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On the Diophantine Equation $(9d^2+1)^x+(16d^2-1)^y=(5d)^z$ Regarding Terai's Conjecture

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Research Article

Abstract — This study proves that the Diophantine equation $\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z$ has a unique positive integer solution $(x,y,z)=(1,1,2)$, for all $d>1$. The proof employs elementary number theory techniques, including linear forms in two logarithms and Zsigmondy's Primitive Divisor Theorem, specifically when d is not divisible by 5. In cases where d is divisible by 5, an alternative method utilizing linear forms in p-adic logarithms is applied.

Keywords *Terai's conjecture, Diophantine equations, primitive divisor theorem*

Mathematics Subject Classification (2020) 11D61, 11D75

1. Introduction

The exponential Diophantine equation $e^x+f^y=g^z$ involves coprime positive integers e, f, and g greater than 1. Solutions (x,y,z) satisfying this equation are referred to as valid solutions to the provided equation [1]. In 1956, Sierpinski [2] demonstrated that by substituting exponential expressions for the sides of the Pythagorean theorem into variables, the exponential Diophantine equation $3^x+4^y=5^z$ has a unique solution, specifically $(2,2,2)$. Furthermore, Jeśmanowicz [3] extended this equation to various Pythagorean triples, affirming that for positive integers e, f, and g that satisfy the exponential Diophantine equation, the unique solution remains $(2,2,2)$.

In 1994, Terai [4] proposed that if the equation $e^k+f^l=g^m$ holds for positive constant integers k, l, and m with $m\geq 2$, multiple known solutions (k,l,m) exist for the equation, except for certain specific sets of triples (e,f,g). This conjecture is proved for many special cases. One of them is as follows:

$$\left(pd^2+1\right)^x+\left(ud^2-1\right)^y=(wd)^z \tag{1.1}$$

This study explores the solutions of the following exponential Diophantine equation

$$\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z \tag{1.2}$$

(1.2) is a specific case derived from (1.1), particularly when the condition $p+u=w^2$ is satisfied. Several specific instances of (1.1) have been explored, confirming the validity of Terai's conjecture. Some of these are as follows:

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i. $(4d^2+1)^a+(5d^2-1)^b=(3d)^c$ [5]

ii. $(d^2+1)^a+(yd^2-1)^b=(zd)^c,\ 1+y=z^2$ [6]

iii. $(12d^2+1)^a+(13d^2-1)^b=(5d)^c$ [7]

iv. $(xd^2+1)^a+(yd^2-1)^b=(zd)^c$, $z|d$ [8]

v. $(xd^2+1)^a+(yd^2-1)^b=(zd)^c$, $d=\mp 1 \pmod 5$ [9]

vi. $(18d^2+1)^a+(7d^2-1)^b=(5d)^c$ [10]

vii. $((x+1)d^2+1)^a+(xd^2-1)^b=(zd)^c,\ 2x+1=z^2$ [11]

viii. $(3xd^2-1)^a+(x(x-3)d^2+1)^b=(xd)^c$ [12]

ix. $(4d^2+1)^a+(21d^2-1)^b=(5d)^c$ [13]

x. $(5pd^2-1)^a+(p(p-5)d^2+1)^b=(pd)^c$ [14]

xi. $(3d^2+1)^a+(bd^2-1)^b=(cd)^c$ [15]

xii. $(4d^2+1)^a+(45d^2-1)^b=(7d)^c$ [16]

xiii. $(6d^2+1)^a+(3d^2-1)^b=(3d)^c$ [17]

xiv. $(c(c-l)d^2+1)^a+(cld^2-1)^b=(cd)^c$ [18]

xv. $(44d^2+1)^a+(5d^2-1)^b=(7d)^c$ [19]

This research is dedicated to exploring and analyzing Terai's conjecture, focusing specifically on investigating the exponential Diophantine equation.

2. Preliminaries

This section presents some basic properties to be required in the following section.

Theorem 2.1. For any positive integer d, (1.2) possesses a sole and distinct positive integer solution, namely, $(x,y,z)=(1,1,2)$.

The proof of this theorem involves several important steps. Firstly, elementary methods, such as congruences and properties of the Jacobi symbol are employed to simplify the solution. Particular attention is given to the case where $x=1$, especially when $d\equiv \pm 2 \pmod 5$. Subsequently, a lower bound for linear forms in two logarithms, as established by Laurent [20], is utilized.

