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Research Article

Pre-service Mathematics Teachers' Modes of Thinking in Linear Algebra: The Case of Linear Transformation*

Matematik Öğretmeni Adaylarının Lineer Cebirde Düşünme Biçimleri: Lineer Dönüşüm Örneği

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ABSTRACT

The aim of this research was to determine the modes of thinking that pre-service mathematics teachers' employ to solve problems related to the concept of linear transformation in linear algebra. A study was conducted with 22 pre-service mathematics teachers' using the case study method - a qualitative research method. The data of the research were collected through four problems defined in the context of the "definition of linear transformation" and "matrix representation of linear transformation". 10 codes were created upon the descriptive analysis of the data collected, and those codes were classified in the context of Sierpinska's (2000) theoretical framework modes of thinking (analytical-structural, analytical-arithmetic, synthetic-geometric). According to the study, pre-service mathematics teachers' had different modes of thinking in "definition" and "matrix representation" but they could not switch between modes of thinking. It was found that analytical-arithmetic thinking was more common than analytical-structural and synthetic-geometric thinking throughout the study. The concept of linear transformation could not be internalized with all its components and it was a challenging process for pre-service teachers' to switch to the matrix representation of linear transformation.

Keywords: Linear algebra, linear transformation, modes of thinking, pre-service mathematics teachers.

ÖZ

Bu araştırmanın amacı matematik öğretmeni adaylarının lineer cebirde, lineer dönüşüm kavramına ilişkin problemleri çözerken sahip oldukları düşünme biçimlerini belirlemektir. Nitel araştırma yöntemlerinden durum çalışması benimsenerek, 22 matematik öğretmeni adayı ile araştırma gerçekleştirilmiştir. Araştırmanın verileri "lineer dönüşümün tanımı" ve "lineer dönüşümün matris temsili" bağlamında hazırlanan dört adet problem aracılığıyla toplanmıştır. Elde edilen verilerin betimsel analize tabi tutulmasıyla 10 adet kod oluşturulmuş ve bu kodlar Sierpinska'nın (2000) düşünme biçimleri (analitik-yapısal, analitik-aritmetik, sentetik-geometrik) kuramsal çerçevesi bağlamında sınıflandırılmıştır. Araştırmanın sonucunda öğretmen adaylarının lineer dönüşüm kavramını "tanım" ve "matris temsili" bağlamında farklı düşünme biçimlerine sahip oldukları ancak düşünme biçimleri arasında geçiş yapamadıkları belirlenmiştir. Tüm süreçte analitik-aritmetik düşünme biçiminin analitik-yapısal ve sentetik-geometrik düşünme biçimine kıyasla daha baskın olduğu belirlenmiştir. Lineer dönüşüm kavramı

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tüm bileşenleri ile içselleştirilememiş ve lineer dönüşümün matris temsiline geçme fikri öğretmen adayları için zorlayıcı bir süreç olduğu belirlenmiştir.

Anahtar Kelimeler: Lineer cebir, lineer dönüşüm, düşünme biçimi, matematik öğretmeni adayı.

INTRODUCTION

The process of learning new concepts is based on pre-formed concepts particularly in advanced mathematics (Villabona et al., 2020). The concept of linear transformation is also one of the advanced algebra concepts, central to linear algebra, and often involves a process that students struggle to grasp, encountering new definitions and theorems along with the concept (Roa-Fuentes & Oktaç, 2010).

Linear transformations are functions defined from one vector space to another that preserve vector addition and scalar multiplication (Bagley et al., 2015). Although this concept contains ideas familiar to students, it is one of the concepts that students find quite challenging (Sierpinska et al., 1999; Sierpinska, 2000). Students may struggle to grasp the concept of linear transformation due to it being a special type of function between vector spaces and their prior exposure to the concept of functions (Oktaç, 2018). The concept of linear transformation, including components such as functions, vector spaces, vector addition, and scalar multiplication, is also related to matrix transformations. As a result of their formal definition, linear transformations contain zero vectors, and this provides an idea about the geometric representation of the transformation, especially in one-dimensional spaces. In one-dimensional spaces, the linearity of a transformation can be inferred by easily seeing whether the graph of a transformation transforms the zero vector into a non-zero vector. Linear transformations can be defined as matrix transformations (Bogomolny, 2006), and students find it complex to conceptualize matrices within the context of linear transformations (Turgut, 2022). The necessity to understand this entire process both algebraically and geometrically, which involves different thinking processes, has made the concept challenging for both students and educators.

Considering the formal structure of the concept of linear transformation and its relationship with matrices, it is possible to state that the existence of different representations of the concept requires transitions between these representations to involve various modes of thinking. Sierpinska (2000) linked students' difficulties in understanding linear algebra concepts to inconsistencies in their modes of thinking and aimed to determine how students think in linear algebra and what the characteristics of these thinking modes are. Sierpinska (2000) tried to determine the students' modes of thinking and the main characteristics of those modes of thinking and has examined it in three categories: Analytical-structural, analytical-arithmetic, and synthetic-geometric. Sierpinska (2000) stated that "the purpose of analytical-structural thinking was to expand knowledge about concepts, and the purpose of analytical-arithmetic thinking was to simplify calculations and ensure their accuracy". An object is defined by a formula that facilitates calculation in analytical-arithmetic thinking, whereas in analytical-structural thinking, an object is best defined by a set of properties (Sierpinska, 2000). Synthetic-geometric thinking is associated with using geometric representations and avoiding definitions related to the concepts used. According to Sierpinska (2000), these three modes of thinking differ in the representations used. Geometric structures are used in synthetic-geometric thinking. In analytical-arithmetic thinking, geometric figures are considered as a set of "ordered n-tuples" of the numbers that fulfill certain conditions. In this mode of thinking, numeric components of geometric objects, such as dots or vectors are important. Analytical-structural thinking, on the other hand, considers algebraic elements of analytical representations as a structural integrity (Sierpinska, 2000).

