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APPLICATION OF NEUTROSOPHIC POISSON DISTRIBUTION SERIES ON HARMONIC CLASSES OF ANALYTIC FUNCTIONS DEFINED BY $q-$DERIVATIVE OPERATOR AND SIGMOID FUNCTION

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Abstract. There are several authors who have obtained various forms of properties for some subclasses of analytic univalent functions related to different distribution series, such as Binomial, Generalized Discrete Probability, Geometric, Mittag-Leffler, Pascal, and Poisson distribution series. The authors, in this paper, proved the inclusion relation of the harmonic analytic function class $H_q^\alpha(\theta, \gamma(s), \Psi)$ established by applying convolution operators regarding neutrosophic distribution series equipped with the Sigmoid function (activation function). The present results are capable of handling both accurate (determinate) data and inaccurate (indeterminate) data.

1. Introduction

Indicate by $\mathcal{A}$ the family of functions analytic in $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$, of the form $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ which fulfill the normalization $f(0) = f'(0) - 1 = 0$ and also indicate by S the subfamily of $\mathcal{A}$ including univalent functions in $\mathbb{U}$. Further, for the function $g(z) = z + b_2 z^2 + \cdots$, the convolution $f * g$ is expressed as

$$(f * g)(z) = z + \Sigma_{n=2}^{\infty} a_n b_n z^n.$$

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A harmonic function is a type of function that arises in various areas of mathematics, including complex analysis, partial differential equations, and physics. The real-valued function $v(x, y)$ is named harmonic in a domain $B \subset \mathbb{C}$ if it has continuous second order partial derivative in B, which fulfills

$$\Delta v := \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

A harmonic mapping f of the simply connected domain B is a complex-valued function of the form $f = \phi + \overline{\lambda}$, where ϕ, λ are analytic and $\phi(0) = \phi'(0) - 1 = 0$, $\lambda(0) = 0$. We call ϕ and λ analytic and co-analytic part of f, respectively. $J_{f(z)} = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = \left|\phi'(z)\right|^2 - \left|\lambda'(z)\right|^2$ is defined as the Jacobian of f. Also, f is locally univalent iff its Jacobian is never zero, and is sense-preserving provided that the Jacobian is positive. To this end, without loss of generality, indicate by H the family of all harmonic functions of the form $f = \phi + \bar{\lambda}$, where

$$\phi(z) = z + \sum_{v=2}^{\infty} a_v z^v, \quad \lambda(z) = \sum_{v=1}^{\infty} b_v z^v \quad (|b_1| < 1) \tag{1}$$

are analytic in $\mathbb{U}$. We further indicate by S_H the family of functions $f = \phi + \bar{\lambda}$ that are harmonic univalent and sense preserving in $\mathbb{U}$. Consider the subfamily S_H^0 of S_H as $S_H^0 = \left\{f = \phi + \bar{\lambda} \in S_H : \lambda'(0) = b_1 = 0\right\}$. A sense-preserving harmonic mapping $f \in S_H^0$ is in the class S^* if the range $f(\mathbb{C})$ is starlike with respect to the origin. A function $f \in S_H^*$ is named a harmonic starlike mapping in $\mathbb{U}$. On the other hand, a function $f \in \mathbb{U}$ is included in K_H if $f \in S_H^0$ and if $f(\mathbb{U})$ is a convex domain. A function $f \in K_H$ is named convex harmonic in $\mathbb{U}$. Analytically, $f \in S_H^*$ iff $\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right) \geq 0$, and $f \in K_H$ iff $\frac{\partial}{\partial\theta}\left\{\arg\left(\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right)\right\} \geq 0$, where $z = re^{i\theta} \in \mathbb{U}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. For further details on the harmonic classes of analytic functions, we may refer to some papers (see [3], [7]– [9], [11]–[13], [17], [18], [22], [24], [26]– [30]) and the relevant literature cited in there.