In cases where $d\equiv 0 \pmod 5$, a result concerning linear forms in p-adic logarithms, as detailed in Bugeaud's study [21], is applied. Conversely, for the case $d\equiv \pm 1 \pmod 5$, an earlier version of the Primitive Divisor Theorem, is attributed to Zsigmondy [22].

Definition 2.2. The expression of the absolute logarithmic height for any non-zero algebraic number α with degree m over $\mathbb{Q}$ is provided by the following

$$h(\alpha)=\frac{1}{m}\left(\log\left(|a_0|+\sum_{i=0}^{m}\log\left(\max\{1,|\alpha^{(i)}|\}\right)\right)\right)$$

Here, the symbol a_0 denotes the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(i)}$ represents the conjugates of α.

The linear form defined by $L=k_1\alpha_1+k_2\alpha_2$ is an expression involving two real algebraic numbers, α_1 and α_2, where the absolute values of both α_1 and α_2 are greater than or equal to 1. The coefficients

k_1 and k_2 are positive integers. The linear form is as follows:

$$\Lambda = k_2 \log \alpha_2 - k_1 \log \alpha_1$$

Let $D = [Q(\alpha_1, \alpha_2) : Q]$. Set

$$k' = \frac{k_1}{D \log K_2} + \frac{k_2}{D \log K_1}$$

where K_1 and K_2 are real numbers greater than 1, satisfying

$$\log K_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \qquad i \in \{1,2\}$$

The following proposition is a specific instance derived from Corollary 2 in [20], with the values $m = 10$ and $C_2 = 25.2$ chosen as indicated in Table 1 [20].

Proposition 2.3. [20] Given the previously defined variables Λ, α_i, D, K_i, and k' where $\alpha_i > 1$, for $i \in \{1,2\}$, and assuming that α_1 and α_2 are not multiplicatively related, the following inequality is valid:

$$\log|\Lambda| \geq -25.2\ D^4 \left(\max\left\{\log k' + 0.38, \frac{10}{D}, 1\right\}\right) \log K_1 \log K_2$$

In this context, a specific case is considered where $y_1 = y_2 = 1$ from Theorem 2 [21], referencing a result from [21]. Prior to investigating this result, it is pertinent to reintroduce some notations. Take an odd prime p and define v_p as the p-adic valuation normalized such that $v_p(p) = 1$. Consider two nonzero integers a_1 and a_2. The smallest positive integer g satisfying the following conditions is identified:

$$v_p({a_1}^g - 1) > 0 \quad \text{and} \quad v_p({a_2}^g - 1) > 0$$

Suppose that there exists a real number E such that

$$v_p({a_1}^g - 1) \geq E > \frac{1}{p-1}$$

The following theorem provides a specific upper bound for the p-adic valuation of

$$\Lambda = {a_1}^{k_1} - {a_2}^{k_2}$$

where k_1 and k_2 are positive integers.

Proposition 2.4. [21] Let $K_1, K_2 > 1$ be real numbers such that

$$\log K_i \geq \max\{\log|a_i|, E \log p\}, \quad i \in \{1,2\}$$

and put

$$t' = \frac{k_1}{\log K_2} + \frac{k_2}{\log K_1}$$

If a_1 and a_2 are multiplicatively independent then, the upper estimates can be expressed as follows

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4} \left(\max\{\log t' + \log(E\log p) + 0.4, 6E\log p, 5\}\right)^2 \log K_1 \log K_2$$

Proposition 2.5. [22] Consider relatively prime integers E and F with $E > F \geq 1$. Define the sequence $\{a_n\}_{n \geq 1}$ as

$$a_n = E^n + F^n$$

For $n > 1$, the sequence a_n has a prime factor not dividing $a_1 a_2 a_3 \cdots a_{n-1}$, except when $(E, F, n) \neq (2, 3, 1)$.

3. Main Results

This section presents the proof of Theorem 2.1, based on a series of lemmas.

Lemma 3.1. If (x, y, z) represents a positive integer solution of (1.2), then it follows that y must be an odd integer.