Sierpinska (2000) contended that the modes of thinking she identified should not be regarded as successive stages in the evolution of algebraic thinking; rather, she suggested that utilizing different modes of thinking in contexts involving various representations is beneficial (Çelik, 2015). She noted that transitions between these thinking modes can provide insight into how the concept is understood in different contexts. There is a limited number of studies on linear transformation in the literature (Andrews-Larson et al., 2017; Bagley et al., 2015; González-Rojas & Roa-Fuentes, 2017; Lamb et al., 2002; Viirman, 2011; Zandieh et al., 2017) and the fact that these studies have not explored students' thinking modes, this research focuses on determining how students think about linear transformations in their various representations. In this context, this research was to determine the modes of thinking that pre-service mathematics teachers employ to solve problems related to the concept of linear transformation in linear algebra. The problems of the research are presented below:

- Which modes of thinking do pre-service mathematics teachers use for the definition of linear transformation?
- Which modes of thinking do pre-service mathematics teachers use for the matrix representation of the linear transformation concept?

A student is expected to have outputs related to knowing the "definition" and "matrix representation" of linear transformation, which is a concept from the course subject of linear algebra. The approach taken in line with these contexts may be a representation of a students' mode of thinking. Knowledge about students' modes of thinking about the basic concepts in linear algebra, such as linear transformation, may be useful for pedagogical purposes. Also, knowledge about how students think can pave the way for a meaningful teaching environment and creating materials based on students' needs. Accordingly, one can say that the studies focusing on students' modes of thinking about the basic concepts of linear algebra are important for developing practices for learning/teaching linear algebra (Çelik, 2015).

METHOD

2.1. Research Design

This research was carried out using case study, a qualitative research method. The case discussed in the research involved an investigation of the modes of thinking used by the pre-service teachers in the context of linear transformation. It is expected that this will offer rich and important perspectives (Brown, 2008) for explaining various issues including how things are interpreted in the context of the modes of thinking, and what arrangements could be made for pedagogic purposes.

2.2. Participants

This research was conducted with 22 pre-service teachers, all of whom were enrolled in the third-year mathematics teaching program at a state university. 14 participants were female and 8 were male. The participants were 20 years old on average, and had a grade point average of 2.92. Pre-service teachers were coded PT1, PT2, … PT22.

2.3. Data Collection Tool

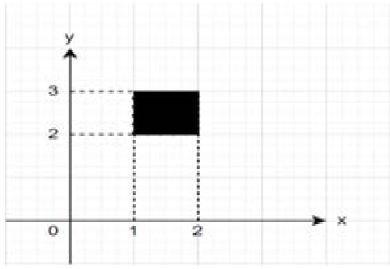
The data of the research were collected with four problems. The problems are given below.

1. Show that $\mathrm{L}: \mathrm{R}^2 \rightarrow \mathrm{R}^2,\ \mathrm{L}(\mathrm{u}_1, \mathrm{u}_2) = (\mathrm{u}_1, -\mathrm{u}_2)$ is a linear transformation. Explain what this transformation means in geometric terms.
2. What can you say about whether the function L given in the form of $\mathrm{L}: \mathrm{R} \rightarrow \mathrm{R},\ \mathrm{L}(\mathrm{u}) = 2\mathrm{u} + 5$ is a linear transformation?

3. $L(1,1) = (3,4)$ and $L(1, -1) = (-1,2)$ are provided for the linear transformation $L: R^2 \rightarrow R^2$. Find the rule of the linear transformation L.
4. $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ being a matrix and $L: R^2 \rightarrow R^2$ a linear transformation,

a. Find the linear transformation L where the representation of R^2 on the natural base is the matrix A.

b. Plot the graph of the area resulting from the application of the transformation graph L to the square area on the right.



The first two problems involve knowing, selecting, and applying the formal definition of linear transformation and its corresponding characterizations. The third problem requires the ability to apply both the formal definition of linear transformation and the matrix representation of linear transformation. The final problem involves deriving the rule of linear transformation from its matrix representation and interpreting a transformation geometrically. Different approaches to the problems are present, and the approaches exhibited by the pre-service teachers will reveal their modes of thinking.

2.4. Data Collection Process

Following the obtainment of legal permits, a meeting was arranged with the participants in a quiet classroom, and they were asked to solve the four problems individually. The data collection process ended when the participants solved the problems within half an hour.

2.5. Data Analysis

The data collected in the research were subjected to descriptive analysis. The aim of this analysis is "to present the findings to the reader in an organized and interpreted manner" (Yıldırım & Şimşek, 2016). The problems solved by the pre-service teachers were analyzed multiple times, and each problem and each pre-service teacher were subject to several interactions with the documents. The answers were then classified by similarity. Then codes were formed based on the answers to each problem. Codes were labeled with expressions representing the solve process for pre-service teachers. The codes were re-examined, and the codes process was terminated upon the researchers' assessment of the analyses. The codes made on the responses given by pre-service teachers to problems, and the descriptions of those codes are shown in Table 1.

