ESKİŞEHİR TECHNICAL UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY
A- APPLIED SCIENCES AND ENGINEERING

Estuscience – Se, 2024, 25 [3] pp. 471-484, DOI: 10.18038/estubtda.1485966

RESEARCH ARTICLE

NUMERICAL SOLUTIONS OF BOUSSINESQ TYPE EQUATIONS BY MESHLESS METHODS

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Abstract

In this paper, two different meshfree method with radial basis functions (RBFs) is proposed to solve Boussinesq-type (Bq) equations. The basic conservative properties of the equation are investigated by computing the numerical values of the motion's invariants. The accuracy of the method is tested using computational tests to simulate solitary waves in terms of L_∞ error norm. The outcomes are contrasted with analytical solution and a few other earlier studies in the literature. The results show that meshless methods are very effective and accurate.

Keywords

Radial basis function,
Collocation method,
Method of lines,
Soliton,
Boussinesq equation

Time Scale of Article

Received :19 May 2024
Accepted : 15 Ağustos 2024
Online date :30 September 2024

1. INTRODUCTION

The Boussinesq equation is a well-known partial differential equation used in fluid mechanics to model the behavior of shallow water waves. It was first introduced by the French mathematician Joseph Boussinesq in the late 19th century [1]. The equation is a simplified version of the Navier-Stokes equations, and it is often used as an approximation for modeling wave propagation in a wide range of applications. The Boussinesq equation:

$$u_{tt} = u_{xx} + (u^2)_{xx} + qu_{xxxx} \tag{1.1}$$

where $u = u(x, t)$ is a sufficiently differentiable function and q is a real constant. When $q = -1$ Boussinesq equation (Bq) is called "good" or "well-posed" (GBq) and when $q = 1$ is called "bad" or "ill-posed" (BBq). An improved version of Boussinesq equation (Bq) is Improved Boussinesq (IBq) equation given as follows:

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxtt} \tag{1.2}$$

The Boussinesq type equatios has been solved by using a variety of numerical methods. In [2], Manoranjan et al. used Petrov-Galerkin method to get numerical solution of (GBqE). Then, Manoranjan et al. [3] examined the interaction of solitary waves and underlined three essential characteristics of

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solitary waves. Bratsos [4] used method of lines approach to get solution of Bq equation. Pani and Saranga [5] employed the finite element Galerkin method, while Ortega and Serna [6] developed the finite difference method to solve the (GBqE). The Boussinesq equation was solved numerically by El-Zoheiry [7] using an implicit finite difference scheme, and Wazwaz [8] suggested the Adomian decomposition method. Ismail and Bratsos [9] predictor-corrector scheme to obtain numerical solutions to the (GBq) and (BBq) equations. Additionally, the third order implicit finite difference method has been used to solve the (GBqE) by Bratsos [10, 11]. He used modified Predictor-Corrector and Predictor-Corrector techniques to solve the equation. For the numerical solution of the Boussinesq equation, Dehghan and Salehi [12] combined the boundary knot method and meshless analog equation approach. In addition, Ucar et al. [13] applied Galerkin FEM to the (BqE) using cubic B-spline basis, and Ismail and Mosally [14] created a fourth-order FDM for approximating solutions to the (GBqE). Kırlı and Irk [15] gave numerical solutions of (GBqE) by using the quartic B-spline Galerkin method. Zhijian [16] investigated the existence and uniqueness of solutions, and non-existence of global solutions to the initial-boundary value problem of a generalized IBq equation both locally and globally in time. In order to discretize the nonlinear partial differential equation in space, Lin et al. [17] used the finite element method with linear B-spline basis functions. They then developed a second-order system using only ordinary derivatives to solve a class of initial-boundary value problems for the improved Boussinesq equation. Iskandar and Jain [18] used numerical analysis to examine the dynamical behavior of the IBq equation. Irk and Dag [19] used two finite difference schemes and two finite element approaches, based on the second- and third-order temporal discretization, to achieve numerical simulations of the improved Boussinesq equation.

The objective of the current work is to use meshless radial basis functions collocation method and meshless kernel based method of lines to obtain the numerical solution of Boussinesq type equations. Thus, to obtain the numerical solution of the Boussinesq type equations, radial basis functions in the mesh free approach will be employed. Different types of radial basis functions can be found in literature. We'll employ the widely used radial basis functions.

