Celal Bayar University Journal of Science
Volume 20, Issue 2, 2024, p 54-60
Doi: 10.18466/cbayarfbe.1479066 

**Celal Bayar University Journal of Science**

# An Alternative Approach to Find the Position Vector of a General Helix

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Received: 6 May 2024
Accepted: 14 June 2024
DOI: 10.18466/cbayarfbe.1479066

**Abstract**

In this paper, we introduce an alternative approach centered around an alternative moving frame for finding the position vector of a general helix given its curvature and torsion. Our methodology begins by formulating a vector differential equation, leveraging the unit principal normal vector of a general helix with the assistance of the alternative moving frame. Then, by solving this differential equation, we obtain the position vector of the general helix. This innovative technique is then applied to ascertain the position vector of a circular helix. To illustrate the effectiveness of our method, we showcase parametric representations of various general helices, each defined by unique curvature and torsion functions.

**Keywords:** Alternative moving frame, curvatures, general helix, position vector

## 1. Introduction

The fundamental theorem for curves states that a space curve can be uniquely determined up to rigid motions by its curvature and torsion [1,2]. The problem of determining the position vector of this curve is known as *solving natural or intrinsic equations* and is usually achieved by solving a certain complex Riccati equation [3]. However, the solution usually cannot be obtained explicitly. Explicit solutions have only been found for some special curves. First known example is an explicit integral formula by Euler for a planar curve, which is one of these special curves [4]. Recently, Ali constructed a vector differential equation according to the Frenet vectors by the aid of Frenet formulae to find the position vector of a space curve. Since this vector differential equation has variable coefficients, he did not find a general solution for an arbitrary curve. For the special cases of general helices and slant helices, he solved the vector differential equations and obtained the parametric representations of these curves in Euclidean 3-space [5,6]. After that, this problem has been investigated for different types of curves such as general helices, circular helices, slant helices, $k$-slant helices, relatively normal-slant helices, and isophote curves. These studies that constitute a vast literature on this subject include the handling of the curves in Euclidean, Minkowski, and Galilean spaces with the help of different moving frames such as Frenet frame, Darboux frame, type-2 Bishop frame, and alternative moving frame [7-16].

There are two important reasons why the problem of finding the position vector of a curve still attracts researchers. Firstly, in Galilean space, this problem has been resolved by Ali in [7], however in Euclidean and Minkowski spaces, it remains unresolved for any curve. Secondly, since the trajectory followed by a particle moving in space can be thought of as a curve, examining the position vector of the curve is an important goal for determining the behavior of the particle.

This paper presents an approach utilizing an alternative moving frame to ascertain the position vector of a general helix given its curvature and torsion. Initially, we delve into the fundamentals of Frenet and alternative moving frames for space curves. Leveraging the derivative formulae of alternative moving frame, we formulate a vector differential equation based on the principal normal vector of the general helix. Solving this equation yields the position vector expressed in terms of the first alternative curvature. Subsequently, we extend this method to find the position vector of a circular helix. Finally, through practical application, we derive parametric representations of various examples of general helices, considering some certain functions for curvature and torsion.

With the method presented in this paper, the problem of determining the position vector of a curve, one of the important problems in the theory of curves, has been solved for both general helices and circular helices. The method presented in this paper is simpler and more practical than the method which is based on the use of Frenet frame in the literature. Moreover, the helix examples obtained by using the method in this paper add diversity to the helix examples in the literature.

## 2. The Frenet and Alternative Moving Frame

In this section, we briefly introduce the Frenet and alternative moving frames of a space curve in Euclidean 3-space denoted by $E^3$ and give some basic concepts of these frames.

