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THE RING $R\{X\}$

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Abstract. Let R be a commutative ring with unity and $W = \{f(X) \in R[X] : f(0) = 1\}$. We define $R\{X\} = W^{-1}R[X]$. We show that the maximal ideals of $R\{X\}$ are of the form $W^{-1}(M, X)$ where M is a maximal ideal of R, and so if R is finite dimensional, then $\dim R\{X\} = \dim R[X]$. We show that $R\{X\}$ is a Prüfer ring if and only if R is a von Neumann regular ring, and so if $R\{X\}$ satisfies one of the Prüfer conditions, it satisfies all of them.

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1. Introduction

Throughout, R will denote a commutative ring with unity and X an indeterminate over R. For each polynomial $f(X) = \sum_{i=0}^{n} f_iX^i \in R[X]$, the content of f, denoted by $c(f)$ is the ideal $(f_0, \dots, f_n)$. Many multiplicative closed subsets of $R[X]$ were defined to reduce an overring of $R[X]$, such as $S = \{f(X) \in R[X] : c(f) = R\}$ and $U = \{f(X) \in R[X] : f \text{ is monic}\}$. The Nagata ring $R(X) = S^{-1}R[X]$ and Serre's conjecture ring $R\langle X\rangle = U^{-1}R[X]$ are widely known and were studied by many mathematicians in the last decades, see for example [1], [3], [8] and for detailed newly bibliography, see [6]. For more multiplicative closed subsets of $R[X]$, see [4] and [5]. Let $W = \{f(X) \in R[X] : f(0) = 1\}$. Then clearly W is a multiplicative closed subset of $R[X]$, and thus we can define an overring for $R[X]$ using this set. Let $R\{X\} = W^{-1}R[X]$. This ring was suggested in [1, page 97] as it has applications in automata theory. We didn't find any mentioning of this ring since then. In this article, we are interested in knowing if R has a certain property whether $R\{X\}$ has this property and conversely. We characterize maximal ideals in $R\{X\}$, we show that there is a one-to-one correspondence between the maximal ideals of R and the maximal ideals of $R\{X\}$ given by $M \leftrightarrow W^{-1}(M, X)$. We also show that there is a one-to-one correspondence between the minimal prime ideals of R and the minimal prime ideals of $R\{X\}$ given by $P \leftrightarrow W^{-1}P[X]$. We show that for each $M \in Max(R)$, we have $R_M\{X\} \approx R[X]_{(M,X)} \approx R\{X\}_{W^{-1}(M,X)R}$. Thus we conclude that if R is a finite dimensional ring, then $\dim R[X] = \dim R\{X\}$. Then we turn to the problem of characterizing when $R\{X\}$ satisfies any of the Prüfer conditions. We show that a ring R is von Neumann regular if and only if $R\{X\}$ is a Prüfer ring. So we conclude if $R\{X\}$ satisfies any one of the Prüfer conditions, it satisfies all of them. There are still a lot of properties to be investigated in this ring.

2. Construction

Let R be a ring, X an indeterminate over R, and let $R[X]$ be the polynomial ring of R. Let $W = \{f(X) \in R[X] : f(0) = 1\}$. Then W is a multiplicative closed subset of $R[X]$, and thus we can define an overring for $R[X]$ using this set. Let $R\{X\} = W^{-1}R[X]$. One notice immediately that $R\{X\} \subseteq R(X) \subseteq T(R[X])$, the total quotient ring of $R[X]$, and so we can use some properties of $R(X)$ to study properties of $R\{X\}$, for instance the idempotents of $R\{X\}$ are those of R, since we have the same case in $R(X)$. Also $Z(R) = Nil(R)$ if and only if $Z(R\{X\}) = Nil(R\{X\})$.

The saturation set of W is $W^* = \{f(X) \in R[X] : f(X) \text{ is a unit in } R\{X\}\} = \{f(X) \in R[X] : f(0) \text{ is a unit in } R\}$, and in this case W^* is the largest multiplicatively closed subset of $R[X]$ containing W such that $W^{-1}R[X] = {W^*}^{-1}R[X]$. Thus $R\{X\} \subset R(X) \subset T(R[X])$.

It is clear that R is an integral domain if and only if so is $R\{X\}$. Similar results are obtained if R is reduced or Noetherian, since $R\{X\}$ is faithful flat over R. Note that if $\dfrac{f(X)}{g(X)} = \dfrac{a}{b} \in R\{X\} \cap T(R)$, then $bf(0) = a$, and so $\dfrac{f(X)}{g(X)} = \dfrac{f(0)}{1} \in R$, that is $R\{X\} \cap T(R) = R$, and so if $R\{X\}$ is integrally closed, then so is R. If R was an integral domain, then the converse would be also true.