2. THE MESHLESS RBF COLLOCATION METHOD

First, let us introduce the Meshless RBF Collocation Method that we will use in this section. This method is a meshless method (MRBFCM) and was first used by Kansa [20, 21]. To apply the method, let us approximate the function $u(x,t)$ in equations (1.1) and (1.2) by a linear combination of radial basis functions:

$$u(x) = \sum_{j=1}^{N} \lambda_j \phi_i(r_j), \quad i = 1,2,\dots,N \tag{2.1}$$

In equation (2.1) $\{\lambda_j\}_j^N$ are the unknown coefficients to be determined and the $\phi_i(r_j)$ are radial basis functions. The formulas of the used basis functions are defined as follows:

Multiquadric (MQ)	$\phi(r_j) = \sqrt{(\varepsilon r_j)^2 + 1}$
Gaussian (GA)	$\phi(r_j) = exp(-r_j^2/\varepsilon^2)$

where ε is a shape parameter that the method automatically calculates for each kernel matrix that is used and $r_j = |x - x_j|$ represents the Euclidean norm between x and x_j. Wendland's functions [22] are a class of compactly supported radial basis function and have the following general form:

$$\phi_{l,k}(r) = (1-r)_+^n p_{l,k}(r)$$

with following conditions:

$$(1-r)_+^n = \begin{cases} (1-r)^n, & \text{if} 0 \le r < 1 \\ 0, & \text{if} r \ge 1 \end{cases}$$

where p is a prescribed polynomial for $k \ge 1$ and l is the dimension number. In our calculations, following form of Wendland's function is used:

$$\phi_{7,5}(r) = (1-r)_+^{12}(9 + 108r + 566r^2 + 1644r^3 + 2697r^4 + 2048r^5)$$

For ease of notation in tables $\phi_{l,k}(r)$ will be used as W.

We write meshless methods the following form equation (1.1) by using $u_t = v$ then we get

$$\begin{aligned} u_t &= v, \\ v_t &= u_{xx} + 2((u_x)^2 + u_{xx}u) + qu_{xxxx}. \end{aligned} \tag{2.2}$$

System (2.2) is discretized by using a forward difference rule for u_t and a Crank–Nicolson scheme for u between successive time levels as follows

$$\frac{u^{n+1}-u^n}{\Delta t} = \frac{v^{n+1}+v^n}{2},$$

$$\frac{v^{n+1}-v^n}{\Delta t} = \frac{u_{xx}^{n+1}+u_{xx}^n}{2} + 2\frac{\left[(u_x^2)^{n+1}+(u_x^2)^n\right]}{2} + 2\frac{\left[(uu_{xx})^{n+1}+(uu_{xx})^n\right]}{2} + q\frac{u_{xxxx}^{n+1}+u_{xxxx}^n}{2}$$

where u_x^2 and uu_{xx} are nonlinear terms and by using Taylor's formula we obtain their linear forms [23],

$$(u_x u_x)^{n+1} = u_x^n u_x^{n+1} + u_x^n u_x^{n+1} - u_x^n u_x^n$$

and

$$(uu_{xx})^{n+1} = u_{xx}^n u^{n+1} + u^n u_{xx}^{n+1} - u^n u_{xx}^n.$$

Thus, we get following linearized difference equation by substituting these linear forms:

$$u^{n+1} - \frac{\Delta t}{2} v^{n+1} = u^n + \frac{\Delta t}{2} v^n,$$

$$v^{n+1} - \frac{\Delta t}{2} u_{xx}^{n+1} - 2\Delta t u_x^{n+1} u_x^n - \Delta t u^n u_{xx}^{n+1} - \Delta t u^{n+1} u_{xx}^n - \frac{q\Delta t}{2} u_{xxxx}^{n+1} = v^n + \frac{\Delta t}{2} u_{xx}^n + \frac{q\Delta t}{2} u_{xxxx}^n$$

For each time iteration, by substituting

$$u(x) = \sum_{j=1}^{N} \lambda_j \phi_i(r_j), \quad i = 1,2,\ldots,N \tag{2.3}$$

and

$$v(x) = \sum_{j=1}^{N} \gamma_j \phi_i(r_j), \quad i = 1,2,\ldots,N \tag{2.4}$$

following form of linear equations system is obtained:

$$\begin{aligned}
&\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i(r_j) - \frac{\Delta t}{2}\sum_{j=1}^{N} \gamma_j^{n+1}\phi_i(r_j) = \sum_{j=1}^{N} \lambda_j^{n}\phi_i(r_j) + \frac{\Delta t}{2}\sum_{j=1}^{N} \gamma_j^{n}\phi_i(r_j),\\
&\sum_{j=1}^{N} \gamma_j^{n+1}\phi_i(r_j) - \frac{\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i''(r_j) - 2\Delta t\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i'(r_j)\sum_{j=1}^{N} \lambda_j^{n}\phi_i'(r_j)\\
&-\Delta t\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i''(r_j)\sum_{j=1}^{N} \lambda_j^{n}\phi_i(r_j) - \Delta t\sum_{j=1}^{N} \lambda_j^{n}\phi_i''(r_j)\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i(r_j) - \frac{q\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i''''(r_j)\\
&= \sum_{j=1}^{N} \gamma_j^{n}\phi_i(r_j) + \frac{\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n}\phi_i''(r_j) + \frac{q\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n}\phi_i''''(r_j)
\end{aligned} \tag{2.5}$$