Let $\alpha = \alpha(s)$ be a curve in $E^3$. $\alpha(s)$ is said to be a unit speed curve if $\|\alpha'(s)\| = 1$, where $s$ is the arc-length parameter of $\alpha$. The Frenet frame of the curve $\alpha$ consists of three mutually orthonormal vectors $\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)$ defined by

$$\boldsymbol{T}(s) = \alpha'(s), \boldsymbol{N}(s) = \frac{1}{\|\boldsymbol{T}'(s)\|}\boldsymbol{T}'(s), \boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s).$$

The vector fields $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ are called unit tangent vector field, unit principal normal vector field and unit binormal vector field, respectively [3]. The derivatives of the Frenet vectors are known as Frenet formulae and can be given as

$$\boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s),$$

$$\boldsymbol{N}'(s) = -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s),$$

$$\boldsymbol{B}'(s) = -\tau(s)\boldsymbol{N}(s),$$

where $\kappa(s)$ and $\tau(s)$ are called the curvature and the torsion of the curve $\alpha$, respectively. They can be found as $\kappa(s) = \|\boldsymbol{T}'(s)\|$ and $\tau(s) = -\langle \boldsymbol{B}'(s), \boldsymbol{N}(s)\rangle$ [3].

With the help of curvature and torsion of a curve, some geometric properties of the curve can be revealed. The curve is a straight line, for instance, if $\kappa = 0$, and the curve is a planar curve if $\tau = 0$ [17,18]. Moreover, thanks to curvature and torsion functions, it can be determined whether a curve is one of the special curves such as general helix, slant helix or spherical curve. For example, if the function $\tau/\kappa$ is a constant, then the curve is a general helix which is defined by the property that tangent vectors along the curve make a constant angle with a fixed vector [3]. The curve is referred to as a circular helix or a W-curve if the curvature and torsion are both non-zero constants [17,19].

The Frenet frame is an important tool for studying differential geometric properties of curves. However, there are other frames besides the Frenet frame that can be used to examine differential geometric properties of a curve. Recently, there has been established a novel frame known as alternative moving frame and started to be used in many areas. Three vectors that are orthonormal to one another make up the alternative moving frame. These vectors are the principal normal vector $\boldsymbol{N}(s)$ that also exists in Frenet frame, the vector $\boldsymbol{C}(s)$ defined by $\boldsymbol{C}(s) = \boldsymbol{N}'(s)/\|\boldsymbol{N}'(s)\|$, and the vector $\boldsymbol{W}(s)$ which is in the direction of the instantaneous rotation vector of the Frenet frame and can be written as $\boldsymbol{W}(s) = \boldsymbol{N}(s) \times \boldsymbol{C}(s)$ [20,21]. The derivative formulae of the alternative moving frame $\{\boldsymbol{N}(s), \boldsymbol{C}(s), \boldsymbol{W}(s)\}$ can be given in matrix form as

$$\begin{bmatrix} \boldsymbol{N}'(s) \\ \boldsymbol{C}'(s) \\ \boldsymbol{W}'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}(s) \\ \boldsymbol{C}(s) \\ \boldsymbol{W}(s) \end{bmatrix} \quad (1)$$

where $f$ and $g$ are referred to as first and second alternative curvatures, respectively [21]. By applying relations between Frenet and alternative moving frames, it is possible to derive the alternative curvatures as follows [21]:

$$f = \sqrt{\kappa^2 + \tau^2}, \quad (2)$$

and

$$g = \frac{\kappa^2}{\kappa^2 + \tau^2}\left(\frac{\tau}{\kappa}\right)'. \quad (3)$$

On the other hand, the alternative curvatures may be used to express the curvatures $\kappa(s)$ and $\tau(s)$ as follows [22]:

$$\kappa(s) = f(s)\cos\left(\int g(s)ds\right), \quad (4)$$

and

$$\tau(s) = f(s)\sin\left(\int g(s)ds\right). \quad (5)$$

**Remark 2.1.** In many studies examining curves with the help of alternative moving frame, it has been observed that some mathematical expressions cannot be produced only in terms of alternative moving frame apparatus and that both Frenet frame apparatus and alternative moving frame apparatus are combined improperly. This problem can be overcome with the help of the relations between the curvatures given by Eq. (4) and Eq. (5).