PROOF. If $z \leq 2$, the solution $(x, y, z) = (1, 1, 2)$ is clearly the only solution to (1.2). However, when assuming $z \geq 3$, taking (1.2) modulo d^2 results in $1 + (-1)^y \equiv 0 \pmod{d^2}$. This implies that y must be odd since $d^2 > 2$. □

Lemma 3.2. In (1.2), if d is even, then x is also even. Conversely, if d is odd, then x is odd as well.

PROOF. Applying modulo d^3 to (1.2), it follows that

$$1 + 9d^2x + (-1) + 16d^2y \equiv 0 \pmod{d^3}$$

and thus

$$9x + 16y \equiv 0 \pmod{d}$$

It can be seen from here that if d is even, then x is also even. Similarly, if d is odd, then x is also odd. □

Lemma 3.3. [23] Consider positive integers p, u, and w and $d > 1$ such that $p + u = w^2$. Suppose a positive integer solution (x, y, z) to the exponential Diophantine equation

$$\left(pd^2+1\right)^x + \left(ud^2-1\right)^y = (wd)^z$$

where $x \geq y$. The following inequalities hold true:

$$\left(2 - \frac{\log\left(\frac{w^2}{p}\right)}{\log(wd)}\right) x < z \leq 2x$$

Moreover, if y is the larger value, then

$$\left(2 - \frac{\log\left(\frac{w^2d^2}{ud^2-1}\right)}{\log(wd)}\right) y < z \leq 2y$$

In particular, when $M = \max\{x, y\} > 1$, it follows that

$$\left(2 - \frac{\log\left(\frac{w^2}{\min\left\{p, u-\frac{1}{d^2}\right\}}\right)}{\log(wd)}\right) M < z < 2M$$

This refined characterization delineates the possible range of values for z based on the parameter M and the given variables.

3.1. The Case $5|d$

This section proves that Theorem 2.1 holds true under the condition $5|d$.

Lemma 3.4. If a positive integer solution (x, y, z) to (1.2) is considered under the assumption that d is congruent to 0 in modulo 5, then the only positive integer solution to (1.2) is $(x, y, z) = (1, 1, 2)$.

PROOF. Certainly, $(1, 1, 2)$ is the unique solution of (1.2) when $M = \max\{x, y\} = 1$. Assume that

$M > 1$. Applying Lemma 3.3 for $d \geq 5$, it follows that

$$1.68M < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(25)}\right) < z \leq 2M$$

Thus, it follows that $z \geq 5$. Given that y is odd, as stated in Lemma 3.1,

$$\Lambda = {\alpha_1}^{s_1} - \alpha^{s_2}$$

is set up where $a_1 = 9d^2 + 1$, $a_2 = 1 - 16d^2$, $s_1 = x$, and $s_2 = y$.

Considering $p = 5$ and setting $g = 1$ satisfies the condition outlined before Proposition 2.4. Therefore, set $E = 2$ and apply Proposition 2.4 to obtain

$$2z \leq \frac{36.1}{8(\log 5)^4}\left(\max\left\{\log s^{'} + \log(2\log 5) + 0.4, 12\log 5, 5\right\}\right)^2 \log\left(9d^2+1\right)\log\left(16d^2-1\right) \tag{3.1}$$

where

$$s^{'} = \frac{x}{\log\left(16d^2-1\right)} + \frac{y}{\log\left(9d^2+1\right)}$$

Since $z \geq 5$, applying modulo d^4 to (1.2) yields $9x + 16y \equiv 0 \pmod{d^2}$. Then, $M \geq \frac{d^2}{25}$. As

$$z \geq \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) M$$

by Lemma 3.3, (3.1), and $r^{'} \leq \frac{M}{\log 3d}$,

$$\begin{aligned}2\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) M \leq & \frac{36.1}{8(\log 5)^4}\left(\max\left\{\log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4, 12\log 5\right\}\right)^2 \\ & \log\left(9d^2+1\right)\log\left(16d^2-1\right)\end{aligned} \tag{3.2}$$

is obtained. Let

$$k = \max\left\{\log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4, 12\log 5\right\}$$

