathbb{Z}$ represent the magic constant at each vertex in G. Then because G is semi-magic, it must be true that the sum of the labelings for all edges incident to v_i $(1 \leq i \leq 5)$ is m. Thus, we have the following 5 linear equations:

$$\begin{cases} e_1 = m \\ e_1 + e_2 + e_4 = m \\ e_2 + e_3 = m \\ e_3 = m \\ e_4 = m. \end{cases}$$

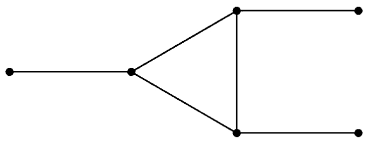
This is a linear system in the variables e_1, e_2, e_3, e_4, and we can use a matrix to solve this system of equations by row-reducing the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & m \\ 1 & 1 & 0 & 1 & m \\ 0 & 1 & 1 & 0 & m \\ 0 & 0 & 1 & 0 & m \\ 0 & 0 & 0 & 1 & m \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m \end{array}\right]$$

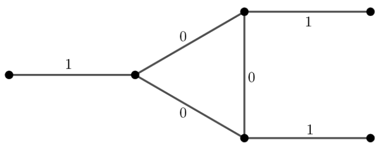
Notice the fifth row of the row reduced augmented matrix contains the statement $0 = m$, which implies that G is only trivially semi-magic. Our assumption that G was non-trivially semi-magic must have been incorrect, thus G (and Figure 6) are not non-trivially semi-magic. □

Zero-divisor graphs with 6 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_3 \times \mathbb{Z}_5$	15	$K_{2,4}$	Yes* - **Prop. 2.3**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_4 \times \mathbb{F}_4$	16	$K_{3,3}$	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_{49}$	49	K_6	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\frac{\mathbb{Z}_7[X]}{(X^2)}$	49	K_6	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	Fig. 7	Yes - **Figure 7**	No - **Prop. 4.1**	No - **Prop. 2.3**



Unlabeled graph.

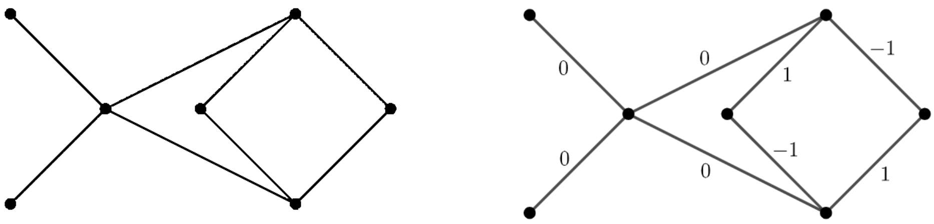


Semi-magic labeling.