When similar steps are applied for equation (1.2) we get

$$\begin{aligned}
&\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i(r_j) - \frac{\Delta t}{2}\sum_{j=1}^{N} \gamma_j^{n+1}\phi_i(r_j) = \sum_{j=1}^{N} \lambda_j^{n}\phi_i(r_j) + \frac{\Delta t}{2}\sum_{j=1}^{N} \gamma_j^{n}\phi_i(r_j),\\
&\sum_{j=1}^{N} \gamma_j^{n+1}\phi_i(r_j) - \frac{\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i''(r_j) - 2\Delta t\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i'(r_j)\sum_{j=1}^{N} \lambda_j^{n}\phi_i'(r_j)\\
&-\Delta t\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i''(r_j)\sum_{j=1}^{N} \lambda_j^{n}\phi_i(r_j) - \Delta t\sum_{j=1}^{N} \lambda_j^{n}\phi_i''(r_j)\sum_{j=1}^{N} \lambda_j^{n+1}\phi_i(r_j) + \sum_{j=1}^{N} \gamma_j^{n+1}\phi_i''(r_j)\\
&= \sum_{j=1}^{N} \gamma_j^{n}\phi_i(r_j) + \frac{\Delta t}{2}\sum_{j=1}^{N} \lambda_j^{n}\phi_i''(r_j) + \sum_{j=1}^{N} \gamma_j^{n}\phi_i''(r_j).
\end{aligned} \tag{2.6}$$

By solving these systems at each time step values of λ and γ are obtained. Substituting computed values of λ_j and γ_j in (2.3) and (2.4) numerical values of $u(x,t)$ and $v(x,t)$ are evaluated.

3. THE MESHLESS KERNEL BASED METHOD OF LINES

The meshless kernel based method of lines (MKBMOL), a second numerical method, will be utilized to find the numerical solution to the Boussinesq type equations. Since this meshles approach generates ordinary differential equations, temporal discretization is not required, and the nonlinear partial differential equation will not be artificially linearized as in the first method. where the kernel function is a radial basis function.

$$u(x,t) = \sum_{j=1}^{N} \alpha_j(t)\phi_i(x), v(x,t) = \sum_{j=1}^{N} \beta_j(t)\phi_i(x)$$

where $\alpha_j(t), \beta_j(t)$ are unknown time-dependent functions to be determined each time level as column vectors and $\phi_i(x)$ are defined by any well-known radial basis functions. Derivatives in Equation (1.1) with respect to time and space variables can be described as:

$$\begin{aligned}
&u_t(x,t) = \sum_{j=1}^{N} \alpha_j'(t)\phi_i(x), \quad v_t(x,t) = \sum_{j=1}^{N} \beta_j'(t)\phi_i(x)\\
&u_x(x,t) = \sum_{j=1}^{N} \alpha_j(t)\phi_i'(x), \quad u_{xx}(x,t) = \sum_{j=1}^{N} \alpha_j(t)\phi_i''(x)\\
&u_{xxxx}(x,t) = \sum_{j=1}^{N} \alpha_j(t)\phi_i''''(x)
\end{aligned} \tag{3.1}$$

By substituting (3.1) and its derivatives in the main (1.1) we get:

$$\sum_{j=1}^{N} \alpha_j'(t)\phi_i(x) = \sum_{j=1}^{N} \beta_j(t)\phi_i(x),$$
$$\sum_{j=1}^{N} \beta_j'(t)\phi_i(x) = \sum_{j=1}^{N} \alpha_j(t)\phi_i''(x) + 2\left(\sum_{j=1}^{N} \alpha_j(t)\phi_i'(x)\right)^2 + 2\sum_{j=1}^{N} \alpha_j(t)\phi_i''(x)\sum_{j=1}^{N} \alpha_j(t)\phi_i(x)$$
$$+q\sum_{j=1}^{N} \alpha_j(t)\phi_i''''(x) \tag{3.2}$$

The system (3.2) can be written with matlab notations

$$V * \alpha'(t) = V * \beta(t),$$
$$V * \beta'(t) = V'' * \alpha(t) + 2(V' * \alpha(t))^2 + 2(V'' * \alpha(t))(V * \alpha(t)) + qV'''' * \alpha(t) \tag{3.3}$$

where the $*$ is the pointwise product. Since matrice V is invertible we get:

$$\alpha'(t) = V^{-1} * (V * \beta(t)),$$
$$\beta'(t) = V^{-1} * (V'' * \alpha(t) + 2(V' * \alpha(t))^2 + 2(V'' * \alpha(t))(V * \alpha(t)) + qV'''' * \alpha(t)) \tag{3.4}$$

When similar steps are applied for equation (1.2) we get

$$\alpha'(t) = V^{-1} * (V * \beta(t)),$$
$$\beta'(t) = (V - V'')^{-1} * \left(V'' * \alpha(t) + 2(V' * \alpha(t))^2 + 2(V'' * \alpha(t))(V * \alpha(t))\right) \tag{3.5}$$

where V is defined as follows:

$$V = \phi_i(x_j).$$

The MATLAB ode solver can be used to solve system 3.4 and 3.5 which are first-order differential equations. We utilized the Adams-Bashforth-Moulton approach i.e ode113 for our computations.