The alternative curvatures have a significant impact on the characterization of curves. This notion is supported by the following theorems.

**Theorem 2.1. ([22])** Let $\alpha$ be a curve in $E^3$, provided $f \neq 0$. The curve $\alpha$ is a general helix if and only if $g = 0$.

**Theorem 2.2.** Let $\alpha$ be a curve in $E^3$, provided $f \neq 0$. The curve $\alpha$ is a circular helix if and only if $g = 0$ and $f$ is a constant.

**Proof.** Let $\alpha$ be a circular helix. The curvature and torsion of this curve are both non-zero constants. From Eqs. (2) and (3), we have $g = 0$ and $f$ is a constant.

Conversely, let $g = 0$ and $f$ be a constant. From Eqs. (4) and (5), it can be seen that κ and τ are both constant. So, the curve $\alpha$ is a circular helix which completes the proof.

## 3. Finding the Position Vector of a General Helix

This section commences with the formulation of a vector differential equation, employing the principal normal vector of a general helix through the utilization of alternative moving frame. Subsequently, solving this equation yields the determination of the position vector of the general helix uniquely up to translation and rotation in $E^3$. In this section, we initially use the derivative formulae of alternative moving frame to help us develop a vector differential equation in terms of the principal normal vector of a general helix. Since the principal normal vector is in Euclidean 3-space, this vector differential equation leads to a system of differential equation consisting of three differential equations. The position vector of the general helix is then obtained uniquely up to translation and rotation in $E^3$ by solving this system. Furthermore, we apply this technique to solve the problem of determining the position vector of a circular helix.

Let $\alpha = \alpha(s)$ be a unit speed curve in $E^3$. Since the unit tangent vector of the curve $\alpha$ is defined by

$$\boldsymbol{T}(s) = \alpha'(s),$$

the curve $\alpha$ can be written as

$$\alpha(s) = \int \boldsymbol{T}(s)ds$$

or can be rewritten by using the Frenet formula as

$$\alpha(s) = \int \left( \int \kappa(s)\boldsymbol{N}(s)ds \right) ds. \tag{6}$$

Substituting Eq. (4) into Eq. (6), we have

$$\alpha(s) = \int \left( \int f(s)cos\left( \int g(s)ds \right) \boldsymbol{N}(s)ds \right) ds. \tag{7}$$

If curvature and torsion functions of a curve are given, alternative curvatures $f$ and $g$ can be found with the help of Eqs. (2) and (3). The only thing required to determine the position vector of the curve is to find the vector $\boldsymbol{N}(s)$ in Eq. (7).

The subsequent theorem establishes a vector differential equation concerning the vector $\boldsymbol{N}(s)$ for a general helix.

**Theorem 3.1.** Let $\alpha = \alpha(s)$ be a unit speed curve in $E^3$. If $\alpha$ is a general helix, then the principal normal vector $\boldsymbol{N}(s)$ of the curve $\alpha$ satisfies the following vector differential equation

$$\boldsymbol{N}''(s) = f'(s)\frac{1}{f(s)}\boldsymbol{N}'(s) - f^2(s)\boldsymbol{N}(s), \tag{8}$$

where $f$ is the first alternative curvature of $\alpha$.

**Proof.** Let $\alpha = \alpha(s)$ be a unit speed general helix in $E^3$. Differentiating the first equation of Eq. (1) and using the second equation of Eq. (1), we have

$$\boldsymbol{N}''(s) = f'(s)\boldsymbol{C}(s) + f(s)\big(-f(s)\boldsymbol{N}(s) + g(s)\boldsymbol{W}(s)\big). \tag{9}$$

We get $g = 0$ from Theorem 2.1 because the curve $\alpha$ is a general helix. Thus the Eq. (9) becomes

$$\boldsymbol{N}''(s) = f'(s)\boldsymbol{C}(s) - f^2(s)\boldsymbol{N}(s). \tag{10}$$

Substituting the first equation in Eq. (1) into Eq. (10), Eq. (8) is obtained which completes the proof.