Table 1

Codes Representing The Pre-Service Teachers' Modes of Thinking, and The Descriptions of These Codes

Code	Description
Formal definition	Represents addressing vector addition and scalar multiplication conditions, which are sufficient and necessary for a function to be a linear transformation, individually.
Characterizations corresponding to formal definition	Represents addressing vector addition and scalar multiplication conditions, which are sufficient and necessary for a function to be a linear transformation, in a single expression.
Zero vector	Represents transformation of the zero vector of V into the zero vector of W by a linear transformation $L: V \rightarrow W$.
Geometric interpretation	Represents making inferences about what a vector or area turns

	into under a certain transformation.
Linear function	Represents the cases where linear functions should also be linear transformations.
Base	Represents the cases where the rule of the linear transformation L is defined exactly based on the conditions that the function $L: R^2 \rightarrow R^2$ is linear and a base image of the space R^2 is given.
Transformation matrix	Represents the cases where a transformation matrix A is addressed with $L: R^2 \rightarrow R^2$ being a linear transformation and $L(u) = Au$.
Trial and error	Represents the cases where a correlation is established between the input and output vectors and there is a linear transformation rule.
Linear combination	Represents the cases where the linear transformation $L: R^2 \rightarrow R^2$ is defined as $L(u_1, u_2) = (au_1 + bu_2, cu_1 + du_2)$
No response	Represents the cases where no explanation was made about the problem/the answer is left unanswered.

As shown in Table 1, 10 codes were formed in line with the answers to the problems. The codes, excluding the “no response” code, were classified to represent the modes of thinking.

According to Sierpinska (2000), both numerical and algebraic representations as well as formulas that allow calculations to be made, and codes containing operational processes are in the analytical-arithmetic thinking (Çelik, 2015). In this context, the codes of “formal definition, trial and error” represent analytical-arithmetic thinking. The code of formal definition require implementing a formula based on “what” a linear transformation is, while the trial and error code requires making numerical calculations rather than implementing a formula. According to Sierpinska (2000), codes where objects are analyzed through theorems and definitions are classified as parts of the analytical-structural thinking (Çelik, 2015). The codes of “characterizations corresponding to formal definition, base, transformation matrix, linear combination, zero vector” represent analytical-structural thinking. Characterizations corresponding to formal definition code requires a strong equivalent expression of the formal definition of linear transformation, and linear combination code requires knowing how to express linear transformations with linear combinations. Base and transformation matrix codes, on the other hand, require using the relevant theorem to find the rule of a linear transformation. Zero vector requires interpretation based on the formal definition. According to Sierpinska (2000), codes containing processes for describing objects rather than defining them are classified as the synthetic-geometric thinking. Moreover, this mode of thinking requires practical thinking as well as dealing only with the geometric properties of shapes. In this regard, the codes of “geometric interpretation, linear function” represent the synthetic-geometric thinking. Linear function code involves interpretation through a line, and the geometric interpretation code involves geometric interpretation of the images under the transformation of shapes.

The researchers examined the answers to each problem individually for each pre-service teacher and assigned them to the codes that represented them most accurately. The fact that the inter-coder reliability is 94% and this rate is above 70% means that the analyses are reliable (Miles & Huberman, 1994). Then the answers were classified by code and mode of thinking, and numerical data were presented based on the descriptive statistical techniques (frequency and percentage). Examples of the pre-service teachers’ answers to the problems were also included. Moreover, since the first and the last problems involved multiple questions, the pre-service teachers’ answers to those problems fell into multiple codes.

2.6. Ethical Procedures

This study was deemed ethically appropriate by the Hacettepe University Ethics Committee in a letter dated 08.05.2023 and numbered E-35853172-300-00002826013.

FINDINGS

The pre-service teachers' approaches to problems are addressed with the codes "formal definition, characterizations corresponding to formal definition, zero vector, geometric interpretation, linear function, base, linear combination, trial and error" in the context of the definition of linear transformation. The codes "transformation matrix, geometric interpretation" are considered in relation to the matrix representation of linear transformation. The codes associated with each problem are presented in the context of the pre-service teachers' modes of thinking.

The pre-service teachers' codes in line with their answers to the first problem are shown in Table 2.

Table 2

Pre-service Teachers' Answers to The First Problem

Code	Pre-service Teachers	f	%
Formal definition	PT1, PT2, PT3, PT4, PT5, PT7, PT8, PT9, PT11, PT12, PT13, PT14, PT15, PT16, PT18, PT19, PT20, PT21, PT22	19	76
Characterizations corresponding to formal definition	PT10	1	4
Geometric interpretation	PT4, PT7	2	8
Zero vector	PT3	1	4
No response	PT6, PT17	2	8

In Table 2, pre-service teachers' were assigned to five codes, i.e. "formal definition" (f=19, 76%), "characterizations corresponding to formal definition" (f=1, 4%), "geometric interpretation" (f=2, 8%), "zero vector" (f=1, 4%) and "no response" (f=2, 8%), according to their answers to the problem 1.

3.1. Analytical-Structural Mode of Thinking

The pre-service teacher PT10 has expressed, in a strong characterization, linear transformation through a single expression, involving vector addition and scalar multiplication. However, PT10 did not complete the problem solving process.

The pre-service teacher PT3 has considered the inclusion of the zero vector as one of the requirements for the linearity of the transformation and has stated that it is required for transformation to encompass the zero vector as well.

3.2 Analytical-Arithmetic Mode of Thinking

Pre-service teachers tended to define linear transformation by the formal definition of linear transformation, which corresponded to the analytical-arithmetic thinking. The pre-service teachers in the formal definition code were classified in the context of their answers to the first problem, as seen in Figure 1.