Suppose

$$k = \log\left(\frac{M}{\log 3d}\right) + \log(2\log 5) + 0.4 \geq 12\log 5$$

The inequality

$$\log M \geq 12\log 5 - \log(2\log 5) - 0.4$$

leads to the conclusion that $M > 50841462$. However, from (3.2)

$$2M \leq (0.68)(\log M + 1.57)^2 \log(225M+1)\log(400M-1)$$

and this implies $M < 8128$. This discrepancy results in a contradiction. If $k = 12\log 5$, then (3.2) takes the form

$$\frac{2d^2}{25}\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) \leq 251\log(9d^2+1)\log(16d^2-1)$$

This implies that $d \leq 629$. Hence,

$$M < \frac{251\log\left(9d^2+1\right)\log\left(16d^2-1\right)}{2\left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right)}$$

$$1.68x < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(25)}\right) x < \left(2 - \frac{\log\left(\frac{25}{9}\right)}{\log(5d)}\right) x < z \leq 2x \tag{3.3}$$

and

$$\begin{aligned} 1.84y < \left(2 - \frac{\log\left(\frac{26}{16}\right)}{\log(25)}\right) y &< \left(2 - \frac{\log\left(\frac{26d^2-26}{16d^2-16}\right)}{\log(5d)}\right) y \\ &< \left(2 - \frac{\log\left(\frac{25d^2}{16d^2-1}\right)}{\log(5d)}\right) y < z \leq 2y \end{aligned} \tag{3.4}$$

(3.3) and (3.4) lead to the conclusion that there are no positive integer solutions for (1.2) when $z \leq 6$. Assuming $z > 6$, an analysis of (1.2) is performed by considering congruences modulo d^4, d^6, and d^8.

i. Applying modulo d^4 to (1.2) results in $9d^2x + 16d^2y \equiv 0 \pmod{d^4}$ which is further expressed as

$$9x + 16y \equiv 0 \pmod{d^2} \tag{3.5}$$

ii. Analysis of (1.2) yields a simplified expression

$$9x + 9^2d^2\frac{x(x-1)}{2} + 16y - 16^2d^2\frac{y(y-1)}{2} \equiv 0 \pmod{d^4} \tag{3.6}$$

iii. The analysis extends to modulo d^8 with a more complex expression

$$\begin{aligned} &9x + 9^2d^2\frac{x(x-1)}{2} + 9^3d^4\frac{x(x-1)(x-2)}{6} \\ &+ 16y - 16^2d^4\frac{y(y-1)}{2} + 16^3d^4\frac{y(y-1)(y-2)}{6} \equiv 0 \pmod{d^6} \end{aligned} \tag{3.7}$$

(3.5)-(3.7) summarize the congruence conditions derived from (1.2) modulo d^2, d^4, and d^6, respectively. These conditions lead to bounds on all the variables x, y, and z. Through an exhaustive search using a Maple program running for several hours, no additional positive integer solutions (d, x, y, z) were discovered for (1.2) beyond the solution $(x, y, z) = (1, 1, 2)$ when $5|d$. Hence, it is confirmed that there are no other positive integer solutions to (1.2). □

3.2. The Case $d \equiv \pm 2 \pmod 5$

This section proves that Theorem 2.1 holds true under the condition $d \equiv \pm 2 \pmod 5$.

Lemma 3.5. For a positive integer solution (x, y, z) to (1.2) where $d \equiv \pm 2 \pmod 5$, it is established that the sole positive integer solution is $(x, y, z) = (1, 1, 2)$.

Let d be even. Thus, x is also even from Lemma 3.2. Applying modulo 5 to (1.2) results in the equation

$$2^x + 3^y \equiv 0 \pmod 5$$

However, this is impossible when x is even and y is odd.

Proceed by first establishing Lemma 3.6 and Lemma 3.7, starting with the assumption that d is odd, which implies that x is also odd as indicated in Lemma 3.2.

Lemma 3.6. If d is odd and $d \equiv \pm 2 \pmod 5$, then $x = 1$ and y is odd.