4. NUMERICAL EXAMPLES

In this section, we give solutions of Boussinesq type equations for proposed methods. The accuracy of the solutions are tested by maximum error L_∞

$$L_\infty = \|u - U_N\|_\infty = \max_j |u_j - U_j|$$

The order of convergence is calculated by following formula:

$$\text{order} = \frac{\log\left|\frac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}}\right|}{\log\left|\frac{\Delta t_i}{\Delta t_{i+1}}\right|}$$

Here $(L_\infty)_{\Delta t_i}$ is the error norm L_∞ for the time step Δt_i.

Since the difference between the figures for both methods is indistinguishable, we have given a single graph for each method.

4.1. Numerical Results For GBqE

We consider the GBqE which is Eq. (1.1) with $q = -1$.

4.1.1. The Single Soliton Wave

The initial conditions from the analytic solution for GBqE is as given:

$$u(x,0) = -Asech^2\left(\sqrt{\frac{A}{6}}(x-\bar{x}_0)\right)$$
$$u_t(x,0) = -2Ac\sqrt{\frac{A}{6}}sech^2\left(\sqrt{\frac{A}{6}}(x-\bar{x}_0)\right)\tanh\left(\sqrt{\frac{A}{6}}(x-\bar{x}_0)\right).$$

The exact solution of this test problem is given by

$$u(x,t) = -Asech^2\left(\sqrt{\frac{A}{6}}(x-ct-\bar{x}_0)\right)-(b+1/2)$$

and boundary conditions can be found with the help of exact solutions. In the exact solution, $c = \sqrt{1-2A/3}$ is the speed of the soliton wave and A is the amplitude of the soliton wave.

We worked over the solution domain $-80 \le x \le 100$ and the time interval $0 \le t \le 30$ for $A = 0.369, b = \frac{-1}{2},\ h = 0.5$ and $0.3, \bar{x}_0 = 0$ and $\Delta t = 0.002$. Table 1. lists the results of meshless approaches employing Multiquadric, Gaussian and Wendland functions. Meshless approaches are thought to offer superior precision. The calculated numerical results are excellent. Figure 1. shows the trajectory of a single solitary wave for GBqE. The error norm L_∞ and rate of convergence for both proposed methods are listed in Table 2. for $[-40,40]$ space interval and $A = 0.5, \bar{x}_0 = 0$, space step $h = 0.1$, and various time steps $\Delta t = 5,2,1,0.5,0.2,0.1$ at time $t = 10$. The results in Tables 1 and 2, shows proposed methods are considerable good in comparison with other methods. It is observed from Table 2 that the orders of the proposed methods converge to 2 for MRBFCM and 0.52 for MKBMOL.

Table 1. The error norm L_∞ of numerical solutions of GBqE.

h	RBF-MQ	RBF-G	RBF-W	MOL-MQ	MOL-G	MOL-W	[15]
0.5	4.2136e-04	9.0293e-08	2.1516e-04	8.0420e-04	7.4523e-10	7.5896e-05	8.4650e-08
0.3	4.3005e-04	9.0397e-08	2.5410e-04	5.3194e-04	1.6384e-09	5.6709e-04	3.6130e-09

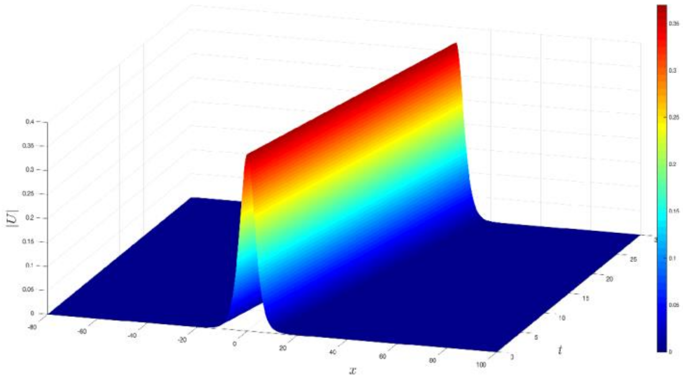

Figure 1. Motion of the single solitary wave for GBqE.

Table 2. The error norm and order of convergence.