If the vector $\boldsymbol{N}(s)$ obtained by solving Eq. (8) is substituted into Eq. (7), the position vector of the general helix can be determined. The following theorem provides the position vector of a general helix in terms of first alternative curvature.

**Theorem 3.2.** Let $\alpha = \alpha(s)$ be a unit speed curve in $E^3$. If $\alpha$ is a general helix, then the position vector $\alpha(s) = \big(\alpha_1(s), \alpha_2(s), \alpha_3(s)\big)$ can be given in terms of first alternative curvature as follows:

$$\begin{cases} \alpha_1(s) = \int \left( \int f(s)cos(c_1)cos\left( \int f(s)ds \right) ds \right) ds, \\ \alpha_2(s) = \int \left( \int f(s)cos(c_1)sin\left( \int f(s)ds \right) ds \right) ds, \\ \alpha_3(s) = \int c_2 ds, \end{cases} \tag{11}$$

where $c_1$ and $c_2$ are real constants.

**Proof.** Given that $\alpha$ is a general helix, it follows that

$$\langle \boldsymbol{T}(s), \boldsymbol{U} \rangle = cos\theta, \tag{12}$$

where $\boldsymbol{U}$ is a constant vector parallel to the axis of the curve $\alpha$ and $\theta$ is a constant angle between $\boldsymbol{T}$ and $\boldsymbol{U}$. Differentiating Eq. (12) and using the first equation of Frenet formulae, we have

$$\langle \boldsymbol{N}(s), \boldsymbol{U} \rangle = 0.$$

The unit principal normal vector can be written with the standard basis of $E^3$ as $\boldsymbol{N}(s)=N_1\boldsymbol{e_1}+N_2\boldsymbol{e_2}+N_3\boldsymbol{e_3}$. We can select $\boldsymbol{e_3}$ as the axis of the curve $\alpha$ without losing generality. So, we have

$$\langle \boldsymbol{N}(s),\boldsymbol{e_3}\rangle = N_3 = 0.$$

Since $\boldsymbol{N}(s)$ is a unit vector, we have the following relation between the components of $\boldsymbol{N}(s)$:

$$N_1^2+N_2^2=1. \tag{13}$$

From Eq. (13), the components $N_1$ and $N_2$ can be written as

$$N_1(s)=\cos[t(s)] \text{ and } N_2(s)=\sin[t(s)],$$

where $t$ is a function of the arc-length parameter $s$. Thus, the vector $\boldsymbol{N}(s)$ can be written according to the function of $t(s)$ as

$$\boldsymbol{N}(s)=(\cos[t(s)],\sin[t(s)],0).$$

Each component of the vector $\boldsymbol{N}(s)$ must meet Eq. (8). It is easy to see that $N_3=0$ satisfies Eq. (8). When the components $N_1$ and $N_2$ are substituted into Eq. (8), we obtain the following differential equations of $t(s)$:

$$(-(t')^2+f^2)cost+\left(-t''+\frac{f'}{f}t'\right)sint=0, \tag{14}$$
$$\left(t''-\frac{f'}{f}t'\right)cost+(-(t')^2+f^2)sint=0. \tag{15}$$

From Eqs. (14) and (15), we have the following differential equations

$$-(t')^2+f^2=0, \tag{16}$$
$$-t''+\frac{f'}{f}t'=0. \tag{17}$$

From Eq. (16), we get

$$t'=f,$$

or

$$t'=-f.$$

Since the above equations satisfy Eq. (17), we have

$$t=\int f(s)ds,$$

or

$$t=-\int f(s)ds.$$

Consequently, the principal normal vector $\boldsymbol{N}(s)$ of the general helix $\alpha$ can be found as

$$\boldsymbol{N}(s)=\left(cos\left(\int f(s)ds\right),sin\left(\int f(s)ds\right),0\right) \tag{18}$$

or

$$\boldsymbol{N}(s)=\left(cos\left(\int f(s)ds\right),-sin\left(\int f(s)ds\right),0\right). \tag{19}$$

Then by substituting Eq. (18) or Eq. (19) into Eq. (7) and by using Theorem 2.1, we have Eq. (11) which completes the proof.