Figure 1

Classification of The Pre-service Teachers in The Code of Formal Definition

Able to implement a formal definition	Able to express a formal definition	Unable to implement a formal definition	Checks only one condition of a formal definition
• Represents the situations where the pre-service teachers' apply the formal definition of a linear transformation completely and accurately. • PT2, PT3, PT4, PT5, PT7, PT15, PT20, PT21	• Represents the situations where the pre-service teachers' can express the formal definition of a linear transformation but left the relevant problem unanswered. • PT8, PT9, PT12, PT14	• Represents the situations where the pre-service teachers' can express the formal definition of a linear transformation but cannot complete the operational processes regarding the problem. • PT1, PT11, PT18, PT22	• Represents the situations where the pre-service teachers' checked only one of the conditions of vector addition and scalar multiplication in the formal definition of linear transformation. • PT13, PT16, PT19

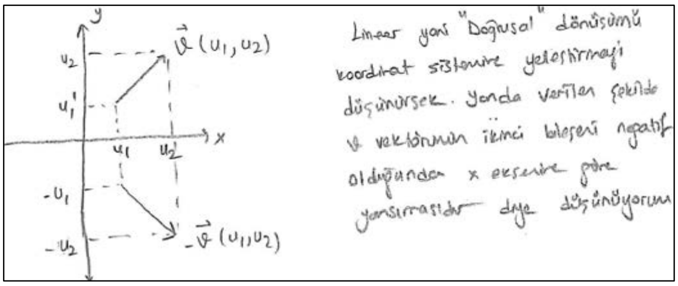
The pre-service teachers in this code were in four different classes, namely (i) able to implement (f=8), (ii) able to express (f=4), (iii) unable to implement (f=4), and (iv) checks only one condition (f=3) with regard to the formal definition of linear transformation.

3.3. Synthetic-Geometric Mode of Thinking

Most of the pre-service teachers ignored this in problem 1 which also included geometric interpretation of a transformation. Pre-service teachers PT4 and PT7 represented the given transformation on a coordinate axis and interpreted it geometrically. Figure 2 shows the answer of pre-service teacher PT4 to the first problem.

Figure 2

The Answer of PT4 to The First Problem



Let us consider putting linear transformation on a coordinate system. Since the second component of the vector v is negative in the figure on the right, I think it is a reflection on the x axis.

The pre-service teacher PT4 stated that the linear transformation given had the function of "reflecting a vector on the x axis". Although the pre-service teacher PT4 associated linear transformation with structures that can be geometrically interpreted through input and output vectors, she emphasizes the necessity of linear transformations being "linear".

The pre-service teachers' codes in line with their answers to the second problem are shown in Table 3.

Table 3

Pre-service Teachers' Answers to The Second Problem

Code	Pre-service Teachers	f	%
Formal definition	PT1, PT2, PT3, PT7, PT10, PT11, PT15	7	32
Linear function	PT4, PT5, PT8, PT9, PT13, PT14, PT16, PT18, PT20	9	41
No response	PT6, PT12, PT17, PT19, PT21, PT22	6	27

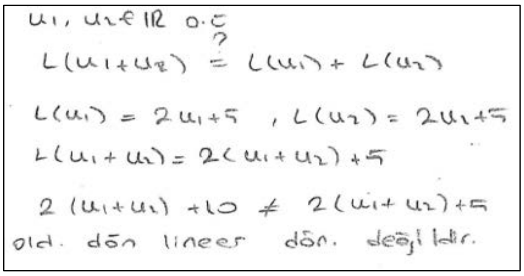
In Table 3, pre-service teachers were assigned to three codes, i.e. “formal definition (f=7, 32%), “linear function” (f=9, 41%) and “no response” (f=6, 27%), according to their answers to the problem 2.

3.4. Analytical-Arithmetic Mode of Thinking

Almost half of the pre-service teachers answers to the problem gave a formal definition of linear transformation. The pre-service teachers in this code checked the conditions required for a function to be a linear transformation, and showed that the given function was not a linear transformation for (i) not fulfilling vector addition (PT2, PT3, PT15), (ii) not fulfilling scalar multiplication (PT10, PT11), (iii) not fulfilling both vector addition and scalar multiplication (PT1, PT7). Figure 3 shows the answer of pre-service teacher PT15 to the second problem.

Figure 3

The Answer of PT15 to The Second Problem



Let $u_1, u_2 \in R$

$L(u_1) = 2u_1 + 5 \quad L(u_2) = 2u_2 + 5$

$L(u_1 + u_2) = 2(u_1 + u_2) + 5$

$2(u_1 + u_2) + 10 \neq 2(u_1 + u_2) + 5$

L is not a linear transformation.

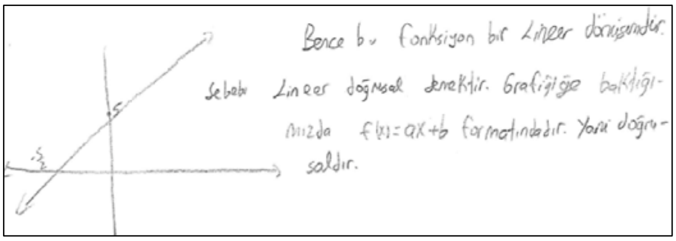
The pre-service teacher PT15 showed that the given function was not a linear transformation by proving that the sum of the two vectors was not equal to the sum of the transformations.

3.5. Synthetic-Geometric Mode of Thinking

The other half of the pre-service teachers answers to the problem associated whether a single-variable and single-value function is a linear transformation with the concept of “linear function”. The pre-service teachers in this code concluded that “the given function is a linear transformation because it is a linear function”. Figure 4 shows the answer of pre-service teacher PT18 to the second problem.