The Nagata ring $R(X) = S^{-1}R[X]$ and Serre's conjecture ring $R\langle X\rangle = U^{-1}R[X]$ are very related to our new ring $R\{X\}$. Since $W \subset S$, $R\{X\}$ is a subring of $R(X)$, while it is incomparable with $R\langle X\rangle$. The three rings share many properties being overrings for $R[X]$, faithfully flat, have the same shape of minimal prime ideals. The ring $R\{X\}$ as $R(X)$ has a concrete shape of maximal ideals $(M, X)R\{X\}$ $(MR(X))$ where $M \in Max(R)$, while this not the only shape of maximal ideals in $R\langle X\rangle$. Since X is not a unit in $R\{X\}$, unlike $R(X)$ and $R\langle X\rangle$, $\dim R\{X\} = \dim R[X]$, while it is $\dim R[X]-1$ for $R(X)$ and $R\langle X\rangle$. This also leads to that $R\{X\}$ is never a Hilbert ring, unlike $R(X)$ and $R\langle X\rangle$.

3. Prime ideals in $R\{X\}$

We try to relate prime ideals of $R\{X\}$ with those of R. We first characterize maximal ideals in $R\{X\}$, and then use it to characterize some prime ideals. In $R(X)$ the maximal ideals are of the form $MR(X)$, where M is a maximal ideal in R, while for the ring $R\langle X\rangle$ the maximal ideals are of the form $MR\langle X\rangle$, where M is a maximal ideal in R, or of the form $QR\langle X\rangle$ for some prime ideal Q of $R[X]$ which is an upper to a non-maximal prime ideal P of R.

Lemma 3.1. *Let $\mathcal{M}$ be a maximal ideal in $R[X]$ with $f(0)\neq 1$ for each $f(X)\in\mathcal{M}$. Then $\mathcal{M}=(M,X)$ for some maximal ideal M of R.*

Proof. Let $M=\{f(0):f(X)\in\mathcal{M}\}$. Then clearly M is a proper ideal of R. Assume N is an ideal of R with $M\subset N\subseteq R$, and let $n\in N-M$. Then $n\notin\mathcal{M}$, and so $nR[X]+\mathcal{M}=R[X]$. Whence $ng(X)+m(X)=1$ for some $g(X)\in R[X]$ and $m(X)\in\mathcal{M}$. So $1=ng(0)+m(0)\in N$, hence M is a maximal ideal of R. But $\mathcal{M}\subseteq(M,X)\subset R[X]$. By maximality of $\mathcal{M}$, we get the result. □

Theorem 3.2. *There is a one-to-one correspondence between the maximal ideals of R and the maximal ideals of $R\{X\}$ given by $M\leftrightarrow W^{-1}(M,X)$.*

Proof. Let $M\in Max(R)$, and let $\mathcal{M}=W^{-1}(M,X)$. Then clearly, $\mathcal{M}$ is a prime ideal in $R\{X\}$. Assume $\mathcal{N}$ is an ideal of $R\{X\}$ with $\mathcal{M}\subset\mathcal{N}\subseteq R\{X\}$. Let $\frac{f}{g}\in\mathcal{N}-\mathcal{M}$. Then $f\notin(M,X)$ and so $f(0)\notin M$. By maximality of M, there exist $a\in R$ and $m\in M$ such that $1=f(0)a+m$, and so $af+m\in W$. But $\frac{af}{g}+\frac{m}{g}\in\mathcal{N}$. Therefore $\mathcal{N}=R\{X\}$ and $\mathcal{M}$ is a maximal ideal in $R\{X\}$.