Indicate by T_H the family of functions in S_H that are expressible as $f = \phi + \bar{\lambda}$, where

$$\phi(z) = z - \sum_{v=2}^{\infty} |a_v| z^v, \quad \lambda(z) = \sum_{v=1}^{\infty} |b_v| z^v \quad (|b_1| < 1). \tag{2}$$

Then, for $0 \leq \nu < 1$, the following geometric representations are possible

$$N_H(\nu) = Re\left\{f \in H : \Re\left[\frac{f'(z)}{z'}\right] \geq \nu, \ \ z = re^{i\theta} \in \mathbb{U}\right\}$$

and

$$R_H(\nu) = Re\left\{f \in H : \Re\left[\frac{f''(z)}{z''}\right] \geq \nu, \ \ z = re^{i\theta} \in \mathbb{U}\right\},$$

where

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}), \ \ z'' = \frac{\partial}{\partial\theta}(z'), \ \ f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}), \ \ f'' = \frac{\partial}{\partial\theta}(f'(z)).$$

Define

$$TN_H(\nu) = N_H(\nu) \cap T_H \quad \text{and} \quad TR_H(\nu) = R_H(\nu) \cap T_H.$$

The classes T_H, $N_H(\nu)$, $TN_H(\nu)$, $R_H(\nu)$ and $TR_H(\nu)$ were defined and investigated in [1], [14], [24], [27].

The $q-$derivative, also known as the Jackson $q-$derivative [15], is a concept from the theory of $q-$calculus, which is a generalization of calculus that incorporates a parameter q (often interpreted as a complex number) and extends various concepts from classical calculus.

Next, for $0 < q < 1$, the Jackson's $q-$derivative of a function $f \in S_H$ is expressed as

$$D_q\phi(z) = \begin{cases} \frac{\phi(z)-\phi(qz)}{(1-q)z}, & z \neq 0 \\ \\ \phi'(0), & z = 0 \end{cases} \tag{3}$$

and

$$D_q\lambda(z) = \begin{cases} \frac{\lambda(z)-\lambda(qz)}{(1-q)z}, & z \neq 0 \\ \\ \lambda'(0), & z = 0 \end{cases}. \tag{4}$$

From (3) and (4), we obtain

$$D_q\phi(z) = 1 + \sum_{v=2}^{\infty} [v]_q a_v z^{v-1}$$

and

$$D_q\lambda(z) = \sum_{v=1}^{\infty} [v]_q b_v z^{v-1}.$$

For more details, we can refer to reference [16].

A harmonic function $f = \phi + \bar{\lambda}$ expressed by (1) is said to be $q-$harmonic, locally univalent, and sense preserving in $\mathbb{U}$ if and only if second dilatation w_q fulfills

$$|w_q(z)| = \left| \frac{D_q\phi(z)}{D_q\lambda(z)} \right| < 1 \quad (z \in \mathbb{U}).$$

Let us indicate this class by S_{H_q}. As $q \to 1^-$, S_{H_q} reduces to the class SH (see [2]).

The concept of neutrosophic theory, a new branch of philosophy as a generalization for the fuzzy logic, and also a generalization of the intrinsic fuzzy logic, introduced by Smarandache [25]. This generalization provided a new foundation for handling with the issues of indeterminate data. The usage of neutrosophic crisp sets theory by means of the classical probability distributions, particularly, Poisson, Exponential and uniform distributions provide a new pathway to deal with issues that follow the classical distribution, and also contain data not specified accurately.

A discrete random variable Y is said to have a neutrosophic Poisson distribution if it has a probability mass function

$$P(Y=v)=m_N^v\frac{e^{-m_N}}{v!},\quad v=0,1,2,\cdots$$

and m_N is the parameter of the distribution. Further,

$$NE(Y)=NV(Y)=m_N$$

where $N=d+I$ is a neutrosophic number [25].