FIGURE 7

Zero-divisor graphs with 7 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	$K_{1,6}$	No - **P. 3.1**	No - **P. 3.2**	No - **P. 3.3**
$\mathbb{F}_4 \times \mathbb{Z}_5$	20	$K_{3,4}$	Yes* - **P. 3.1**	No - **P. 3.2**	No - **P. 3.3**
$\mathbb{Z}_3 \times \mathbb{Z}_4$	12	Fig. 8	Yes - **Figure 8**	No - **P. 4.1**	No - **P. 2.3**
$\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	12	Fig. 8	Yes - **Figure 8**	No - **P. 4.1**	No - **P. 2.3**
$\mathbb{Z}_{16}$	16	Fig. 9	Yes - **Figure 9**	No - **P. 4.1**	No - **P. 2.3**
$\frac{\mathbb{Z}_2[X]}{(X^4)}$	16	Fig. 9	Yes - **Figure 9**	No - **P. 4.1**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^2+2)}$	16	Fig. 9	Yes - **Figure 9**	No - **P. 4.1**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^2+2X+2)}$	16	Fig. 9	Yes - **Figure 9**	No - **P. 4.1**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^3-2,2X^2,2X)}$	16	Fig. 9	Yes - **Figure 9**	No - **P. 4.1**	No - **P. 2.3**
$\frac{\mathbb{Z}_2[X,Y]}{(X^3,XY,Y^2)}$	16	Fig. 10	Yes - **Figure 10**	No - **P. 5.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_8[X]}{(2X,X^2)}$	16	Fig. 10	Yes - **Figure 10**	No - **P. 5.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^3,2X^2,2X)}$	16	Fig. 10	Yes - **Figure 10**	No - **P. 5.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X,Y]}{(X^2-2,XY,Y^2,2X,2Y)}$	16	Fig. 10	Yes - **Figure 10**	No - **P. 5.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^2+2X)}$	16	Fig. 11	Yes - **Figure 11**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_8[X]}{(2X,X^2+4)}$	16	Fig. 11	Yes - **Figure 11**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2-XY)}$	16	Fig. 11	Yes - **Figure 11**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2-XY,XY-2,2X,2Y)}$	16	Fig. 11	Yes - **Figure 11**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY-2,2X,2Y)}$	16	Fig. 12	Yes - **Figure 12**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2)}$	16	Fig. 12	Yes - **Figure 12**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_4[X]}{(X^2)}$	16	Fig. 12	Yes - **Figure 12**	No - **P. 4.2**	No - **P. 2.3**
$\frac{\mathbb{Z}_2[X,Y,Z]}{(X,Y,Z)^2}$	16	K_7	Yes - **P. 2.3**	Yes - **P. 2.3**	Yes - **P. 3.3**
$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY,2X,2Y)}$	16	K_7	Yes - **P. 2.3**	Yes - **P. 2.3**	Yes - **P. 3.3**
$\frac{\mathbb{F}_8[X]}{(X^2)}$	64	K_7	Yes - **P. 2.3**	Yes - **P. 2.3**	Yes - **P. 3.3**
$\frac{\mathbb{Z}_4[X]}{(X^3+X+1)}$	64	K_7	Yes - **P. 2.3**	Yes - **P. 2.3**	Yes - **P. 3.3**

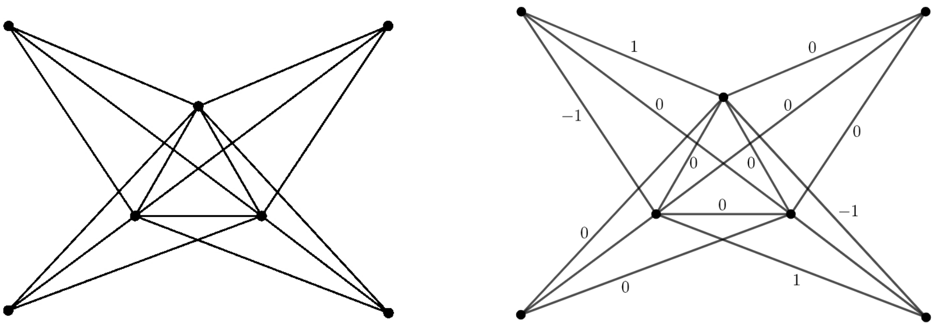


Unlabeled graph. Semi-magic labeling.

FIGURE 8

Unlabeled graph. Semi-magic labeling.

FIGURE 9



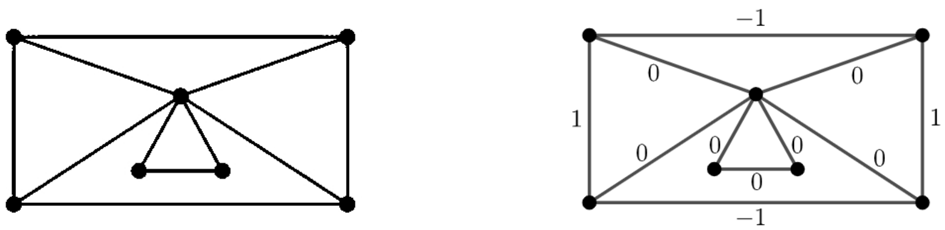
Unlabeled graph. Semi-magic labeling.