Proof. With reference to Lemma 3.2, our focus is directed specifically towards the scenario where $d > 2$ is an odd number. Additionally, as implied by Lemma 3.1, it is established that y is an odd integer. Consequently,

$$\left(\frac{9d^2+1}{16d^2-1}\right)=\left(\frac{25d^2}{16d^2-1}\right)=1$$

and

$$\begin{aligned}\left(\frac{5d}{16d^2-1}\right)&=\left(\frac{5}{16d^2-1}\right)\left(\frac{d}{16d^2-1}\right)\\&=\left(\frac{16d^2-1}{5}\right)\left(\frac{16d^2-1}{d}\right)\\&=\left(\frac{3}{5}\right)\left(\frac{16d^2-1}{d}\right)(-1)^{\frac{16d^2-2}{2}\frac{d-1}{2}}\\&=(-1)(-1)^{\frac{d-1}{2}}(-1)^{(8d^2-1)\frac{d-1}{2}}\\&=(-1)(-1)^{\frac{d-1}{2}(8d^2-1+1)}\\&=-1\end{aligned}$$

Using the Jacobi symbol notation $\left(\frac{*}{*}\right)$ deduce that z is an even integer. Suppose that $x\geq 3$. Applying modulo 8 to (1.2)

$$2^x+(-1)^y\equiv 1 \pmod 8$$

and thus

$$2^x\equiv 2 \pmod 8$$

This implies that x must be equal to 1. □

Consequently, (1.2) transforms into the following

$$9d^2+1+\left(16d^2-1\right)^y=(5d)^z \tag{3.8}$$

Lemma 3.7. $y\geq\frac{1}{16}\left(d^2-9\right)$

Proof. As $y\geq 3$ and $x=1$, (1.2) leads to

$$(5d)^z\geq 9d^2+1+\left(16d^2-1\right)^3>(5d)^3$$

Applying modulo d^4 to equation (3.8) yields

$$9d^2+1+16d^2y-1\equiv 0 \pmod{d^4}$$

and thus

$$9+16y\equiv 0 \pmod{d^2}$$

Having established this claim, the subsequent step involves deriving an upper bound for y. □

Lemma 3.8. $y<2521\log 5d$

Proof. Consider

$$\left(9d^2+1\right)^x+\left(16d^2-1\right)^y=(5d)^z \tag{3.9}$$

If $y=1$, then clearly $z=2$. Assume that $y\geq 3$. Then, $z>2$ from (3.9). For simplicity set the following notation $p=9d^2+1$, $q=16d^2-1$, and $r=5d$ and consider the linear form of two logarithms

$$\Lambda=z\log r-y\log q$$

Since

$$0<\Lambda<e^{\Lambda}-1=\frac{r^z}{q^y}-1=\frac{p}{q^y} \tag{3.10}$$

then

$$\log\Lambda<\log p-y\log q \tag{3.11}$$

From Proposition 2.3,

$$\log\Lambda\geq -25.2D^4\left(\max\left\{\log t'+0.38,10\right\}\right)^2\log q\log r \tag{3.12}$$

where

$$t'=\frac{y}{\log r}+\frac{z}{\log q}$$

and

$$q^{y+1}-r^z=qq^x-r^z=q\left(r^z-p\right)-r^z=(q-1)r^z-pq>\left(16d^2-2\right)25d^2-\left(9d^2+1\right)\left(16d^2-1\right)>0$$

Since $z>2$, then $q^{y+1}>r^z$. Therefore, $t'<\frac{2y+1}{\log r}$. Write $M=\frac{y}{\log r}$, and thus

$$t'<2M+\frac{1}{\log r}$$

Combining (3.11) and (3.12),

$$y\log q<\log p+25.2\left(\max\left\{\log\left(2M+\frac{1}{\log r}\right)+0.38,10\right\}\right)^2\log q\log r$$

Since $\frac{\log p}{\log q\log r}<1$ and $\log r=\log 5d>2$, for $d\geq 3$, the inequality can be expressed as follows:

$$M<1+25.2\left(\max\{\log(2M+0.5)+0.38,10\}\right)^2$$

If $\log(2M+0.5)+0.38>10$, then $M\geq 7532$. However, the inequality

$$M<1+25.2\left(\log(2M+0.5)+0.38\right)^2$$

implies that $M\leq 1867$. Thus, $\max\left\{\log(2M+0.5)+0.38,10\right\}=10$ implies $M<2521$. Hence, $x<2521\log 5d$. By combining Lemma 3.7 and Lemma 3.8,