Recently, Alhabib et al. [4] studied a power series of neutrosophic Poisson, which was further exploited in [5] via coefficient inequalities defined by the power series

$$K(m_N,z)=z+\sum_{v=2}^{\infty}\frac{m_N^{v-1}}{(v-1)!}e^{-m_N}z^v\quad (z\in\mathbb{U})$$

and by ratio test, the radius of convergence of the above series was shown to be infinite.

Now for $m_{N1},m_{N2}>0$, we establish the operator $\Theta(m_{N1},m_{N2})$ for $f\in S_H$ as

$$\begin{aligned}Y(f) &= Y(m_{N1},m_{N2})f(z)\\ &= K(m_{N1},z)*\phi(z)+\overline{K(m_{N2},z)*\lambda(z)}\\ &= \Phi(z)+\overline{\Omega(z)},\end{aligned}$$

where

$$\Phi(z)=z+\sum_{v=2}^{\infty}\frac{m_{N1}^{v-1}}{(v-1)!}e^{-m_{N1}}a_vz^v,\quad \Omega(z)=b_1z+\sum_{v=2}^{\infty}\frac{m_{N2}^{v-1}}{(v-1)!}e^{-m_{N2}}b_vz^v \tag{5}$$

for $f=\phi+\overline{\lambda}\in H$.

The present investigation builds on the foundational works of Smarandache and Khalid [25], Oladipo [21], and the recent contributions by Frasin and Lupas [14]. This study explores the innovative application of neutrosophic Poisson distribution series, augmented with an artificial neural network (Sigmoid function), to analyze harmonic data. This approach effectively handles both determinate (accurate) and indeterminate (inaccurate) data, offering a robust method for dealing with uncertainty in mathematical and statistical analyses.

We define and study the class $H_q^\alpha(\theta,\gamma(s),\Psi)$ of the function of the form (1) that fulfills the condition

$$\Re\left\{\frac{\alpha(1+e^{i\theta})}{2\gamma(s)}[z(D_q\phi(z))'+z(D_q\lambda(z))']+[D_q\phi(z)+D_q\lambda(z)]\right\}>\Psi \tag{6}$$

for $\alpha\geq 0,\ 0\leq\Psi<1,\ -\pi<\theta\leq\pi,\ q\in(0,1)$ and $\gamma(s)=\frac{2}{1+e^{-s}},\ s\geq 0$ (real) is modified Sigmoid functions studied in [6], [19], [20].

By suitably specializing the parameters, the class $H_q^\alpha(\theta,\gamma(s),\Psi)$ reduces to the various subclasses of harmonic univalent functions:

(i) $H_q^\alpha(\theta,\gamma(s),\Psi)=H^\alpha(\theta,\gamma(s),\Psi)$ as $q\to 1^-$.

(ii) $H_q^\alpha(\theta,\gamma(s),\Psi)=H^\alpha(0,\gamma(0),\Psi)=H^\alpha(\Psi)$ as $q\to 1^-$ [28].
(iii) $H_q^\alpha(0,0,\gamma(0))=H^\alpha$ as $q\to 1^-$ [10].
(iv) $H_q^\alpha(\theta,\gamma(0),\Psi)=H_q^\alpha(\theta,\Psi)$.
(v) $H_q^\alpha(0,\gamma(s),\Psi)=H_q^\alpha(\gamma(s),\Psi)$.

The aim of this paper is to present some inclusion properties of the harmonic class $H_q^\alpha(\theta,\gamma(s),\Psi)$ and its related classes.

2. Preliminary Lemmas

Before presenting our main outcomes, we need to state some lemmas that will be used in the sequel.