Conversely, let $\mathcal{M}\in Max(R\{X\})$ and let $M=\{f(0):\frac{f}{g}\in\mathcal{M}\}$. Then M is a proper ideal of R since $1\notin M$. Assume N is an ideal of R with $M\subset N\subseteq R$, and let $n\in N-M$. Then $n\notin\mathcal{M}$, and so $nR\{X\}+\mathcal{M}=R\{X\}$. Thus $1=\frac{nf}{\alpha}+\frac{m}{\beta}$ with $\frac{f}{\alpha}\in R\{X\}$ and $\frac{m}{\beta}\in\mathcal{M}$, which implies that $\alpha\beta=nf\beta+m\alpha$. Thus $1=\alpha(0)\beta(0)=nf(0)\beta(0)+m(0)\alpha(0)\in N$, i.e., $M\in Max(R)$. But $\mathcal{M}\subseteq W^{-1}(M,X)\subset R\{X\}$, and so by maximality of $\mathcal{M}$, we have $\mathcal{M}=W^{-1}(M,X)$. □

For the case of minimal prime ideals, we have a one-to-one correspondence between the minimal prime ideals of R and the minimal prime ideals of $R(X)$ ($R\langle X\rangle$) given by $P\leftrightarrow PR(X)$ ($P\leftrightarrow PR\langle X\rangle$). A similar result is also true for $R\{X\}$.

Theorem 3.3. *There is a one-to-one correspondence between the minimal prime ideals of R and the minimal prime ideals of $R\{X\}$ given by $P \leftrightarrow W^{-1}P[X]$.*

Proof. Let $P \in Min(R)$. Then $W^{-1}P[X]$ is a prime ideal of $R\{X\}$. If $Q \subseteq W^{-1}P[X]$ is a prime ideal of $R\{X\}$, then $Q = W^{-1}I$ for some prime ideal I of $R[X]$. Clearly, $P_0 = I \cap R$ is a prime ideal of R with $P_0 \subseteq P$. By minimality of P, we must have $P_0 = P$. So $P[X] = P_0[X] \subseteq I \subseteq P[X]$. Thus $Q = W^{-1}P[X]$.

Conversely, let $\mathcal{P} \in Min(R\{X\})$ and let I be a prime ideal of $R[X]$ with $\mathcal{P} = W^{-1}I$. The ideal $P = I \cap R$ is a prime ideal in R with $P[X] \subseteq I$. Thus $W^{-1}P[X] \subseteq W^{-1}I = \mathcal{P}$. By minimality of $\mathcal{P}$, we have $\mathcal{P} = W^{-1}P[X]$. Now if P_0 is a prime ideal of R with $P_0 \subseteq P$, then $W^{-1}P_0[X] \subseteq W^{-1}P[X] = \mathcal{P}$, and so $W^{-1}P_0[X] = W^{-1}P[X]$. If $a \in P$, then $\frac{a}{1} \in W^{-1}P[X] = W^{-1}P_0[X]$, and so $\frac{a}{1} = \frac{f}{g}$ with $f \in P_0[X]$ and $g \in W$. Thus $a = ag(0) = f(0) \in P_0$. Hence $P \in Min(R)$. □

The following result can not be found in $R\,(X)$ nor $R\,\langle X\rangle$, since in these rings X is a unit.

Theorem 3.4. *If $\mathcal{Q}$ is a prime ideal in $R\{X\}$ with $X \in \mathcal{Q}$, then $\mathcal{Q} = W^{-1}(P, X)$ for some prime ideal P of R.*

Proof. Let Q be a prime ideal of $R[X]$ such that $\mathcal{Q} = W^{-1}Q$, and let $P = Q \cap R$. Then we have $P[X] \subset (P, X) \subseteq Q$. Thus $Q = (P, X)$, since the prime ideal P has at most two prime ideals of $R[X]$ lying over it, see [2, Corollary 30.2]. □

Corollary 3.5. *If Q is a P-primary ideal in R, then*

(1) $W^{-1}(Q, X)$ *is* $W^{-1}(P, X)$*-primary in* $R\{X\}$.
(2) $W^{-1}Q$ *is* $W^{-1}P$*-primary in* $R\{X\}$.

For any maximal ideal M of R, we have $R_M(X) \approx R[X]_{M[X]} \approx R(X)_{MR(X)}$, while if $\mathcal{M}$ is a maximal ideal of $R\,\langle X\rangle, Q = \mathcal{M} \cap R[X]$ and $P = Q \cap R$, then $R\,\langle X\rangle_{\mathcal{M}} \approx R[X]_Q \approx R_P[X]_{Q_{R\setminus P}}$.

Theorem 3.6. *For each $M \in Max(R)$, we have*

$$R_M\{X\} \approx R[X]_{(M,X)} \approx R\{X\}_{W^{-1}(M,X)}.$$

Proof. Let $\varphi_1\colon R[X]_{(M,X)} \longrightarrow R_M\{X\}$ be defined by $\varphi_1\left(\frac{f}{g}\right) = \frac{\frac{f}{g(0)}}{\frac{g}{g(0)}}$. Then clearly, φ_1 is a monomorphism.