$$\frac{1}{16}\left(d^2-9\right)<2521\log 5d$$

This implies $d\leq 566$. From (3.10),

$$\frac{z}{y}-\frac{\log q}{\log r}<\frac{p}{yq^y\log r}$$

Thus,

$$\left|\frac{\log q}{\log r}-\frac{z}{y}\right|<\frac{p}{yq^y\log r}$$

which further implies

$$\left|\frac{\log q}{\log r}-\frac{z}{y}\right|<\frac{1}{2y^2}$$

Thereby, $\frac{z}{y}$ is a convergent in the simple continued fraction expansion to $\frac{\log q}{\log r}$. Consider $\frac{z}{y}=\frac{a_n}{b_n}$ where $\frac{a_n}{b_n}$ represents the n-th convergent of the simple continued fraction expansion of $\frac{\log q}{\log r}$. Since $\gcd(a_n,b_n)=1$, it follows that $b_n\leq y$. Hence, an upper bound for b_n is given by $b_n<2521\log 5d$ according to Lemma 3.8. Any such convergent $\frac{a_n}{b_n}$ satisfies

$$\frac{1}{b_n\left(b_n+b_{n+1}\right)}<\left|\frac{\log q}{\log r}-\frac{a_n}{b_n}\right|$$

By setting $b_{n+1}=u_{n+1}b_n+b_{n-1}$,

$$\frac{1}{(b_n)^2(b_n+b_{n+1})}<\left|\frac{\log q}{\log r}-\frac{a_n}{b_n}\right|<\frac{p}{yq^y\log r}<\frac{p}{b_nq^{b_n}\log r}$$

where u_n is the n-th partial quotient of the simple continued fraction expansion of $\frac{\log q}{\log r}$ refer to [24]. Therefore, b_n and u_{n+1} satisfy

$$u_{n+1}+2>\frac{q^{b_n}\log r}{pb_n} \tag{3.13}$$

As a final step, a short computer program in Maple was utilized to verify that no convergents $\frac{a_n}{b_n}$ of $\frac{\log q}{\log r}$ satisfy equation (3.13) when $b_n<2521\log(5d)$, for $1<d\leq 566$. This process took only a few seconds to complete, concluding the proof. Therefore, Lemma 3.5 is also proven. □

3.3. The Case $d\equiv\pm1 \pmod 5$

This section proves that Theorem 2.1 holds true under the condition $d\equiv\pm1 \pmod 5$.

Lemma 3.9. (1.2), with d being a positive integer such that $d\equiv\pm1 \pmod 5$, possesses a unique positive integer solution $(x,y,z)=(1,1,2)$.

Proof. Consider the positive integers k_1 and k_2 and a positive integer d satisfying $d\equiv\pm1 \pmod 5$. (1.2) is expressed as follows:

$$9d^2+1=5^{k_1}A, \qquad \left(9d^2+1\right)^x=5^{k_1x}A^x \tag{3.14}$$

$$16d^2-1=5^{k_2}B, \qquad \left(16d^2-1\right)^y=5^{k_2y}B^y \tag{3.15}$$

where A and B are nonzero integers not congruent to 0 modulo 5. Then, (1.2) can be rewritten as

$$5^{k_1x}A^x+5^{k_2y}B^y=(5m)^z \tag{3.16}$$

Firstly, consider the case $k_1x>k_2y$. This implies

$$5^{k_2y}\left(5^{k_1x-k_2y}A^x+B^y\right)=5^zm^z$$

which leads to

$$k_2y=z \tag{3.17}$$

Substituting (3.17) back into (3.16),

$$\left(9d^2+1\right)^x=\left((5d)^{k_2}\right)^y-\left(16d^2-1\right)^y \tag{3.18}$$

Applying Proposition 2.5 [22], $y=1$ is found. Therefore, (3.15) simplifies to

$$\left(16d^2-1\right)^y=5^{k_2y}B^y=5^{k_2}B \tag{3.19}$$

Substituting (3.17) into (3.19) with $y=1$,

$$16d^2=5^zB+1 \tag{3.20}$$

Delve into the case $z=3$, for (1.2). This transforms into

$$\left(9d^2+1\right)^x+16d^2-1=(5d)^3$$

However, when $x\geq2$, it leads to

$$(5d)^3>\left(9d^2+1\right)^x\geq\left(9d^2+1\right)^2>9^2d^4$$

which results in the contradiction $5^3>9^2d$ since $d>1$.