Lemma 1. *A function f of the form (1) belongs to class $H_q^\alpha(\theta,\gamma(s),\Psi)$ if and only if*

$$\begin{aligned}&\textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]|a_v|\\ &+\textstyle\sum_{v=1}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]|b_v|\le 2\gamma(s)(1-\Psi).\end{aligned}\tag{7}$$

Proof. Assume $f\in H_q^\alpha(\theta,\gamma(s),\Psi))$. From (6), we note that

$$\Re\left\{1-\textstyle\sum_{v=2}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|a_v|\,z^{v-1}+\sum_{v=1}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|b_v|\,z^{v-1}\right\}>\Psi.$$

Choosing z to be real and lettting $z\to 1^-$, we arrive at

$$1-\sum_{v=2}^{\infty}[v]_q\left[\tfrac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|a_v|+\sum_{v=1}^{\infty}[v]_q\left[\tfrac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|b_v|>\Psi,$$

which is equivalent to (7). Conversely, assume that (7) is true, then

$$\begin{aligned}&\left|\frac{\alpha(1+e^{i\theta})}{2\gamma(s)}\left[z(D_q\phi(z))'+z(D_q\lambda(z))'\right]+\left[D_q\phi(z)+D_q\lambda(z)\right]\right|\\ &<\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]|a_v|\\ &+\sum_{v=1}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]|b_v|\\ &\le 2\gamma(s)(1-\Psi),\end{aligned}$$

which implies that $f\in H_q^\alpha(\theta,\gamma(s),\Psi)$.

When $f\in H_q^\alpha(\theta,\gamma(s),\Psi)$, then

$$|a_v|\le\frac{2\gamma(s)(1-\Psi)}{[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]},\quad v\ge 2$$

and

$$|b_v|\le\frac{2\gamma(s)(1-\Psi)}{[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]},\quad v\ge 1.$$

As $q \to 1^-$, we arrive at

$$|a_v| \leq \frac{2\gamma(s)(1-\Psi)}{v\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \geq 2$$

and

$$|b_v| \leq \frac{2\gamma(s)(1-\Psi)}{v\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \geq 1.$$

□

Lemma 2. *A function f of the form (2) belongs to class $TN_H(\nu)$ if and only if*

$$\sum_{v=2}^{\infty} v\,|a_v| + \sum_{v=1}^{\infty} v\,|a_v| \leq 1-\nu.$$

Then

$$|a_v| \leq \frac{1-\nu}{v}, \quad v \geq 2, \quad |b_v| \leq \frac{1-\nu}{v}, \ v \geq 1.$$

Lemma 3. *A function f of the form (2) belongs to class $TR_H(\nu)$ if and only if*

$$\sum_{v=2}^{\infty} v^2\,|a_v| + \sum_{v=1}^{\infty} v^2\,|a_v| \leq 1-\nu.$$

Then

$$|a_v| \leq \frac{1-\nu}{v^2}, \quad v \geq 2, \quad |b_v| \leq \frac{1-\nu}{v^2}, \quad v \geq 1.$$

Lemma 4. *Consider $f \in S_H^*$, where the function f is of the form (1) and $b_1 = 0$, then*

$$|a_v| \leq \frac{(2v+1)(v+1)}{6}, \qquad |b_v| \leq \frac{(2v-1)(v-1)}{6}.$$

Lemma 5. *Consider $f \in K_H$, where the function f is of the form (1) and $b_1 = 0$, then*

$$|a_v| \leq \frac{(v+1)}{2}, \qquad |b_v| \leq \frac{(v-1)}{2}.$$

For easy handling throughout the sequel, we designate the notations:

$$\textstyle\sum_{v=2}^{\infty} \frac{m_{N1}^{v-1}}{(v-1)!} = e^{m_{N1}} - 1, \qquad \sum_{v=2}^{\infty} \frac{m_{N2}^{v-1}}{(v-1)!} = e^{m_{N2}} - 1,$$

$$\textstyle\sum_{v=j}^{\infty} \frac{m_{N1}^{v-1}}{(v-j)!} = m_{N1}^{j-1} e^{m_{N1}}, \quad \sum_{v=j}^{\infty} \frac{m_{N2}^{v-1}}{(v-j)!} = m_{N2}^{j-1} e^{m_{N2}}, \ (j \geq 2).$$