Figure 4

The Answer of PT18 to The Second Problem



I think this function is a linear transformation. The reason is that the concept of linear involves a line. The graph is in the format of $f(x) = ax + b$. In other words, it is linear.

The pre-service teachers PT18 stated that the given function was a linear transformation for making a linear graph in the coordinate axis.

The pre-service teachers’ codes in line with their answers to the third problem are shown in Table 4.

Table 4

Pre-service Teachers' Answers to The Third Problem

Code	Pre-service Teachers	f	%
Formal definition	PT2	1	4
Trial and error	PT2, PT7, PT11, PT15	4	17
Base	PT8, PT16	2	9
Linear combination	PT3, PT9	2	9
Transformation matrix	PT20, PT21	2	9
No response	PT1, PT4, PT5, PT6, PT10, PT12, PT13, PT14, PT17, PT18, PT19, PT22	12	52

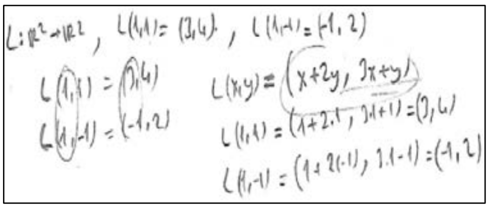
In Table 4, pre-service teachers were assigned to six codes, i.e. “formal definition” (f=1, 4%), “trial and error” (f=4, 17%), “base” (f=2, 9%), “linear combination” (f=2, 9%), “transformation matrix” (f=2, 9%), and “no response” (f=12, 52%), according to their answers to the problem 3.

3.6. Analytical-Arithmetic Mode of Thinking

About half of the pre-service teachers answers to the problem had the analytical-arithmetic thinking since they were in the formal definition and trial and error codes. Figure 5 shows the answer of pre-service teacher PT7 to the third problem.

Figure 5

The Answer of PT7 to The Third Problem



The pre-service teacher PT7 established a correlation among the input and output vectors of the given transformation to find the rule of the transformation. Similarly, the pre-service teacher PT2 who found the transformation rule questioned whether the transformation they found was linear, and showed that the transformation maintained the rules of vector addition and scalar multiplication.

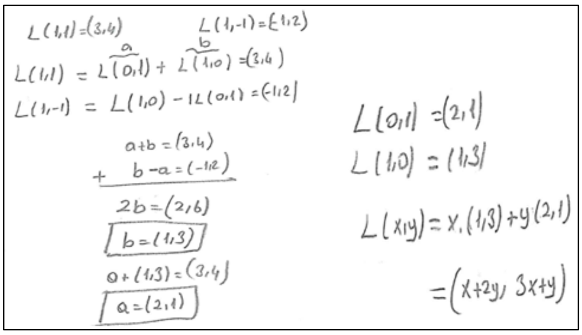
3.7. Analytical-Structural Mode of Thinking

The pre-service teachers in the linear combination code defined linear transformation as $L(u_1, u_2) = (au_1 + bu_2, cu_1 + du_2)$, and found the rule of the linear transformation by finding the variables of a, b, c, d through given transformations.

The starting point of the pre-service teachers in the base code was that the function $L: R^2 \rightarrow R^2$ was linear and that it was possible to find the rule of the transformation since the R^2 space gave the image of a base. Figure 6 shows the answer of pre-service teacher PT8 to the third problem.

Figure 6

The Answer of PT8 to The Third Problem

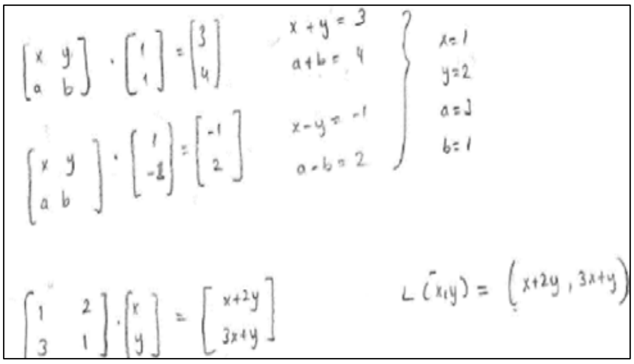


The pre-service teacher PT8 associated the vectors $(1,1)$ and $(1, -1)$ with the possibility of writing them as a linear combination of the vectors of $\{(1,0), (0,1)\}$ which was the natural base of the R^2 space, finding the rule of the transformation using the properties of linear transformation.

Pre-service teachers in the transformation matrix code set out from the fact that the multiplication of a matrix and a vector was equivalent to a linear transformation to establish the rule of the linear transformation. Figure 7 shows the answer of pre-service teacher PT21 to the third problem.

Figure 7

The Answer of PT21 to The Third Problem



The pre-service teacher PT21 created 2×2 matrix of the form $\begin{bmatrix} x & y \\ a & b \end{bmatrix}$ using matrix transformation and applied the given transformations to find the elements of the matrix. They multiplied the matrix they found by a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ from R^2 to find the linear transformation rule.

The pre-service teachers' code in line with their answers to the last problem are shown in Table 5.

Table 5

Pre-service Teachers' Answers to The Fourth Problem

Code	Pre-service Teachers	f	%
Transformation matrix	PT2, PT7, PT15	3	13
Geometric interpretation	PT7, PT15	2	8
No response	PT1, PT3, PT4, PT5, PT6, PT8, PT9, PT10, PT11, PT12, PT13, PT14, PT16, PT17, PT18, PT19, PT20, PT21, PT22	19	79

In Table 5, pre-service teachers were assigned to three codes, i.e. "transformation matrix" (f=3, 13%), "geometric interpretation" (f=2, 8%) and "no response" (f=19, 79%), according to their answers to the last problem. The modes of thinking of the three pre-service teachers who solved to this problem (PT2, PT7, PT15) encompass the processes associated with the analytical-structural and synthetic-geometric. In addition, since the option (b) of the problem is linked to the option (a), the inability to answer option (a) resulted in an inability to answer option (b).