Let $\dfrac{\sum \frac{a_i}{\alpha_i}x^i}{\sum \frac{b_i}{\beta_i}x^i} \in R_M\{X\}, a_j^{'} = \dfrac{a_j}{\alpha_j}\alpha, b_j^{'} = \dfrac{b_j}{\beta_j}\beta$, where $\alpha = \prod \alpha_i$ and $\beta = \prod \beta_i$, and note that $b_0^{'} = \beta$. Now,

$$\varphi_1\left(\frac{\beta}{\alpha}\frac{\sum a_i^{'}x^i}{\sum b_i^{'}x^i}\right) = \frac{\frac{1}{\alpha}\sum a_i^{'}x^i}{\frac{1}{\beta}\sum b_i^{'}x^i} = \frac{\sum \frac{a_i}{\alpha_i}x^i}{\sum \frac{b_i}{\beta_i}x^i}.$$

Thus $R_M\{X\} \approx R[X]_{(M,X)}$.

Note that we can write

$$R_M\{X\} = \left\{\frac{f}{1+xg} : f, g \in R_M[X]\right\}.$$

Let $\varphi_2\colon\ R_M\{X\} \longrightarrow R\{X\}_{W^{-1}(M,X)}$ be defined by $\varphi_2\left(\frac{f}{1+xg}\right) = \frac{\frac{f}{1}}{\frac{1+xg}{1}}$. Then clearly, φ_2 is a monomorphism.

Let $\frac{\frac{f}{1+xg}}{\frac{h}{1+xk}} \in R\{X\}_{W^{-1}(M,X)}$. Then $h(0) \notin M$, and so $\frac{\frac{1}{h(0)}f(1+xk)}{\frac{1}{h(0)}h(1+xg)} \in R_M\{X\}$, thus we have $\varphi_2\left(\frac{\frac{1}{h(0)}f(1+xk)}{\frac{1}{h(0)}h(1+xg)}\right) = \frac{\frac{\frac{1}{h(0)}f(1+xk)}{1}}{\frac{\frac{1}{h(0)}h(1+xg)}{1}} = \frac{\frac{f}{1+xg}}{\frac{h}{1+xk}}$.

Hence $R_M\{X\} \approx R\{X\}_{W^{-1}(M,X)}$. □

We end up this section with calculating the Krull dimension of $R\{X\}$.

Theorem 3.7. *If R is a finite dimensional ring, then* $\dim R\{X\} = \dim R[X]$.

Proof. Let $\mathcal{M}$ be a maximal ideal in $R[X]$ of maximal height. Then $M = \mathcal{M} \cap R$ is a maximal ideal in R. By [2, page 368] and [7, page 25], we may find a chain of maximal length in $R[X]$ of the form $P \subset \cdots \subset M[X] \subset \mathcal{M}$, and so $P \subset \cdots \subset M[X] \subset (M,X)$ is a chain of maximal length too, since there are no prime ideals properly between $M[X]$ and (M,X). Thus $\dim R\{X\} = \dim R[X]$. □

It was shown in [8, Theorem 2.1] that for a finite dimensional ring, $\dim R\langle X\rangle = \dim R[X] - 1$, and so for a Noetherian ring, $\dim R\langle X\rangle = \dim R$. A similar result is also true for the ring $R(X)$. Thus we can conclude the following corollary.

Corollary 3.8. *Let R be a Noetherian ring. Then*

$$\dim R + 1 = \dim R(X) + 1 = \dim R\langle X\rangle + 1 = \dim R[X] = \dim R\{X\}.$$

A ring R is called a Hilbert ring if any prime ideal of R is the intersection of all maximal ideals containing it. It was shown in [1, Lemma 4.1] that $R(X)$ is Hilbert if and only if R is Hilbert and every prime ideal of R is the extension of a prime ideal of R. Here $R\{X\}$ is never a Hilbert ring, since if P is a prime ideal of R, $W^{-1}P[X]$ is a prime ideal in $R\{X\}$ that is not an intersection of maximal ideals, since $X \notin W^{-1}P[X]$.

4. Prüfer conditions

In this section, we characterize the case at which the ring $R\{X\}$ is a Prüfer ring. But first we give some definitions and facts. The six well known Prüfer conditions:

(1) R is a Prüfer ring (every finitely generated regular ideal in R is invertible).
(2) R is a strongly Prüfer ring (every finitely generated dense ideal in R is locally principal).
(3) R is a Gaussian ring (for every $f, g \in R[X], c(fg) = c(f)c(g)$).
(4) R is an arithmetical ring (every finitely generated ideal of R is locally principal).
(5) $w.\dim(R) \leq 1$ (every finitely generated ideal of R is flat).
(6) R is semihereditary (every finitely generated ideal of R is projective).