	MRBFCM		MKBMOL	
Δt	L_∞	Order	L_∞	Order
5	1.36×10^{-1}	1.21	2.46×10^{-7}	0.40
2	4.49×10^{-2}	1.76	2.46×10^{-7}	0.52
1	1.32×10^{-2}	2.06	2.46×10^{-7}	0.52
0.5	3.17×10^{-3}	2.01	2.46×10^{-7}	0.52
0.2	5.01×10^{-4}	2.00	2.46×10^{-7}	0.52
0.1	1.25×10^{-4}		2.46×10^{-7}	

4.1.2. Interaction of Two Soliton Waves

The following initial conditions are used to study the problem of two soliton waves interaction for GBqE

$$u(x,0) = u_1(x,0) + u_2(x,0),$$
$$v(x,0) = v_1(x,0) + v_2(x,0),$$

where

$$u_i(x,0) = -A_i sech^2\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$

$$v_i(x,0) = -2A_i c_i\sqrt{\frac{A_i}{6}}sech^2\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right]\tanh\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$

$$c_i = \pm\left(1 - \frac{2A_i}{3}\right)^{\frac{1}{2}}, \quad i = 1,2.$$

By selecting the parameters $x_1^0 = -x_2^0 = -50, A_1 = A_2 = 0.369,\ c_1 = -c_2 = \sqrt{1 - 2A/3}, h = 0.1$, and $\Delta t = 0.01$ the computations are performed. These parameters produce two separate soliton waves located at $x_1^0 = -50$ and $x_2^0 = 50$, respectively. The program runs over the range $x \in [-100,100]$ up until time $t = 120$. Figure 1 shows the interaction of two soliton waves. The figure illustrates how the waves collide and appear as a single wave around $t = 60$.

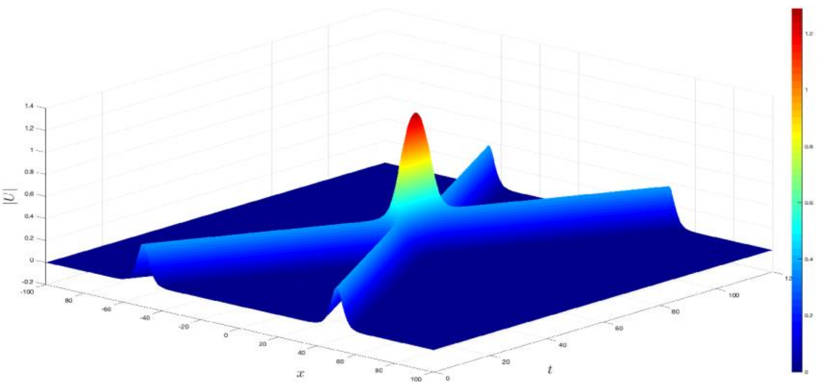

Figure 2. Interaction of two solitons for GBqE.

4.2. Numerical Results For BBqE

We consider the BBqE which is Eq. (1.1) with $q = 1$.

4.2.1. The Single Soliton Wave

The initial conditions from the analytic solution for BBqE is as given:

$$u(x,0) = Asech^2\left(\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right)$$

$$u_t(x,0) = 2Ac\sqrt{\frac{A}{6}}sech^2\left(\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right)\tanh\left(\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right)$$

For various space and time steps, numerical solutions of the single wave with amplitude $A = 0.369$ are obtained. The error norms L_∞ is computed for $A = 0.369$, $h = 4$ and $x \in [-80,100]$. The obtained results with the MQ, G, and Wendland's functions are compared with the Ref. [13] and presented in Table 2. We run the program till $t = 72$ to demonstrate the motion of a single wave. As can be observed from the Figure 2, there are secondary waves whose amplitudes increase and fluctuating with each time step.

Table 3. The error norm L_∞ of numerical solutions of BBqE.

Δt	t	RBF-MQ	RBF-G	RBF-W	MOL-MQ	MOL-G	MOL-W	[13]
0.1	36	0.041749	0.038566	0.044717	0.041761	0.038649	0.045754	0.0118
	72	0.059383	0.061030	0.065906	0.064723	0.062128	0.066795	0.0207
0.01	36	0.041454	0.038890	0.045719	0.042825	0.038649	0.045684	0.0118
	72	0.058355	0.061452	0.066536	0.060705	0.071248	0.066795	0.0207

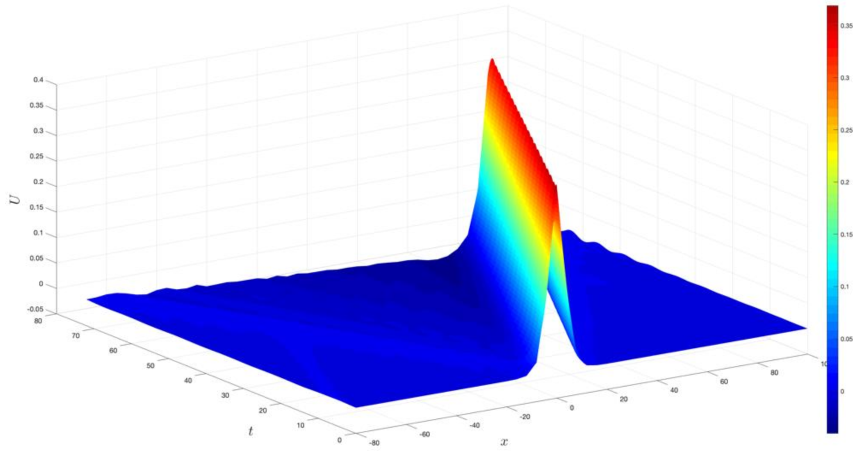

Figure 3. Motion of the single solitary wave for BBqE.