3. Main Results

Theorem 1. *Assume* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1,\ q \in (0,1)$. *If*

$$\begin{aligned} &2\alpha(1+\cos\theta)(m_{N1}^4 + m_{N2}^4) + [21\alpha(1+\cos\theta) + 4\gamma(s)]m_{N1}^3 + 5[4\alpha(1+\cos\theta) + 5\gamma(s)]m_{N1}^2 \\ &+6[5\alpha(1+\cos\theta) + 8\gamma(s)]m_{N1} + 8\gamma(s)[1 - e^{-m_{N1}}] + [15\alpha(1+\cos\theta) + 4\gamma(s)]m_{N2}^3 \\ &+6[4\alpha(1+\cos\theta) + 3\gamma(s)]m_{N2}^2 + 6[\alpha(1+\cos\theta) + 2\gamma(s)]m_{N2} \leq 12\gamma(s)(1-\Psi), \end{aligned} \tag{8}$$

then $Y(S_H^*) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f = \phi + \bar{\lambda} \in S_H^*$ such that ϕ and λ are represented by (1) with $b_1 = 0$. We aim to establish that $Y(f) = \Phi + \Omega \in H_q^\alpha(\theta, \gamma(s), \Psi)$, where Φ and Ω are analytic functions in $\mathbb{U}$ as shown by (5) with $b_1 = 0$. According to Lemma 1, we need to show that

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned} \Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \textstyle\sum_{v=2}^\infty [v]_q \left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right] \left|\frac{m_{N1}^{v-1} e^{-m_{N1}}}{(v-1)!} a_v\right| \\ &+ \textstyle\sum_{v=2}^\infty [v]_q \left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right] \left|\frac{m_{N2}^{v-1} e^{-m_{N2}}}{(v-1)!} b_v\right|. \end{aligned}$$

Applying the inequalities from Lemma 1 and letting $q \to 1^-$, we obtain

$$\begin{aligned} &\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \\ &\leq \tfrac{1}{6}\left[\textstyle\sum_{v=2}^\infty v(2v+1)(v+1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1} e^{-m_{N1}}}{(v-1)!}\right|\right] \\ &+\tfrac{1}{6}\left[\textstyle\sum_{v=2}^\infty v(2v-1)(v-1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1} e^{-m_{N2}}}{(v-1)!}\right|\right] \\ &= \tfrac{1}{6}\left[\textstyle\sum_{v=2}^\infty \left\{2\alpha[1+cos\theta)v^4 + [4\gamma(s) + \alpha(1+cos\theta)]v^3 + Q_1\right\} \frac{m_{N1}^{v-1} e^{-m_{N1}}}{(v-1)!}\right] \\ &+\tfrac{1}{6}\left[\textstyle\sum_{v=2}^\infty \left\{2\alpha[1+cos\theta)v^4 + [4\gamma(s) - 5\alpha(1+cos\theta)]v^3 + Q_1\right\} \frac{m_{N2}^{v-1} e^{-m_{N2}}}{(v-1)!}\right] \end{aligned} \tag{9}$$

where

$$Q_1 = 2[2\gamma(s) - \alpha(1+cos\theta)]v^2 + [2\gamma(s) - \alpha(1+cos\theta)]v$$

and

$$Q_2 = 2[2\alpha(1+cos\theta) - 3\gamma(s)]v^2 + [2\gamma(s) - \alpha(1+cos\theta)]v.$$

Setting

$$v = (v-1)+1, \qquad v^2 = (v-1)(v-2) + 3(v-1) + 1,$$
$$v^3 = (v-1)(v-2)(v-3) + 6(v-1)(v-2) + 7(v-1) + 1,$$