FIGURE 10

Proposition 5.2. *Figure 10 is not magic.*

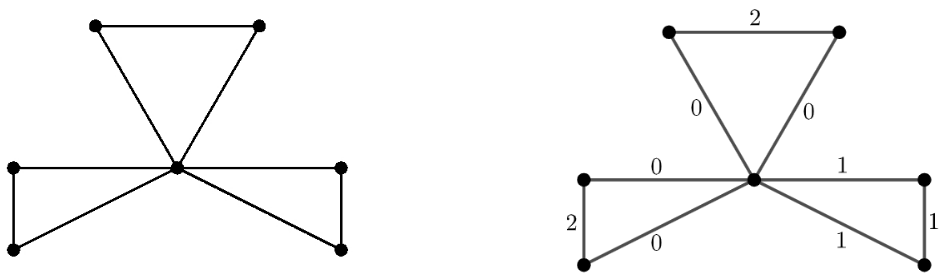
Proof. Suppose for the sake of contradiction that Figure 10 has a magic labeling. Let $m \in \mathbb{N}$ be the magic constant for Figure 10. We will refer to the outer

12 edges as $e_1, e_2, \dots, e_{11}, e_{12}$ and the inner three edges that form a triangle as e_{13}, e_{14}, e_{15}. Then, the outer four vertices of Figure 10 have incident edges that sum to m, so $\sum_{i=1}^{12} f(e_i) = 4m$. Since the inner three vertices that form a triangle have incident edges that sum to m, $\sum_{i=1}^{12} f(e_i) + [f(e_{13}) + f(e_{14}) + f(e_{15})] = 3m$, this implies that $4m = 3m - [f(e_{13}) + f(e_{14}) + f(e_{15})]$. Thus, $[f(e_{13}) + f(e_{14}) + f(e_{15})] = -m < 0$. This is a contradiction since $[f(e_{13}) + f(e_{14}) + f(e_{15})] > 0$ and $f(e_{13}), f(e_{14}), f(e_{15}) \in \mathbb{N}^*$ by definition. Therefore Figure 10 is not magic. $\square$



Unlabeled graph. Semi-magic labeling.

FIGURE 11

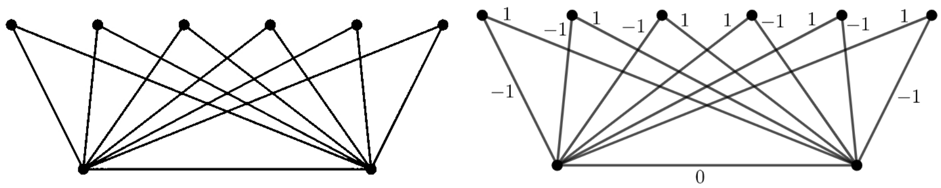


Unlabeled graph. Semi-magic labeling.

FIGURE 12

Zero-divisor graphs with 8 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{F}_8$	16	$K_{1,7}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_3 \times \mathbb{Z}_7$	21	$K_{2,6}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	$K_{4,4}$	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_{27}$	27	Fig. 13	Yes - **Figure 13**	No - **Prop. 5.3**	No - **Prop. 2.3**
$\frac{\mathbb{Z}_9[X]}{(3X,X^2-3)}$	27	Fig. 13	Yes - **Figure 13**	No - **Prop. 5.3**	No - **Prop. 2.3**
$\frac{\mathbb{Z}_9[X]}{(3X,X^2-6)}$	27	Fig. 13	Yes - **Figure 13**	No - **Prop. 5.3**	No - **Prop. 2.3**
$\frac{\mathbb{Z}_3[X]}{(X^3)}$	27	Fig. 13	Yes - **Figure 13**	No - **Prop. 5.3**	No - **Prop. 2.3**
$\frac{\mathbb{Z}_3[X,Y]}{(X,Y)^2}$	27	K_8	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_9[X,Y]}{3,X)^2}$	27	K_8	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{F}_9[X]}{(X^2)}$	81	K_8	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_9[X]}{(X^2+1)}$	81	K_8	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**



Unlabeled graph. Semi-magic labeling.