3.8. Analytical-Structural Mode of Thinking

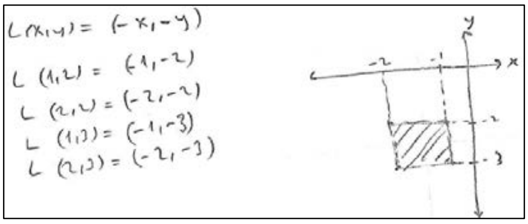
The pre-service teachers PT2, PT7 and PT15 in the code of transformation matrix established the rule of the transformation by creating a relationship between the elements of the matrix and a vector from R^2 in the context of matrix-vector multiplication.

3.9. Synthetic-Geometric Mode of Thinking

The pre-service teachers PT7 and PT15 in the geometric interpretation code, after finding the linear transformation, made the image of a square area under that transformation. Figure 8 shows the answer of pre-service teacher PT7 to the fourth problem.

Figure 8

The Answer of PT7 to The Fourth Problem



The pre-service teacher PT7 first found the corners of the square area in the coordinate axis, and found the image of these points under the transformation that they got. Then they set those points on the coordinate axis to get a new square area.

DISCUSSION, CONCLUSION AND RECOMMENDATIONS

In this research, we investigated how pre-service mathematics teachers thought while solving problems related to the concept of linear transformation, based on the theoretical framework of modes of thinking proposed by Sierpinska (2000). Pre-service teachers answer to the problems about linear transformation were used to set 10 codes; the codes, excluding the "no response" code, have been evaluated in the context of the theoretical framework on modes of thinking. The analytical-arithmetic thinking is represented by two codes (formal definition, trial and error); the analytical-structural thinking is represented by five codes (characterizations corresponding to formal definition, base, transformation matrix, linear combination, zero vector); and the synthetic-geometric thinking is represented by two codes (geometric interpretation, linear function). These modes of thinking were discussed in connection with the "definition of linear transformation" and the "matrix representation of linear transformation".

Pre-service teachers defined linear transformation predominantly by the analytical-arithmetic mode of thinking. Linear transformation was addressed as a set of operational calculations for testing whether a function fulfills certain conditions in the context of formal definition. Since it is algebraically and operationally easier to check these conditions individually, linear transformation supersedes using its different characterizations. In addition, the fact that some pre-service teachers checked only one condition of a formal definition made it necessary to question the definitions. Even though linear transformation is considered through vector addition and scalar multiplication, the fact that characterizations corresponding to formal definition are not realized indicates that an analytical-structural mode of thinking cannot be used.

A key property of linear transformation is that it transforms the zero vector of the definition set to the zero vector of the value set. This is a strong property that provides information about whether a transformation is linear; however, this property was not recognized by pre-service teachers. Essentially an outcome of the formal definition, it is an indication that conclusions regarding formal definition cannot be made. In this regard, Andrews-Larson et al. (2017) highlighted the necessity of interpreting linear transformation as a mathematical asset that transforms input vectors into output vectors.

Pre-service teachers in the synthetic-geometric thinking concluded that "a linear function is a linear transformation". It is though that the pre-service teachers considered linear transformation and linear function as equivalent. Therefore, they applied their interpretation based on linear functions to the concept of linear transformation. Interestingly enough, the pre-service teachers defined linear transformation by a formal definition but did not make sense of the geometric representation of linear transformation. In this respect, it can be said that the pre-service teachers were unable to internalize the concept. This situation is also supported by the fact that pre-service teachers avoid the geometric representation of linear transformation and its applications.

It was indicated that the pre-service teachers were not familiar with making a connection between the concepts of linear transformation and matrix, or with switching from a linear transformation to a matrix or from a matrix to a linear transformation. Encountering a similar outcome, Andrews-Larson et al., (2017) designed a set of tasks to help students learn matrices linear transformations. Pre-service teachers use the matrix representation of linear transformation with arguments related to both analytical-arithmetic thinking and analytical-structural thinking. Pre-service teachers in the analytical-arithmetic mode of thinking went through a process that involved operational calculations in the form of trial and error. This does not emphasize a linear transformation but rather setting a pattern rule for finding the rule of any function or a transformation. The key properties of the analytical-structural mode of thinking include considering definitions and definition-related properties as a whole, and eliminating the dominance of the numerical and algebraic calculations (Çelik, 2015). This is reflected by the codes of base, linear combination, and transformation matrix reflect. There are gaps in the arguments of the pre-service teachers about how to switch from a linear transformation to matrix representation. This supports the argument of Dorier et al. (2000) that students lacked knowledge of how to calculate the matrix representation of a linear transformation. Therefore, one can say that pre-service teachers fail to make sense of the matrix representation of linear transformation. The idea that a matrix represents a transformation may be challenging (Bagley et al., 2015).

In the light of these conclusins, it is fair to say that pre-service teachers had different modes of thinking in "definition" and "matrix representation" but they could not switch between modes of thinking. The fact that the analytical-arithmetic mode of thinking was more common than analytical-structural and synthetic-geometric thinking is attributable to the fact that the operational process that requires making calculations in that mode of thinking is more dominant. The concept of linear transformation could not be internalized with all its components, and no meaning could be ascribed to the geometric representation of linear transformation in particular. It was thought that it was a challenging process for pre-service teachers to switch to the matrix representation of linear transformation. This might be attributed to such reasons as the lack of knowledge about the reason for transitioning to the matrix representation of linear transformation and finding the algorithmic structure of that transition challenging.