4.2.2. Interaction of Two Soliton Waves

The following initial conditions are used to study the problem of two soliton waves interaction for BBqE

$$u(x,0) = u_1(x,0) + u_2(x,0),$$
$$v(x,0) = v_1(x,0) + v_2(x,0),$$

where

$$u_i(x,0) = A_i sech^2\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$

$$v_i(x,0) = 2A_i c_i \sqrt{\frac{A_i}{6}} sech^2\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right] \tanh\left[\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$

$$c_i = \pm\left(1 - \frac{2A_i}{3}\right)^{\frac{1}{2}}, \quad i = 1,2.$$

We examine the interaction of two waves travelling toward each other by using A_i as the wave amplitudes, x_i^0 as the beginning locations, and $c_i(c_1 = c_2)$ as the wave speeds. The waves are situated at $x_1^0 = -40$ and $x_2^0 = 40$, with the interval being taken as $x \in [-150,150]$. Figure 9 shows the results for waves with same amplitudes $A_1 = A_2 = 0.369$ and $h = 4$, $\Delta t = 0.1$, $t = 72$. We can infer that the waves interact and combine to generate a single wave whose amplitude is greater than the sum of its components.

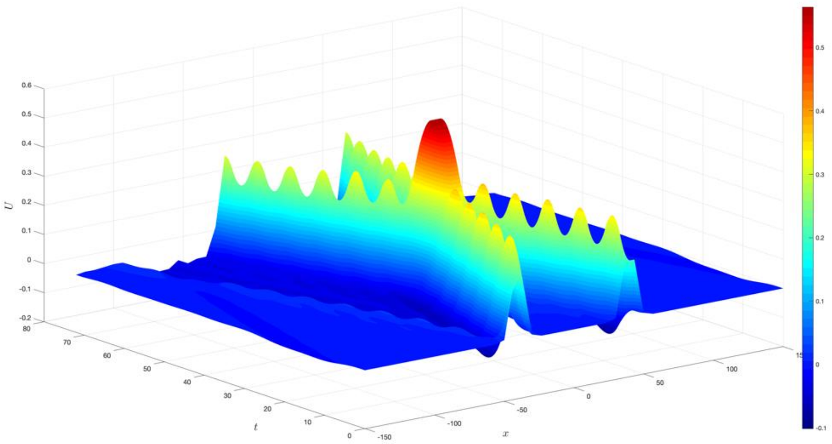

Figure 4. Motion of the two solitary wave interaction for BBqE.

4.3. Numerical Results For IBqE

We consider the IBqE which is Eq. (1.2).

4.3.1. The Single Soliton Wave

The initial conditions for IBqE is as given:

$$u(x,0) = Asech^2\left(\frac{1}{c}\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right)$$

$$u_t(x,0) = 2Ac\sqrt{\frac{A}{6}} sech^2\left(\frac{1}{c}\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right) \tanh\left(\frac{1}{c}\sqrt{\frac{A}{6}}(x - \bar{x}_0)\right)$$

The exact solution of this test problem is given by

$$u(x,t) = Asech^2\left(\frac{1}{c}\sqrt{\frac{A}{6}}(x - ct - \bar{x}_0)\right); \quad c = \pm\left(1 + \frac{2A}{3}\right)^{\frac{1}{2}}$$

and boundary conditions can be found from the exact solutions. In the exact solution, c is the speed and A is the amplitude of the soliton wave. Table 3 shows error norm L_∞ with mesh size $h = 0.25$ for amplitudes $A = 0.25$ and $A = 0.5$. Table 4 shows comparition of error norm L_∞ with [24] for mesh size $h = 0.5$ and amplitudes $A = 0.25$ and $A = 0.5$. Figure 4 demonstrates the solutions of the solitary wave for $h = 0.5$, $\Delta t = 0.01$ and $A = 0.5$ at different time levels. The graphic makes it clear that, as time goes on, the single wave advances steadily to the right while maintaining a nearly constant amplitude.

Table 4. The error norm L_∞ of numerical solutions of IBqE with $h = 0.25$.