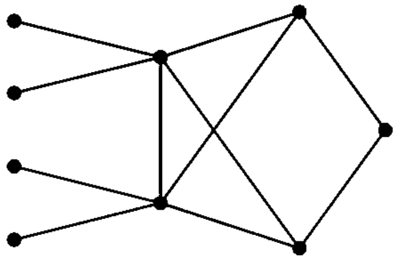
FIGURE 13

Proposition 5.3. *Figure 13 is not magic.*

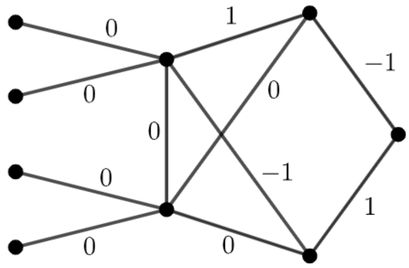
Proof. For notational convenience, we will refer to the horizontal edge at the bottom as e_{13} and each of the other edges as $e_1, e_2, \ldots, e_{11}, e_{12}$. Suppose for the sake of contradiction that Figure 13 has a magic labeling. Let $m \in \mathbb{N}$ be the magic constant for Figure 13. Then, each of the vertices on the top of Figure 13 has incident edges that sum to m, so $\sum_{i=1}^{12} f(e_i) = 6m$ and since the two vertices at the bottom of Figure 13 have incident edges that sum to m, $\sum_{i=1}^{12} f(e_i) + f(e_{13}) = 2m$. That implies $6m = 2m - f(e_{13})$. Thus $f(e_{13}) = -4m < 0$. This is a contradiction since $f(e_{13}) > 0$ by definition, therefore Figure 13 is not magic. $\square$

Zero-divisor graphs with 9 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{F}_9$	18	$K_{1,8}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_3 \times \mathbb{F}_8$	24	$K_{2,7}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_4 \times \mathbb{Z}_7$	28	$K_{3,6}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12	Fig. 14	Yes - **Figure 14**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_4 \times \mathbb{F}_4$	16	Fig. 15	Yes - **Figure 15**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$	16	Fig. 15	Yes - **Figure 15**	No - **Prop. 4.1**	No - **Prop. 2.3**

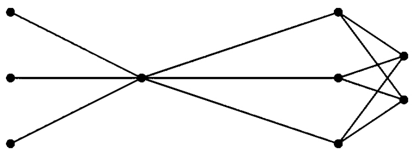


Unlabeled graph.

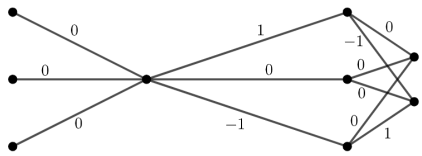


Semi-magic labeling.

FIGURE 14



Unlabeled graph.



Semi-magic labeling.

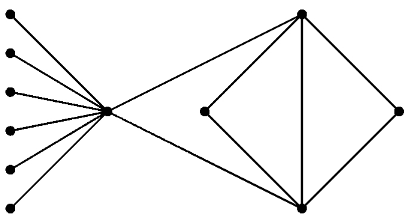
FIGURE 15

Zero-divisor graphs with 10 vertices.

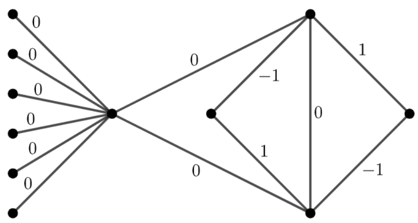
R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_3 \times \mathbb{F}_9$	27	$K_{2,8}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_4 \times \mathbb{F}_8$	32	$K_{3,7}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_5 \times \mathbb{Z}_7$	35	$K_{4,6}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_{121}$	121	K_{10}	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_{11}[X]/(X^2)$	121	K_{10}	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**

Zero-divisor graphs with 11 vertices.