In conclusion, pre-service teachers were unable to switch between different representations of linear transformation. Çelik (2015) made a similar conclusion for the concepts of linear dependent/independent. Dubinsky (1997) and Harel (1987), on the other hand, suggested that flexibility in various representations of a specific concept might help students abstract it. Linear algebra, by its nature, features a lot of abstract concepts, and students lack flexibility among different modes of thinking, which has a negative effect on learning and teaching linear algebra (Sierpinska, 2000). In this sense, it is advisable to design teaching experiments that will help students switch between the representations of the linear algebra concepts and conduct the process of teaching with appropriate materials.

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GENİŞLETİLMİŞ ÖZ

Giriş

Lineer dönüşümler vektörel toplamayı ve skaler ile çarpmayı koruyan, bir vektör uzayından diğerine tanımlı fonksiyonlardır (Bagley vd., 2015). Bu kavram öğrencilerin aşina oldukları kavramları içermesine rağmen öğrencilerin oldukça zorlandıkları kavramlardan biridir (Sierpinska vd., 1999; Sierpinska, 2000). Lineer dönüşümün vektör uzayları arasında özel bir tür fonksiyon olması ve öğrencilerin daha önce oluşturulmuş fonksiyon kavramı nedeniyle (Oktaç, 2018) lineer dönüşüm kavramını algılamada zorlanabilmektedirler. Kavramın formal yapısında fonksiyon, vektör uzayı, vektörel toplama ve skaler ile çarpma bileşenlerinin yer almasına ek olarak; kavram matris dönüşümleri ile de ilişkilidir. Lineer dönüşümler birer matris dönüşümü olarak tanımlanabilmekte (Bogomolny, 2006) ve öğrenciler matrisleri birer lineer dönüşüm bağlamında kavramsallaştırabilmeyi karmaşık bulmaktadırlar (Turgut, 2022). Tüm bu sürecin cebirsel ve geometrik olarak anlamlandırılmasının farklı düşünme süreçlerini içeriyor olması kavramı hem öğrenci hem eğitmen açısından zor kılmıştır.

Lineer dönüşüm kavramının formal yapısı ve matrislerle olan ilişkisi göz önüne alındığında; kavramın farklı temsillerinin bulunması, bu temsiller arasındaki geçişin farklı düşünme biçimlerini gerektirdiğini söylemek mümkündür. Sierpinska (2000) öğrencilerin lineer cebir kavramlarını anlamlandırmada zorluk yaşamalarını öğrencilerin düşünme biçimleri arasındaki tutarsızlıkla ilişkilendirmiş ve öğrencilerin lineer cebirdeki düşünme biçimlerinin nasıl olduğunu ve bu düşünme biçimlerinin özelliklerinin ne olduğunu belirlemeye çalışmıştır. Öğrencilerin lineer cebirdeki düşünme biçimlerini analitik-yapısal, analitik-aritmetik ve sentetik-geometrik olmak üzere üç başlıkta değerlendirmiştir. Sierpinska (2000) analitik-yapısal düşünmenin amacını "kavramlara yönelik bilgiyi genişletme", analitik-aritmetik düşünmenin amacının "hesaplamaları basitleştirme ve doğru yapma" olduğunu ifade etmiştir. Analitik-aritmetik düşünme biçiminde bir nesne, hesaplama yapmaya imkan veren bir formül ile tanımlanırken; analitik-yapısal düşünme biçiminde bir nesne en iyi bir dizi özellik tarafından tanımlanır (Sierpinska, 2000). Sentetik-geometrik düşünme biçimi ise geometrik temsillerin kullanımı ve kullanılan kavramlarla ilgili tanımlara yer verilmemesi ile ilgilidir.

Öğrencilerin lineer dönüşüm kavramını anlamalarına ilişkin sınırlı sayıda çalışmanın (Andrews-Larson vd., 2017; Bagley vd., 2015; González-Rojas & Roa-Fuentes, 2017; Lamb vd., 2002; Viirman, 2011; Zandieh vd., 2017) olması ve bu çalışmaların öğrencilerin düşünme biçimini araştırmamış olması; bu araştırmanın lineer dönüşümün farklı temsillerinde öğrencilerin düşünme biçimlerinin nasıl olduğunun belirlenmesini konu edinmiştir. Bir öğrencinin lineer dönüşüm ile ilgili temel olarak lineer dönüşüm kavramının "tanımını" ve "matris temsilini" bilmeye ilişkin çıktılara sahip olması beklenmektedir. Bu bağlamlar doğrultusunda sergilenen yaklaşım öğrencinin düşüne biçiminin bir temsili olabilir.

Bu araştırmanın amacı matematik öğretmeni adaylarının lineer cebirde, lineer dönüşüm kavramına ilişkin problemleri çözerken sahip oldukları düşünme biçimlerini incelemektir. Araştırmanın problemleri aşağıda sunulmuştur:

- Matematik öğretmeni adaylarının lineer dönüşüm kavramının tanımını ilişkin düşünme biçimleri nasıldır?
- Matematik öğretmeni adaylarının lineer dönüşüm kavramının matris temsiline ilişkin düşünme biçimleri nasıldır?

Yöntem

Araştırma nitel araştırma desenlerinden durum çalışması doğrultusunda, bir devlet üniversitesinin matematik öğretmenliği programının üçüncü sınıfında öğrenimine devam eden 22 öğretmen adayı ile gerçekleştirilmiştir.