A	Δt	RBF-MQ	RBF-G	RBF-W	MOL-MQ	MOL-G	MOL-W	[24]
0.25	0.025	2.0333e-05	4.1073e-05	2.0896e-05	1.7517e-04	3.7115e-05	5.1494e-04	5.5570e-06
	0.05	8.3072e-05	8.7266e-05	8.8111e-05	1.7517e-04	3.7115e-05	5.1494e-04	2.2962e-05
0.5	0.025	8.5668e-05	1.5781e-04	9.6307e-05	4.0970e-04	8.9967e-06	2.3852e-05	4.1959e-05
	0.05	3.6089e-04	3.6788e-04	3.6753e-04	4.0970e-04	8.9967e-06	2.3852e-05	1.6799e-04

Table 5. The error norm L_∞ of numerical solutions of IBqE with $h = 0.5$.

A	Δt	RBF-MQ	RBF-G	RBF-W	MOL-MQ	MOL-G	MOL-W	[24]
0.25	0.025	6.7022e-05	2.1038e-05	5.1600e-05	9.2647e-06	2.1765e-05	6.5628e-06	4.1670e-06
	0.05	9.0387e-05	8.4104e-05	8.6177e-05	9.2647e-06	2.1765e-05	6.5628e-06	2.1512e-05
0.5	0.025	1.0382e-04	9.1404e-05	1.0049e-04	2.9753e-05	5.1133e-06	9.2057e-08	3.4386e-04
	0.05	3.2376e-04	3.6552e-04	3.7008e-04	2.9753e-05	5.1133e-06	9.2057e-08	1.6019e-04

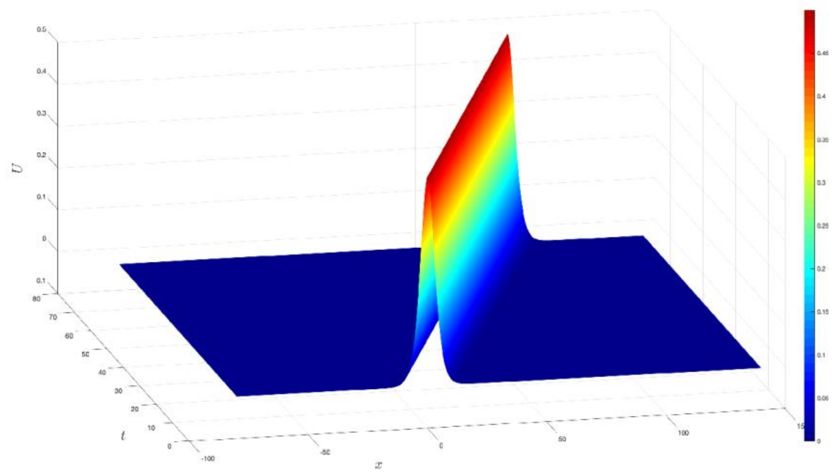

Figure 5. Motion of the single solitary wave for IBqE.

4.3.2. Interaction of Two Soliton Waves

The following initial conditions are used to study the problem of two soliton waves interaction for IBqE

$$u(x,0) = u_1(x,0) + u_2(x,0),$$
$$v(x,0) = v_1(x,0) + v_2(x,0),$$

where

$$u_i(x,0) = A_i sech^2\left[\frac{1}{c_i}\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$
$$v_i(x,0) = 2A_i\sqrt{\frac{A_i}{6}}sech^2\left[\frac{1}{c_i}\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right]\tanh\left[\frac{1}{c_i}\sqrt{\frac{A_i}{6}}\left(x - x_i^0\right)\right],$$
$$c_i = \pm\left(1 + \frac{2A_i}{3}\right)^{\frac{1}{2}}, \quad i = 1,2.$$

In this example we study the interaction of two waves travelling toward each other by using A_i as the wave amplitudes, x_i^0 as the beginning locations, and $c_i(c_1 = c_2)$ as the wave speeds. The waves are situated at $x_1^0 = -20$ and $x_2^0 = 30$, with the interval being taken as $x \in [-80,150]$. Figure 9 shows the results for waves with same amplitudes $A_1 = A_2 = 2$ and $h = 0.25$, $\Delta t = 0.001$, $t = 72$. We can infer that the waves interact and combine to generate a single wave whose amplitude is greater than the sum of its components.

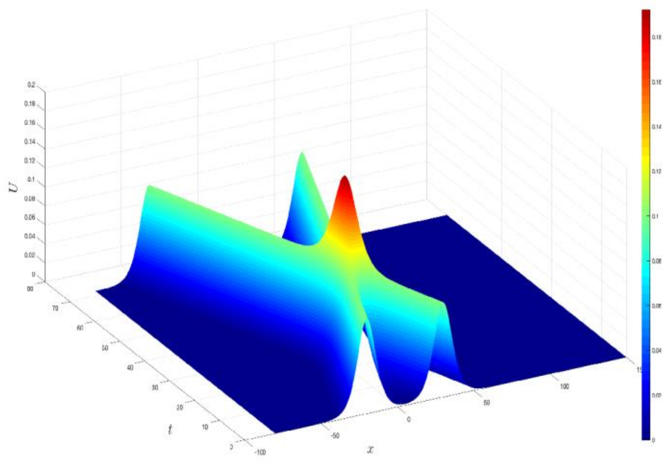

Figure 6. Motion of the two solitary wave interaction for IBqE.