R	$\lvert R\rvert$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{Z}_{11}$	22	$K_{1,10}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_4 \times \mathbb{F}_9$	36	$K_{3,8}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_5 \times \mathbb{F}_8$	40	$K_{4,7}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_9$	18	Fig. 16	Yes - **Figure 16**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[X]}{(X^2)}$	18	Fig. 16	Yes - **Figure 16**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_5 \times \mathbb{Z}_4$	20	Fig. 17	Yes - **Figure 17**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_5 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	20	Fig. 17	Yes - **Figure 17**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \mathbb{Z}_8$	16	Fig. 18	Yes - **Figure 18**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^3)}$	16	Fig. 18	Yes - **Figure 18**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[X]}{(2X, X^2-2)}$	16	Fig. 18	Yes - **Figure 18**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2}$	16	Fig. 19	Yes - **Figure 19**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[X]}{(2,X)^2}$	16	Fig. 19	Yes - **Figure 19**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_4 \times \mathbb{Z}_4$	16	Fig. 20	Yes - **Figure 20**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	16	Fig. 20	Yes - **Figure 20**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\frac{\mathbb{Z}_2[X]}{(X^2)} \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	16	Fig. 20	Yes - **Figure 20**	No - **Prop. 4.1**	No - **Prop. 2.3**

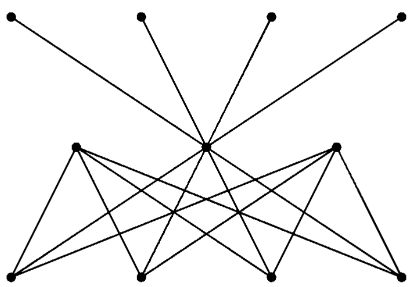


Unlabeled graph.

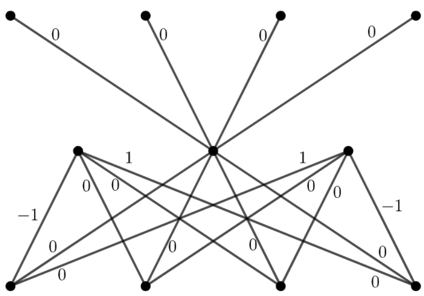


Semi-magic labeling.

FIGURE 16

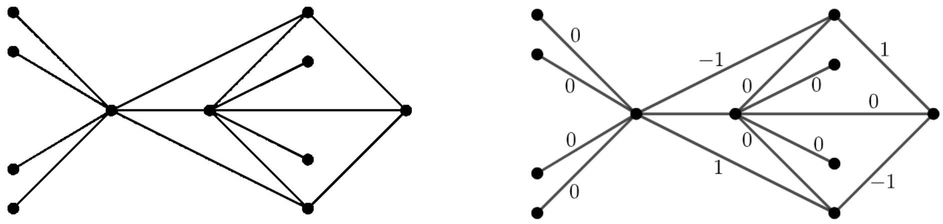


Unlabeled graph.



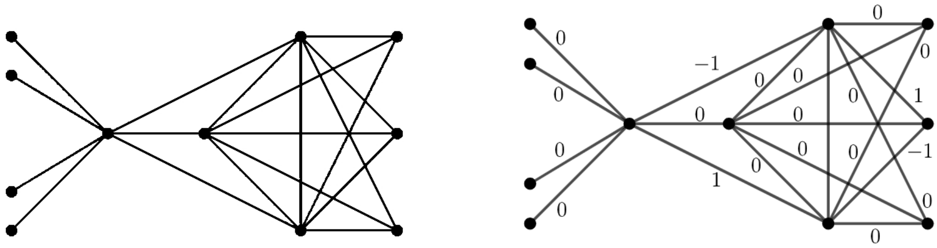
Semi-magic labeling.

FIGURE 17



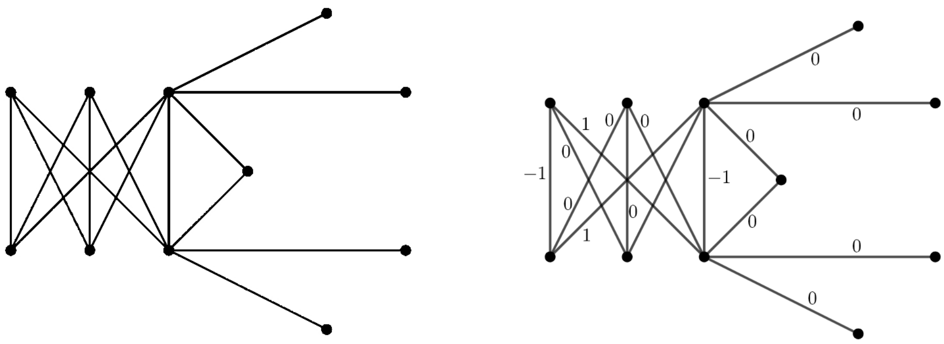
Unlabeled graph. Semi-magic labeling.