Substituting Eq. (2) and Eq. (4) into Eq. (11), the position vector of a general helix can be given in terms of curvatures of Frenet frame as in the following corollary.

**Corollary 3.1.** Let $\alpha=\alpha(s)$ be a unit speed curve in $E^3$. If $\alpha$ is a general helix, then the position vector $\alpha(s)=\big(\alpha_1(s),\alpha_2(s),\alpha_3(s)\big)$ is expressed as

$$\begin{cases}\alpha_1(s)=\int\left(\int\kappa(s)cos\left(\int\sqrt{\kappa^2(s)+\tau^2(s)}\,ds\right)ds\right)ds,\\ \alpha_2(s)=\int\left(\int\kappa(s)sin\left(\int\sqrt{\kappa^2(s)+\tau^2(s)}\,ds\right)ds\right)ds,\\ \alpha_3(s)=\int c_2ds,\end{cases} \tag{20}$$

where $c_2$ is a constant.

From Theorem 2.2 and Theorem 3.2, the position vector of a circular helix can be given in terms of first alternative curvature as in the following corollary.

**Corollary 3.2.** Let $\alpha=\alpha(s)$ be a unit speed curve in $E^3$. If $\alpha$ is a circular helix, then the position vector $\alpha(s)=\big(\alpha_1(s),\alpha_2(s),\alpha_3(s)\big)$ can be computed as

$$\begin{cases}\alpha_1(s)=-\frac{1}{f}\cos(c_1)\cos(fs),\\ \alpha_2(s)=-\frac{1}{f}\cos(c_1)\sin(fs),\\ \alpha_3(s)=fcos(c_1)c_2s,\end{cases}$$

where $c_1$ and $c_2$ are real constants.

Since the curvature $\kappa$ and torsion $\tau$ of a circular helix are both non-zero constants, Eq. (20) which gives the position vector of a general helix in terms of curvatures of Frenet frame can be adapted for the position vector of a circular helix as in the following corollary.

**Corollary 3.3.** Let $\alpha=\alpha(s)$ be a unit speed curve in $E^3$. If $\alpha$ is a circular helix, then the position vector $\alpha(s)=\big(\alpha_1(s),\alpha_2(s),\alpha_3(s)\big)$ can be given as

$$\begin{cases} \alpha_1(s) = -\dfrac{\kappa}{\kappa^2+\tau^2} cos\left(\sqrt{\kappa^2+\tau^2}\, s\right), \\ \alpha_2(s) = -\dfrac{\kappa}{\kappa^2+\tau^2} sin\left(\sqrt{\kappa^2+\tau^2}\, s\right), \\ \alpha_3(s) = \kappa c_2 s, \end{cases}$$

where $c_2$ is a real constant.

## 4. Examples

In this section, we obtain position vectors for some examples of general helices given some special functions for curvature and torsion with the help of the alternative approach described in the previous section. In the process of finding the parametric representations, we choose the integral constants in Eq. (11) or Eq. (20) as some real numbers so that the general helices are curves with unit speed. The axes of all the general helices in the following examples are chosen as parallel to $\boldsymbol{e}_3$.

**Example 4.1.** Let curvature and torsion functions be given as $\kappa = 1/s$ and $= 1/s$ , respectively. From Eqs. (2) and (3), we have the alternative curvatures as $f = \sqrt{2}/s$ and $g = 0$. By using Eq. (11), the position vector of the general helix can be obtained in the parametric representation as

$$\alpha(s) = \Big(\frac{1}{3\sqrt{2}}\big(s\sin(\sqrt{2}\ln s) - \sqrt{2}\, s\cos(\sqrt{2}\ln s)\big), \\ -\frac{1}{3\sqrt{2}}\big(s\cos(\sqrt{2}\ln s) + \sqrt{2}\, s\sin(\sqrt{2}\ln s)\big), \frac{1}{\sqrt{2}}\, s\Big).$$

An illustration of the curve $\alpha$ is given in Figure 4.1.