Indeed, when $y=1$ is set and $x=1$ in (1.2), it simplifies to

$$9d^2+1+16d^2-1=(5d)^3$$

However, this results in a contradiction under the condition $d\equiv\pm1 \pmod 5$.

When $z\geq 4$, investigating (1.2) in modulo d^4 results in the inference that $y=1$. This deduction is made by employing Proposition 2.5 in [22]. This simplifies the equation to

$$9d^2x+16d^2\equiv 0 \pmod{d^4}$$

and thus

$$9x+16\equiv 0 \pmod{d^2}$$

It can be observed that

$$d^2\leq 9x+16 \tag{3.21}$$

Substituting (3.20) into (3.21),

$$5^zB\leq 144x+255 \tag{3.22}$$

Since $x<z$, (3.22) turns into (3.23):

$$5^zB\leq 144z+255 \tag{3.23}$$

As a result, there are no positive integer solutions, for $z>4$, and $z=4$, the equation does not have any positive integer solutions for appropriate values of x and y. Similarly, by employing analogous procedures when $k_2y>k_1x$, it can be deduced that there exist no positive integer solutions for $z\geq 3$.

Finally, investigate the scenario $k_1x=k_2y$. Summing up (3.14) and (3.15),

$$25d^2=5^{k_1}A+5^{k_2}B$$

Analyze this equation based on the positive integers k_1 and k_2:

i. $k_1=2$ and $k_2\geq 3$

If $k_1=2$, then it is observed that k_2 must be even while y is odd. Thus,

$$2x=k_2y \tag{3.24}$$

and there is a positive integer such that k_3 satisfies $2k_3=k_2$. Putting it into the (3.24), $x=k_3y$ is acquired. Then, (1.2) becomes

$$\left(\left(9d^2+1\right)^{k_3}\right)^y+\left(16d^2-1\right)^y=(5d)^z$$

Apply Proposition 2.5 [22], $y=1$ is seen. Consequently, there are no solutions for $x>2$.

ii. $k_1\geq 3$ and $k_2=2$

$$\frac{k_1}{k_2}=\frac{y}{x}$$

since $k_1x=k_2y$. Note that $\gcd(x,y)=1$. Indeed, if there exists an odd prime $p\geq 1$ such that $p|x$ and $p|y$, then by Zsigmondy Theorem [22] there is no solution, for x and y. Hence, it is clear that $x=2$ and $k_2=2$ where y is odd. Therefore,

$$y=k_1\geq 3 \qquad \text{and} \qquad x=k_2=2$$

(3.16) becomes

$$5^{k_1x}A^x+5^{k_2y}B^y=(5d)^z$$

and thus

$$5^{2y}\left(A^2+B^y\right) = (5d)^z$$

If $5 \nmid (A^2+B^y)$, then $2y = z$. Then, (1.2) becomes

$$\left(9d^2+1\right)^x = \left((5d)^2\right)^y - \left(16d^2-1\right)^y$$

Applying Proposition 2.5 from Zsigmondy's theorem, it follows that $y = 1$. However, this leads to a contradiction. Therefore, there exist no positive integer solutions, for x and y. Thus, $z \leq 2$.

If $5 | (A^2+B^y)$, by (3.17) and (3.19),

$$16d^2 - 1 = 5^{k_2}B = 25B$$

and thus

$$9d^2 + 1 = 5^{k_1}A$$

If add the above equations side by side, then

$$25^2 = 5^{k_1} + 25B \tag{3.25}$$

When taking (3.25) modulo 5,

$$1 \equiv \ B \pmod 5$$

In conclusion, no positive integer A can be found that satisfies the condition $5 \mid (A^2+B^y)$.

□

4. Conclusion

This research investigates the equation (1.2) with specific parameters $(p,u,w) = (9,16,5)$ and determines the unique solution $(x,y,z) = (1,1,2)$ when $d > 1$. Particularly, it addresses an unexplored area in the literature by considering the case where u is a positive even integer and p is an odd integer in the equation

$$\left(pd^2+1\right)^x + \left(ud^2-1\right)^y = (wd)^z$$

In doing so, it guides future research in solving equations where the coefficient u is a positive even integer and contributes to the existing knowledge in this field. The aim is to take a step towards finding and generalizing many equations, leading to a generalized equation.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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