It is known that if R is an integral domain, then (1) to (6) are all equivalent, but if R is not an integral domain, then (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), while the reverse implications are all false.

One of the main questions raised for the rings $R(X)$ and $R\langle X\rangle$ were when they satisfy one of the Prüfer conditions. Full characterizations can be found in [1] and [6]. We now use [6, Remark 2.1] to study when $R\{X\}$ satisfies the Prüfer conditions. We first recall the correspondent results for $R(X)$ and $R\langle X\rangle$.

Proposition 4.1. ([1, Theorem 3.2]) *Let R be a commutative ring with* 1.

(1) *$R(X)$ is a Prüfer ring if and only if R is strongly Prüfer.*
(2) *$R\langle X\rangle$ is a Prüfer ring if and only if R is strongly Prüfer,* $\dim R \leq 1$, *and R_P is a field for every non-maximal prime ideal P of R.*

Lemma 4.2. *Let I be an ideal of a ring R. Then I is finitely generated and locally principal if and only if $W^{-1}I$ is finitely generated and locally principal.*

Proof. The result follows easily by Theorem 3.6, since for any $M \in Max(R)$, we have $I_M = I_{W^{-1}(M,X)}$. □

Theorem 4.3. *R is von Neumann regular if and only if $R\{X\}$ is a Prüfer ring.*

Proof. ($\Rightarrow$) If R is von Neumann regular, then $R[X]$ is a Prüfer ring, and so is its localization $R\{X\}$.

($\Leftarrow$) Assume now that $R\{X\}$ is a Prüfer ring. Then $R(X)$ Prüfer being a localization of $R\{X\}$ and so R is strongly Prüfer. We want to show that R_M is a field for each $M \in Max(R)$. So let $M \in Max(R)$, $m \in M - \{0\}$ and $I = (m, X)$. Then I is a finitely generated regular ideal in $R[X]$, and so $IR\{X\}$ is invertible. Let

$\mathcal{M} = W^{-1}(M, X)_W$. Then $I_{(M,X)} = W^{-1}I_{\mathcal{M}}$ is principal, since $W^{-1}I$ is invertible. But $R[X]_{(M,X)} \approx R\{X\}_{W^{-1}(M,X)}$ is Prüfer with (M, X) is the unique regular maximal ideal of $R[X]_{(M,X)}$, and so it is a valuation ring (i.e., for any ideals A and B of $R[X]_{(M,X)}$, we have $A \subseteq B$ or $B \subseteq A$). Thus $I_{(M,X)} = (X)_{(M,X)}$, since we can not have $(X)_{(M,X)} \subseteq (m)_{(M,X)}$. So there exist $f, g \in R[X]$, with $g \notin (M, X)$ with $\frac{m}{1} = X\frac{f}{g}$, and hence $mgh = Xfh$ for some $h \notin (M, X)$. Thus we have $mg(0)h(0) = 0$, and so $\frac{m}{1} = \frac{0}{1}$ in R_M, since $g(0)$ and $h(0)$ are units in R_M. This yields that $M_M = 0$ in R_M, and so R_M is a field for each $M \in Max(R)$. □

Corollary 4.4. *If $R\{X\}$ satisfies any of the Prüfer conditions, then it satisfies all the Prüfer conditions.*

Proof. If $R\{X\}$ satisfies any of the Prüfer conditions, then it is Prüfer, and so R is a von Neumann regular ring. Thus it follows by [6, Remark 2.1] that $R[X]$ is semihereditary, which implies that $R\{X\}$ is semihereditary, hence the result. □

Note that if $R\{X\}$ satisfies any of the Prüfer conditions, then so are $R(X)$ and $R\langle X\rangle$, because in this case $R[X]$ is semihereditary, which implies that $R(X)$ and $R\langle X\rangle$ are semihereditary, being localizations of $R[X]$. On the other hand since the ring of integers $\mathbb{Z}$ is semihereditary and $\mathbb{Z}_{(0)} = \mathbb{Q}$ is a field, the integral domains $\mathbb{Z}(X)$ and $\mathbb{Z}\langle X\rangle$ are semihereditary, but $\mathbb{Z}\{X\}$ is not Prüfer, since $\mathbb{Z}$ is not a von Neumann regular ring.

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