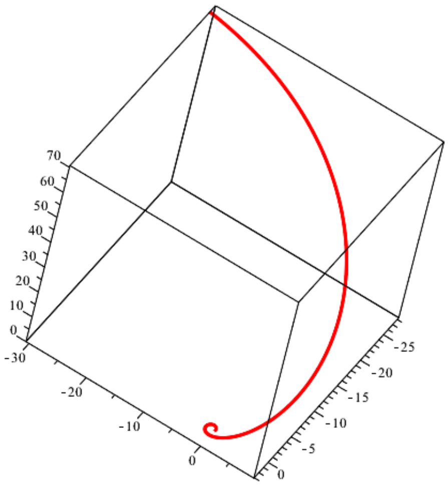


**Figure 4.1.** The general helix with $\kappa = 1/s$ and $\tau = 1/s$

**Example 4.2.** If we take $\kappa = \frac{1}{2\sqrt{2s}}$ and $= \frac{1}{2\sqrt{2s}}$ , then we get $f = \frac{1}{2\sqrt{s}}$ and $g = 0$. By using Eq. (11), the position vector $\alpha(s)$ of the general helix is expressed as

$$\alpha(s) = \Big(\sqrt{2}\left(sin(\sqrt{s}) - \sqrt{s}\, cos(\sqrt{s})\right), \\ -\sqrt{2}\left(\sqrt{s}\, sin(\sqrt{s}) + cos(\sqrt{s})\right), \frac{1}{\sqrt{2}}\, s\Big).$$

The shape of the curve $\alpha$ is given in Figure 4.2.

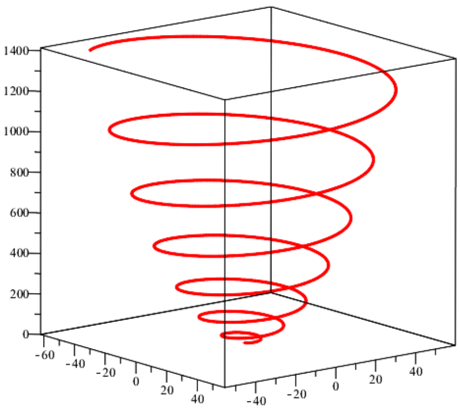


**Figure 4.2.** The general helix with $\kappa = \frac{1}{2\sqrt{2s}}$ and $\tau = \frac{1}{2\sqrt{2s}}$

**Example 4.3.** Let consider the general helix $\alpha(s)$ with curvature $\kappa = \frac{1}{\sqrt{2-2s^2}}$ and torsion $\tau = \frac{1}{\sqrt{2-2s^2}}$. Without finding the alternative curvatures, by using Eq. (20), the position vector of the general helix is obtained as follows:

$$\alpha(s) = \left(\frac{s^2}{2\sqrt{2}}, -\frac{s\sqrt{1-s^2} + sin^{-1}(s)}{2\sqrt{2}}, \frac{1}{\sqrt{2}}s\right).$$

One can see the shape of the curve $\alpha$ in Figure 4.3.

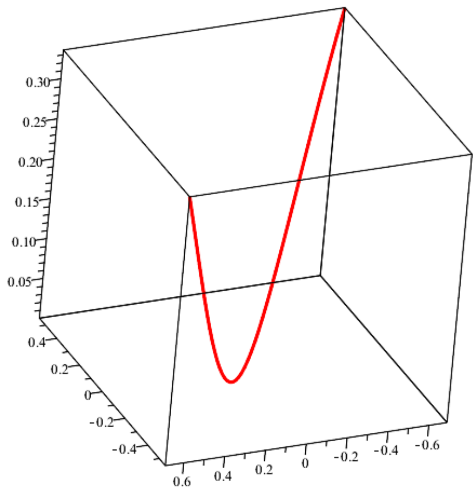


**Figure 4.3.** The general helix with $\kappa = \frac{1}{\sqrt{2-2s^2}}$ and $\tau = \frac{1}{\sqrt{2-2s^2}}$

**Example 4.4.** The position vector $\alpha(s)$ of the general helix with $\kappa = \frac{1}{\sqrt{2}(s^2+1)}$ and $\tau = \frac{1}{\sqrt{2}(s^2+1)}$ can be found in parametric form as