FIGURE 18



Unlabeled graph. Semi-magic labeling.

FIGURE 19

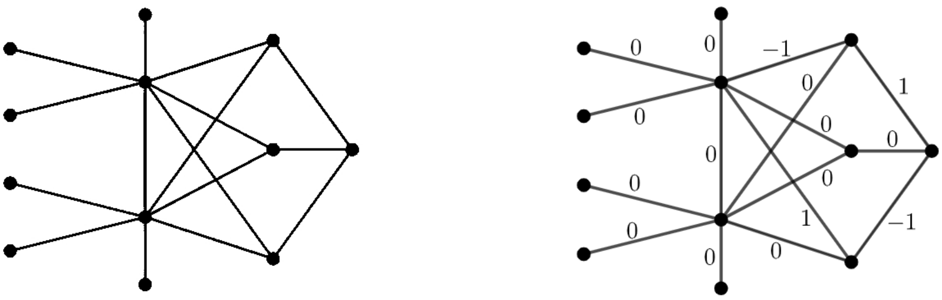


Unlabeled graph. Semi-magic labeling.

FIGURE 20

Zero-divisor graphs with 12 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_3 \times \mathbb{Z}_{11}$	33	$K_{2,10}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_5 \times \mathbb{F}_9$	45	$K_{4,8}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_7 \times \mathbb{Z}_7$	49	$K_{6,6}$	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$	16	Fig. 21	Yes - **Figure 21**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_{169}$	169	K_{12}	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**
$\frac{\mathbb{Z}_{13}[X]}{(X^2)}$	169	K_{12}	Yes - **Prop. 2.3**	Yes - **Prop. 3.2**	No - **Prop. 3.3**

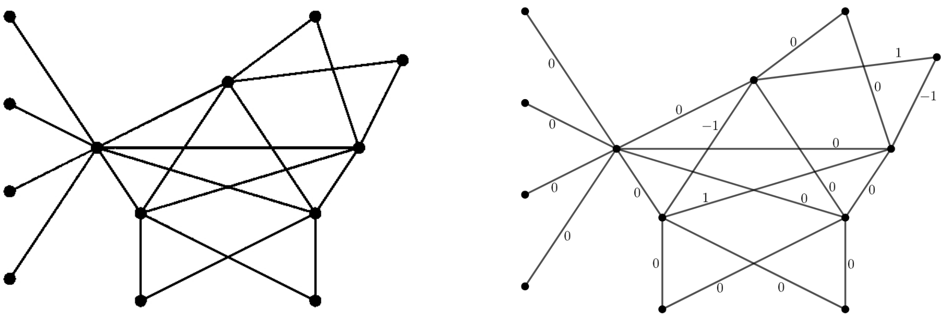


Unlabeled graph. Semi-magic labeling.

FIGURE 21

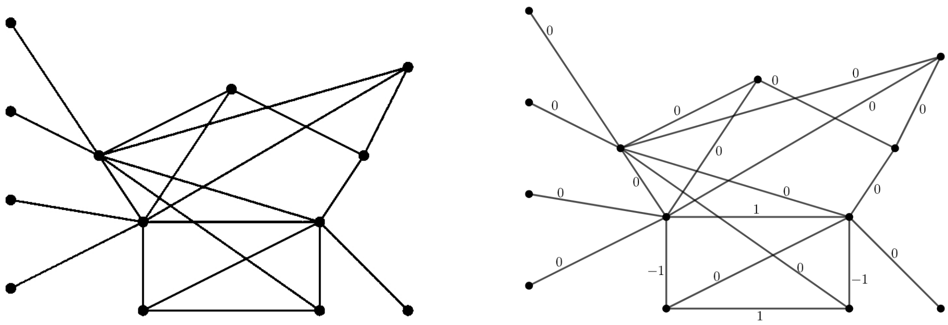
Zero-divisor graphs with 13 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_2 \times \mathbb{Z}_{13}$	26	$K_{1,12}$	No - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_4 \times \mathbb{Z}_{11}$	44	$K_{3,10}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_7 \times \mathbb{F}_8$	56	$K_{6,7}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	18	Fig. 22	Yes - **Figure 22**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	16	Fig. 23	Yes - **Figure 23**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$	16	Fig. 23	Yes - **Figure 23**	No - **Prop. 4.1**	No - **Prop. 2.3**



Unlabeled graph. Semi-magic labeling.