5. ACKNOWLEDGEMENTS

The authors are grateful to Eskişehir Technical University Scientific Research Council (BAP No: 23ADP113).

CONFLICT OF INTEREST

The author(s) stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Murat Arı: Formal analysis, Writing - original draft, Visualization, Software.
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REFERENCES

[1] Boussinesq J. Theorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans un canal des vitesses sensiblement pareilles de la surface au fond. J. Math. Pures Appl. 1872; 7: 55–108.

[2] Manoranjan VS, Mitchell AR, Morris JL. Numerical solutions of the good Boussinesq equation. SIAM Journal on Scientific and Statistical Computing 1984; 5(4): 946-957.

[3] Manoranjan VS, Ortega T, Sanz-Serna JM. Soliton and antisoliton interactions in the "good" Boussinesq equation. J. Math. Phys. 1988; 29: 1964–1968.

[4] Bratsos AG. The solution of the Boussinesq equation using the method of lines. Comput. Methods Appl. Mech. Engrg. 1998; 157: 33–44.

[5] Pani AK, Saranga H. Finite element Galerkin method for the "good" Boussinesq equation. Nonlinear Analysis: Theory, Methods and Applications, 1997; 29: 937-956.

[6] Ortega T, Sanz-Serna JM. Nonlinear stability and convergence of finite difference methods for the "good" Boussinesq equation. Numerische Mathematik, 1990; 58: 215-229. doi: 10.1007/BF01385620

[7] El-Zoheiry H. Numerical investigation for the solitary waves interaction of the "good" Boussinesq equation. Applied Numerical Mathematics, 2003; 45: 161-173.

[8] Wazwaz AM. Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method. Chaos, Solitons and Fractals, 2001; 12: 549-1556.

[9] Ismail MS, Bratsos AG. A predictor-corrector scheme for the numerical solutions of the Boussinesq equation. Journal of Applied Mathematics and Computing, 2003; 13: 11-27.

[10] Bratsos AG, A second-order numerical scheme for the solution of the one-dimensional Boussinesq equation. Numer. Algorithms 46, 2007; 45–58.

[11] Bratsos AG. Solitary-wave propagation and interactions for the 'good' Boussinesq equation. Int. J. Comput. Math. 85, 2008; 1431–1440.

[12] Dehghan M, Salehi RA. Meshless based numerical technique for traveling solitary wave solution of Boussinesq equation. Applied Mathematical Modelling, 2012; 36(5): 1939-1956.

[13] Ucar Y, Esen A, Karaagac B. Numerical solutions of Boussinesq equation using Galerkin finite element method. Numerical Methods for Partial Differential Equations, 2020; 3782: 1612-1630.

[14] Ismail MS, Mosally F. A fourth order finite difference method for the good Boussinesq equation. Abstract and Applied Analysis, 2014; 323260: 10 pages.

[15] Kirli E, Irk D. A Fourth Order One Step Method for Numerical Solution of Good Boussinesq Equation. Turkish Journal of Mathematics, 2021; 45(5): 2693-2703.

[16] Zhijian Y. Existence and non-existence of global solutions to a generalized modification of the improved Boussinesq equation. Math. Methods Appl. Sci. 1998; 21: 1467.

[17] Lin Q, Wu YH, Loxton R. Linear B-Spline Finite Element Method for the Improved Boussinesq Equation. J. Comput. Appl. Math. 2009; 224: 658.

[18] Iskandar L, Jain PC, Proc. Indian Acad. Sci. Math. Sci. 89, 1980, 171.

[19] Irk D, Dağ I, Numer. Methods Partial Differential Equations, 2009, doi:10.1002/num.20492.

[20] Kansa EJ. Multiquadrics–A scattered data approximation scheme with applications to computational fluid dynamics. I., Comput. Math. Appl. 1990; 19: 127–145.

[21] Kansa EJ. Multiquadrics– A scattered data approximation scheme with applications to computational fluid dynamics. II., Comput. Math. Appl. 1990; 19: 147–161.

[22] Wendland H. Scattered Data Approximation. Cambridge Monographs on Applied and Computational Mathematics, Cambridge: Cambridge University Press, 17, 2005.

[23] Rubin SG, Graves RA. A Cubic Spline Approximation for Problems in Fluid Mechanics (NASA TR R-436, Washington, DC, 1975).

[24] Karaagac B, Ucar Y, Esen A. Numerical Solutions of the Improved Boussinesq Equation by the Galerkin Quadratic B-Spline Finite Element Method. Filomat 2018; 32:16: 5573–5583.