$$\alpha(s) = \left(\frac{\sqrt{1+s^2}}{\sqrt{2}}, -\frac{sinh^{-1}(s)}{\sqrt{2}}, \frac{1}{\sqrt{2}}s\right).$$

The shape of the curve $\alpha$ is given in Figure 4.4.

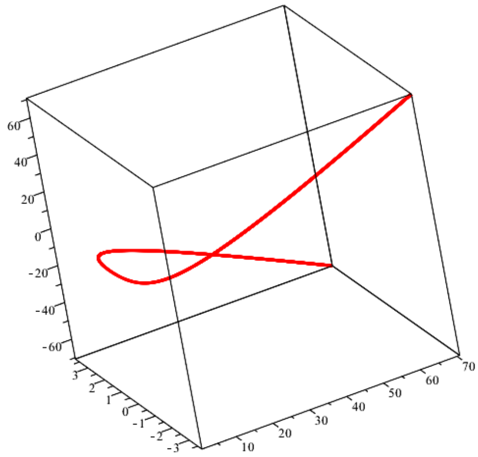


**Figure 4.4.** The general helix with $\kappa = \frac{1}{\sqrt{2}(s^2+1)}$ and $\tau = \frac{1}{\sqrt{2}(s^2+1)}$

## 5. Conclusions

The problem of determining the position vector of an arbitrary curve given its curvature and torsion, known as solving natural or intrinsic equations, is still an open problem in Euclidean 3-space. In this study, we proposed a method based on alternative moving frame to solve this problem for general helices. Using the derivative formulae of alternating moving frame, we first built a vector differential equation correspond to a system of three differential equations in terms of the principal normal vector of a general helix. Then, by solving this system, we found the principal normal vector of the general helix, which allows us to find the position vector of the general helix. We gave the position vector of the general helix, which depends only on the first alternative curvature, in parametric form as in Eq. (11). By using the relation between the first alternative curvature and curvatures of Frenet frame as given in Eq. (2), we also gave the parametric representation of the general helix in terms of curvatures of Frenet frame as in Eq. (20). So, we have two ways to find the position vector of a general helix given its curvature and torsion. The first is to use Eq. (20) directly, and the second is to use Eq. (11) after finding the first alternative curvature with the help of Eq. (2). Moreover, we adapted these two ways to find the position vector of a circular helix.

The problem discussed in the present paper was solved by a method based on the use of the Frenet frame in [5]. In that paper, a vector differential equation was constructed in terms of unit tangent vector by the aid of Frenet formulae and solved to find the position vector of a general helix. It can be said that the method based on the alternative moving frame used in the present paper for determining the position vector of a general helix is simpler and more practical compared to the method in [5]. While parameter change is required to obtain the vector differential equation for finding the position vector of a general helix in [5], it is another advantage of the present paper that this vector differential equation can be constructed according to the arc-length parameter of the curve without any parameter change.

Since helices are used in many different fields such as biology, chemistry, mechanical engineering, and computer-aided geometric design, we hope that this study will contribute scientifically to the relevant fields. Furthermore, we expect that the examples of general helices obtained in the previous section can add variety to the examples of general helices in the literature.

## Author's Contributions

This paper is derived from the first author's master's thesis supervised by the second author. Both authors contributed to the writing of the manuscript.

**Gizem Güzelkardeşler**: conducted the literature review, performed mathematical operations, and found the results.

**Burak Şahiner**: guided and supervised the whole process and interpreted the results.

## Ethics

There are no ethical issues after the publication of this manuscript.

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