$v^4=(v-1)(v-2)(v-3)(v-4)+10(v-1)(v-2)(v-3)+25(v-1)(v-2)+15(v-1)+1$
and using these equalities in (9), we can obtain

$$\Gamma_q(m_{N1},m_{N2},\gamma(s),\theta)$$

$$\leq \frac{1}{6}\Bigg[\sum_{v=2}^{\infty}\{2\alpha(1+cos\theta)(v-1)(v-2)(v-3)(v-4)+Q_3+Q_4+Q_5$$

$$+12\gamma(s)\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\Bigg]+\frac{1}{6}\Bigg[\sum_{v=2}^{\infty}\{2\alpha(1+cos\theta)(v-1)(v-2)(v-3)(v-4)$$

$$+Q_6+Q_7+Q_8\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\Bigg],$$

where

$$Q_3=[21\alpha(1+cos\theta)+4\gamma(s)](v-1)(v-2)(v-3),$$

$$Q_4=5[9\alpha(1+cos\theta)+5\gamma(s)](v-1)(v-2),$$

$$Q_5=6[5\alpha(1+cos\theta)+8\gamma(s)](v-1),$$

$$Q_6=[15\alpha(1+cos\theta)+4\gamma(s)](v-1)(v-2)(v-3),$$

$$Q_7=6[4\alpha(1+cos\theta)+3\gamma(s)](v-1)(v-2),\quad Q_8=6[\alpha(1+cos\theta)+2\gamma(s)](v-1).$$

Thus

$$\Gamma_q(m_{N1},m_{N2},\gamma(s),\theta)\leq\frac{1}{6}\Bigg[2\alpha(1+cos\theta)\sum_{v=5}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-5)!}$$

$$+[21\alpha(1+cos\theta)+4\gamma(s)]\sum_{v=4}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-4)!}+12\sum_{v=1}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\Bigg]$$

$$+\frac{1}{6}\Bigg[5[9\alpha(1+cos\theta)+5\gamma(s)]\sum_{v=3}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-3)!}+6[5\alpha(1+cos\theta)+8\gamma(s)]\sum_{v=2}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-2)!}\Bigg]$$

$$+\frac{1}{6}\Bigg[2\alpha(1+cos\theta)\sum_{v=5}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N2}}}{(v-5)!}+[15\alpha(1+cos\theta)+4\gamma(s)]\sum_{v=4}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-4)!}\Bigg]$$

$$+\frac{1}{6}\Bigg[6[4\alpha(1+cos\theta)+3\gamma(s)]\sum_{v=3}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-3)!}+6[\alpha(1+cos\theta)+2\gamma(s)]\sum_{v=2}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-2)!}\Bigg]$$

$$=\frac{1}{6}\left[2\alpha(1+cos\theta)(m_{N1}^4+m_{N2}^4)+[21\alpha(1+cos\theta)+4\gamma(s)]m_{N1}^3+5[4\alpha(1+cos\theta)+5\gamma(s)]m_{N1}^2\right]$$

$$+\frac{1}{6}\left[6[5\alpha(1+cos\theta)+8\gamma(s)]m_{N1}+8\gamma(s)[1-e^{-m_{N1}}]+[15\alpha(1+cos\theta)+4\gamma(s)]m_{N2}^3\right]$$

$$+\frac{1}{6}\left[6[4\alpha(1+cos\theta)+3\gamma(s)]m_{N2}^2+6[\alpha(1+cos\theta)+2\gamma(s)]m_{N2}\right].$$

This expression is bounded by $2\gamma(s)(1-\Psi)$ if condition (8) holds. □

Theorem 2. *Let* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$\alpha(1+\cos\theta)[m_{N1}^3 + m_{N2}^3] + 2[3\alpha(1+\cos\theta)+\gamma(s)][m_{N1}^2 + m_{N2}^2]$$
$$+6[\alpha(1+\cos\theta)+\gamma(s)]m_{N1} + 2[\alpha(1+\cos\theta)+3\gamma(s)]m_{N2} \tag{10}$$
$$+2\gamma(s)[2 - e^{-m_{N1}} - e^{-m_{N2}}] \leq 4\gamma(s)(1-\Psi),$$

then $Y(K_H) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f = \phi + \bar{\lambda} \in K_H$ such that w and φ are given by (1) with $b_1 = 0$. We need to establish that $Y(f) = \Phi + \Omega \in H_q^\alpha(\theta, \gamma(s), \Psi)$, where Φ and Ω are analytic functions in $\mathbb{U}$ as shown by (5) with $b_1 = 0$. According to Lemma 1, we must show that