Araştırmanın veri toplama aracını araştırmacılardan tarafından hazırlanan dört adet problem oluşturmuştur. İlk iki problem lineer dönüşüm kavramının formal tanımını ve denk karakterizasyonlarını bilmeyi, seçmeyi ve uygulamayı; üçüncü problem hem lineer dönüşümün formal tanımı hem de lineer dönüşümün matris temsili bağlamında uygulama yapabilmeyi; son problem ise lineer dönüşümün matris temsilden lineer dönüşümün kuralını bulabilmeyi ve bir dönüşümü geometrik olarak yorumlayabilmeyi içermektedir.

Öğretmen adaylarının yaklaşık yarım saat içerisinde problemlere yanıt vermesi ile veri toplama süreci sonlandırılmıştır. Araştırmanın verileri ise nitel veri analiz tekniklerinden betimsel analiz ile çözümlenmiştir.

Bulgular, Tartışma ve Sonuç

Öğretmen adaylarının lineer dönüşüm ile ilgili problemlere verdikleri yanıtlar doğrultusunda 10 adet kod oluşturulmuştur. İki adet kod (formal tanım, deneme-yanılma) analitik-aritmetik düşünme biçimini, beş adet kod (formal tanıma denk karakterizasyonlar, taban, dönüşüm matrisi, lineer birleşim, sıfır vektörü) analitik-yapısal düşünme biçimini, iki adet kod (geometrik yorum, lineer fonksiyon) ise sentetik-geometrik düşünme biçimini temsil etmektedir. Bu düşünme biçimleri "lineer dönüşüm kavramının tanımı" ve "lineer dönüşüm kavramının matris temsili" bağlamında ele alınmıştır.

Öğretmen adayları lineer dönüşüm kavramını ağırlıklı olarak analitik-aritmetik düşünme biçimi ile tanımlamaktadırlar. Lineer dönüşüm, formal tanım bağlamında bir fonksiyonun belirli şartları sağlayıp sağlamadığını test edici işlemsel hesaplamalar olarak ele alınmıştır. Sentetik-geometrik düşünme biçiminde yer alan öğretmen adaylarının "bir fonksiyon doğrusal ise lineer dönüşümdür" şeklinde çıkarımları mevcuttur. Öğretmen adayları "doğrusal" ve "lineer" kavramlarının eş olması sebebiyle lineer dönüşüm ve doğrusal fonksiyon kavramlarını bir olarak gördükleri düşünülmektedir. Dolayısıyla doğrusal fonksiyon üzerinden yaptıkları yorumu lineer dönüşüm kavramına da yüklemişlerdir.

Öğretmen adaylarının lineer dönüşüm ve matris kavramları arasında ilişki kurmada yabancı oldukları; bir lineer dönüşümden matrise veya matristen lineer dönüşüme geçme fikrine aşina olmadıkları belirlenmiştir. Benzer bir sonuç ile karşılaşan Andrews-Larson ve diğerleri (2017) öğrencilerin matrisleri lineer dönüşümler olarak öğrenmesini desteklemek için bir görev dizisi tasarlamışlardır. Öğretmen adayları lineer dönüşümün matris temsilini hem analitik-aritmetik hem analitik-yapısal düşünme biçimlerine ilişkin argümanlar ile kullanmaktadırlar. Analitik-aritmetik düşünme biçiminde yer alan öğretmen adayları deneme-yanılma şeklinde işlemsel hesaplamaları içeren bir süreçten geçmişlerdir. Analitik-yapısal düşünme biçiminin en önemli özellikleri, tanımların ve tanımla ilgili özelliklerin bir bütün olarak ele alınması, sayısal ve cebire dayalı hesaplamaların baskınlığını yitirmesidir (Çelik, 2015). Taban, lineer birleşim ve dönüşüm matrisi kodları bu durumu yansıtır niteliktedir. Öğretmen adaylarının lineer dönüşümün matris temsiline nasıl geçiş yapılacağına ilişkin argümanlarda eksiklikler bulunmaktadır. Bu sonuç Dorier ve diğerlerinin (2000) öğrencilerin bir lineer dönüşümün matrisinin nasıl hesaplayacağını bilmediğini belirtmeleri ile paralellik göstermektedir. Dolayısıyla öğretmen adaylarının lineer dönüşümün matris temsillini anlamlandıramadığı söylenebilir.

Tüm bu sonuçlar ışığında öğretmen adaylarının lineer dönüşüm kavramına ilişkin "tanım" ve "matris temsili" bağlamında farklı düşünme biçimlerine sahip oldukları ancak düşünme biçimleri arasında geçiş yapamadıkları söylenebilir. Süreçte analitik-aritmetik düşünme biçiminin, analitik-yapısal ve sentetik-geometrik düşünme biçimine kıyasla daha baskın olması ise bu düşünme biçiminde hesaplama yapmayı gerektiren işlemsel sürecin baskın olması ile ilişkilendirilebilir. Lineer dönüşüm kavramı tüm bileşenleri ile içselleştirilememiş; özellikle lineer dönüşümün geometrik temsiline anlam yüklenememiştir. Lineer dönüşümün matris temsiline geçme fikri öğretmen adayları için zorlayıcı bir süreç olduğu düşünülmektedir. Bu durum lineer dönüşümün matris temsiline neden geçiş yapılmasının gerekliliğinin bilinmemesi, bu geçiş sürecinin algoritmik yapısının zorlayıcı bulunması gibi sebeplerle ilişkilendirilebilir.