FIGURE 22

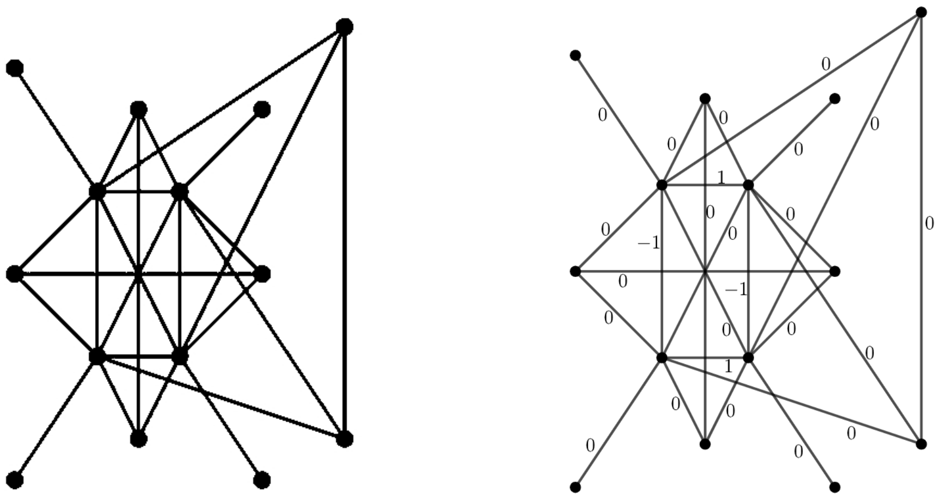


Unlabeled graph. Semi-magic labeling.

FIGURE 23

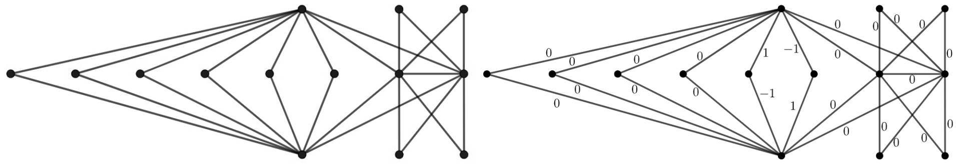
Zero-divisor graphs with 14 vertices.

R	$\|R\|$	Graph	Semi-magic	Magic	Super-magic
$\mathbb{Z}_3 \times \mathbb{Z}_{13}$	39	$K_{2,12}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_5 \times \mathbb{Z}_{11}$	55	$K_{4,10}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{Z}_7 \times \mathbb{F}_9$	63	$K_{6,8}$	Yes* - **Prop. 3.1**	No - **Prop. 3.2**	No - **Prop. 3.3**
$\mathbb{F}_8 \times \mathbb{F}_8$	64	$K_{7,7}$	Yes - **Prop. 2.3**	Yes - **Prop. 2.3**	Yes - **Prop. 3.3**
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	16	Fig. 24	Yes - **Figure 24**	No - **Prop. 4.1**	No - **Prop. 2.3**
$\mathbb{Z}_3 \times \mathbb{Z}_9$	27	Fig. 25	Yes - **Figure 25**	No - **Prop. 5.4**	No - **Prop. 2.3**
$\mathbb{Z}_3 \times \frac{\mathbb{Z}_3[X]}{(X^2)}$	27	Fig. 25	Yes - **Figure 25**	No - **Prop. 5.4**	No - **Prop. 2.3**



Unlabeled graph. Semi-magic labeling.