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right| \\ &+ \textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.\end{aligned}$$

Applying Lemma 5 and the condition $q \to 1^-$, we obtain

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq \frac{1}{2}\left[\sum_{v=2}^{\infty} v(v+1)\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right|\right] \\ &+ \frac{1}{2}\left[\sum_{v=2}^{\infty} v(v-1)\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right|\right] \\ &= \frac{1}{2}\left[\sum_{v=2}^{\infty}\left[\alpha(1+cos\theta)v^3 + 2\gamma(s)v^2 - \alpha(1+cos\theta)v\right]\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right] \\ &+ \frac{1}{2}\Bigg[\sum_{v=2}^{\infty}[\alpha(1+cos\theta)v^3 + 2(\gamma(s) - \alpha(1+cos\theta)v^2 + \alpha(1+cos\theta)v] \\ &\times \frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\Bigg].\end{aligned}$$

Next, we have

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta)$$
$$\leq \frac{1}{2}\left[\sum_{v=2}^{\infty}\left\{\alpha(1+cos\theta)(v-1)(v-2)(v-3) + K_1 + Q_2 + 2\gamma(s)\right\}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right]$$
$$+ \frac{1}{2}\left[\sum_{v=2}^{\infty}\left\{\alpha(1+cos\theta)(v-1)(v-2)(v-3) + K_3 + Q_4 + 2\gamma(s)\right\}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right],$$

where

$$K_1 = 2[3\alpha(1+cos\theta)+\gamma(s)](v-1)(v-2), \quad K_2 = 6[\alpha(1+cos\theta)+\gamma(s)](v-1),$$

$$K_3 = 2[2\alpha(1+cos\theta)+\gamma(s)](v-1)(v-2), \quad K_4 = 2[\alpha(1+cos\theta)+3\gamma(s)](v-1).$$

Thus

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq \frac{1}{2}\big[\alpha(1+cos\theta)[m_{N1}^3+m_{N2}^3]+2[3\alpha(1+cos\theta)+\gamma(s)]\\ &\times [m_{N1}^2+m_{N2}^2]\big]+\frac{1}{2}\Big[6[\alpha(1+cos\theta)+\gamma(s)]m_{N1}\\ &+ 2[\alpha(1+cos\theta)+3\gamma(s)]m_{N2}+2\gamma(s)[2-e^{-m_{N1}}-e^{-m_{N2}}]\Big].\end{aligned}$$

The last relation is bounded by $2\gamma(s)(1-\Psi)$ provided (10) holds. □

Theorem 3. *Assume $m_{N1}, m_{N2} > 0$ and $0 \leq \Psi < 1$. If*

$$(1-\nu)[\alpha(1+\cos\theta)(m_{N1}+m_{N2})+2\gamma(s)(2-e^{-m_{N1}}-e^{-m_{N2}})]+b_1 \leq 2\gamma(s)(1-\Psi),$$

then $Y(TN_H(\nu)) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f \in TN_H(\nu)$. In view of Lemma 1, we need to establish that

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned}\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right|+b_1\\ &+\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.\end{aligned}$$