FIGURE 24



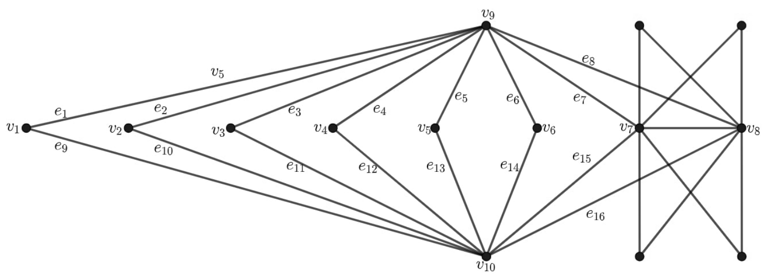
Unlabeled graph. Semi-magic labeling.

FIGURE 25

Proposition 5.4. *Figure 25 is not magic.*

Proof. Suppose for the sake of contradiction that the graph G of Figure 25 is magic with injective edge labeling $f : E \to \mathbb{N}^+$. Then there exists a positive integer m as the magic constant for G.

First consider vertices v_9 and v_{10} as shown in the partially labeled diagram below:



Because the sum at any given vertex for a magic graph is m, for vertex v_9 we can write $m = f(e_1) + f(e_2) + \cdots + f(e_7) + f(e_8)$. Similarly for v_{10} we have $m = f(e_9) + f(e_{10}) + \cdots + f(e_{15}) + f(e_{16})$. Adding these two results gives us $2m = f(e_1) + f(e_2) + \cdots + f(e_{15}) + f(e_{16})$.

Next consider vertices $v_1, \ldots, v_6$. Looking at the edges incident to these vertices, we have $6m = f(e_1) + f(e_2) + f(e_3) + f(e_4) + f(e_5) + f(e_6) + f(e_9) + f(e_{10}) + f(e_{11}) + f(e_{12}) + f(e_{13}) + f(e_{14})$ (notice edges $7, 8, 15, 16$ are missing). If we take this equation and subtract the equation at the end of the previous paragraph, we get $4m = -(f(e_7) + f(e_8) + f(e_{15}) + f(e_{16}))$. However, $f(e_7) + f(e_8) + f(e_{15}) + f(e_{16})$ is a positive number implying that m is negative. This contradicts the assumption that G is magic, thus we conclude that G is not magic. $\square$

Questions 5.1. *We conclude with a few open questions which seem to follow from the patterns arising in these tables with small numbers of vertices. It seems most exotic (non complete or complete bipartite) zero-divisor graphs are all semi-magic,* *but never magic or super-magic for these small examples. We wonder if some patterns we have identified here continue for larger zero-divisor graphs? We formalize a few of these questions below.*

(1) *Are there any zero-divisor graphs which are not non-trivially semi-magic besides those of the form $K_{1,n}$ for $n > 1$ or Figure 6?*
(2) *Are there infinitely many integers n for which every zero-divisor graph on n vertices is semi-magic? We note that $\mathbb{Z}_2 \times \mathbb{F}_n$ (where $\mathbb{F}_n$ is finite field of order n) will have zero-divisor graph $K_{1,|n|-1}$ which has n vertices. Every finite field has order p^m for some integer m. Thus for every integer which is a power of a prime, there will always be a zero-divisor graph which is a star graph and not semi-magic. Thus, to prove this it seems like the most likely place to look to generate integers where every zero-divisor graph is semi-magic would be the squarefree semi-primes ($n = pq$ for distinct primes p, q).*
(3) *Are there any zero-divisor graphs which are magic that are not of of the form K_n for $n = 2$ or $n \geq 5$ or $K_{m,n}$ for $m = n > 2$?*
(4) *Are there any zero-divisor graphs which are super-magic which are not of the form K_n with $n = 2$ or $n > 5$ and $n \not\equiv 0 \pmod 4$ or $K_{m,n}$ with $m = n > 2$?*
(5) *Are there any zero divisor graphs which are magic but not super magic? It is known that there are many graphs which are magic but not supermagic. K_5 is one such graph, but K_5 is not the zero-divisor graph of some ring.*
(6) *Are there any magic or super-magic zero-divisor graphs which are not regular?*

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