Application of Lemma 2 and the condition $q \to 1^-$ yields

$$\begin{aligned}\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq (1-\nu)\left[\sum_{v=2}^{\infty}\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right]\\ &+(1-\nu)\left[\sum_{v=2}^{\infty}\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right]+b_1\\ &= (1-\nu)[\alpha(1+\cos\theta)(m_{N1}+m_{N2})+2\gamma(s)(2-e^{-m_{N1}}-e^{-m_{N2}})]\\ &+b_1\\ &\leq 2\gamma(s)(1-\Psi),\end{aligned}$$

which completes the proof. □

Theorem 4. *Assume* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$(1-\nu)\left[\alpha(1+cos\theta)(2-e^{-m_{N1}}-e^{-m_{N2}})+\tfrac{1}{m_{N1}}(1-e^{-m_{N1}}-m_{N1}e^{-m_{N1}})\right]$$
$$+(1-\nu)\left[\tfrac{1}{m_{N2}}(1-e^{-m_{N2}}-m_{N2}e^{-m_{N2}})\right] \leq 2\gamma(s)(1-\Psi),$$

then $Y(TR_H(\nu)) \subset H_q^{\alpha}(\theta, \gamma(s), \Psi)$.

Proof. Assume $f \in TR_H(\nu)$. In view of Lemma 1, we need to establish that

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) = \sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right|$$
$$+ b_1 + \sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.$$

From Lemma 3, we have

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq (1-\nu)\left[\sum_{v=2}^{\infty}\left[\alpha(1+\cos\theta)+\frac{2\gamma(s)-\alpha(1+\cos\theta)}{v}\right]\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right]$$

$$+ (1-\nu)\left[\sum_{v=2}^{\infty}\left[\alpha(1+\cos\theta)+\frac{2\gamma(s)-\alpha(1+\cos\theta)}{v}\right]\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right] + b_1$$
$$= (1-\nu)\left[\alpha(1+\cos\theta)(2-e^{-m_{N1}}-e^{-m_{N2}})+\frac{1}{m_{N1}}(1-e^{-m_{N1}}-m_{N1}e^{-m_{N1}})\right]$$
$$+ (1-\nu)\left[\frac{1}{m_{N2}}(1-e^{-m_{N2}}-m_{N2}e^{-m_{N2}})\right] \leq 2\gamma(s)(1-\Psi).$$

□

Theorem 5. *Let* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$e^{-m_{N1}} + e^{-m_{N2}} \leq 1 + \frac{b_1}{2\gamma(s)(1-\Psi)},$$

then $Y(H_q^{\alpha}(\theta, \gamma(s), \Psi)) \subset H_q^{\alpha}(\theta, \gamma(s), \Psi)$.

Proof. From Lemma 1, we established that

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta)$$

$$\leq 2\gamma(s)(1-\Psi)\left[\sum_{v=2}^{\infty} \frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!} + \sum_{v=2}^{\infty} \frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right] + b_1$$

$$= 2\gamma(s)(1-\Psi)[2 - e^{-m_{N1}} - e^{-m_{N2}}] + b_1$$

$$= 2\gamma(s)(1-\Psi)[2 - e^{-m_{N1}} - e^{-m_{N2}}] + b_1 \leq 2\gamma(s)(1-\Psi).$$

□

4. Conclusion

In this paper, we have established the inclusion relations for the harmonic analytic function class $H_q^\alpha(\theta, \gamma(s), \Psi)$ by applying convolution operators associated with the neutrosophic distribution series and incorporating the Sigmoid activation function. Our results extend the existing body of knowledge on analytic univalent functions, which previously encompassed distributions such as Binomial, Generalized Discrete Probability, Geometric, Mittag-Leffler, Pascal, and Poisson.

The innovative approach of utilizing the Sigmoid function within the framework of neutrosophic distribution series has demonstrated the potential to handle both accurate (determinate) and inaccurate (indeterminate) data effectively. This dual capability is particularly significant in applications where data uncertainty and variability are prevalent.

Our findings contribute to the broader understanding of harmonic analytic functions and offer new pathways for future research in the domain of mathematical analysis, particularly in the context of univalent functions and their applications. Further exploration may involve extending these results to other classes of functions and distributions, as well as investigating the practical implications of these theoretical advancements in real-world scenarios.

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