Mathematical Modelling and Numerical Simulation with Applications, 2024, 4(2), 149–164

https://dergipark.org.tr/en/pub/mmnsa
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.1487545

RESEARCH PAPER

Approximate solution of integral equations based on generalized sampling operators

Fuat Usta [1,*,‡]

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, 81620 Düzce, Türkiye
[*] Corresponding Author
[‡] fuatusta@duzce.edu.tr (Fuat Usta)

Abstract

In this manuscript, we present and test a numerical scheme with an algorithm to solve Volterra and Abel's integral equations utilizing generalized sampling operators. Illustrative computational examples are included to indicate the validity and practicability of the proposed technique. All of the computational examples in this research have been computed on a personal computer implementing some programs coded in MATLAB.

Keywords: Generalized sampling operators; integral equations; numerical results

AMS 2020 Classification: 65R20; 94A20; 45D05; 47N20

1 Introduction

Integral equations have an important place in the application of mathematical analysis to today's problems. Since integral equations are a vast field of research, a theory that will include all integral equations cannot be established. Therefore, they are examined separately according to their characteristics. Considering these separate examinations, integral equations are divided into Volterra and Fredholm integral equations. In the conducted studies, Volterra integral equations are in the foreground and the relationship between differential equations is established in detail. A variable or constant coefficient differential equation with initial conditions can be converted to a Volterra integral equation or an integral equation can be converted to a differential equation. Therefore, an integral equation can also be considered as a boundary value problem of the differential equation provided for the initial conditions.

It is known that differential equations are not enough to define a problem by itself. Therefore, initial or boundary conditions must be added to the problem in a differential equation. Similarly, initial or boundary conditions are necessary for the problems defined by integral equations. In other words, integral equations include the initial conditions through the Green functions. Thus, it

➤ **Received:** 22.05.2024 ➤ **Revised:** 25.06.2024 ➤ **Accepted:** 29.06.2024 ➤ **Published:** 30.06.2024

shows similar aspects of integral and differential equations. In addition, integral equations require integral over the domain space which is defined according to the nature of integral equations. It means that the value of the unknown function at a point is found in terms of expressions containing the integral of that function over the domain space.

The first studies known with integral equations were performed in the first half of the 19th century. Previously, systematic research has not been conducted [1]. However, more methodical researches were carried out towards the end of this century and some results started to be obtained. It is known for the first time that Abel came across an integral equation when he dealt with a mechanical problem of the *tautochrone* in 1823 [2, 3]. Abel presented the general formula for the mechanical problems he worked on as follows:

$$\varphi(\eta) = \int_0^{\alpha} \frac{\phi(\xi)}{(\eta-\xi)^2} d\xi, \quad \varphi(0) = 0, \quad \alpha \in (0,1),$$

and gave the solution to this problem in 1826 [2]. In this equation, if $\alpha = 0$ and $\alpha = 1/2$, the original equation that Abel encountered was obtained, and the famous *tautochrone* problem related to this equation was first solved by Huygens [4].

In some cases, it may not be possible to find the analytical solution of the integral equation due to their nature. In situations like this, it becomes necessary to investigate the existence of a numerical solution of the integral equation. In order to solve Volterra integral equations numerically, there are a number of proposed techniques in literature such as Taylor-series expansion method, Legendre wavelet method, Adomian decomposition method, Sinc-collection method and power series method [5–13]. In addition to this, in [14], the authors introduced a numerical technique for solving Volterra integral equation of second kind, first kind and even singular types of these equations by using the Bernstein Approximation method. Afterward, Usta et.al. [15] introduced the numerical solution of both second and first-kind Volterra integral equations with the aid of Szasz-Mirakyan operators. Other numerical approaches can be found in [16, 17].

On the other hand, approximation theory is one of the fundamental topics of mathematical analysis. One of the main problems of the approximation theorem is to show the given function f in the form of a series representation of functions that have better properties than itself. In 1885, Weierstrass was the researcher who made the first studies on the approximation theorem. After the famous theorem of Weierstrass, a number of studies have been conducted on the approximation theorem, [18], such as those involving Bernstein approximations. Furthermore, one of the most significant of those studies is the sampling theorem. The main theorem of generalized sampling theorem was introduced to the literature by Butzer and his colleagues at RWTH Aachen in the late 1970s and has been studied by a number of mathematicians as of this date [19–22]. One of the most important superior features of the generalized sampling theorem is that it converges in an infinite interval rather than converging in a closed interval $[0,1]$ like Bernstein operators. In more recent times, generalized sampling theory is a popular subject in approximation theory owing to its great variety of applications, especially in image and signal processing.

In this study, computational solutions of integral equations, which are crucial application areas in several disciplines, are given by making use of the superior features of the generalized sampling theorem. Additionally, we show the applicability and efficiency of the proposed technique both theoretically and numerically. Of course, this work is not a completely new methodology or a new method for the numerical solution of integral equations. However, in the light of existing collocation methods such as projection methods [23, pp. 49-50] which uses the basis functions and unknown constants, it is presented to the attention of the readers as a different alternative. Besides, using the nonideal instantaneous sampling theory and Fourier analysis, another work in

[24] presents a new methodology to solve Fredholm integral equations. In this context, one can argue that the presented technique is the first time that the generalized sampling theory is used to solve Volterra integral equations and can be readily generalized in that direction.
The contents of the article consist of five sections with this section. Section 2 discusses the preliminaries of both integral equations and the generalized sampling theorems. Section 3 explains the construction of the proposed method for the numerical solution of various integral equations via the proposed method. Section 4 provides several computational experiment results to validate the presented technique. Finally, Section 5 summarizes the paper by adding some conclusions and further research.

2 Fundamental facts

In this section, we review some fundamental definitions and theorems that we will benefit from the construction of the proposed method. Therefore, it would be more favorable to give them in two parts systematically.

Integral equations

As mentioned in the previous sections, it will be useful for us to categorize the integral equations. Since we focus on Volterra and Abel integral equations in this study, it will be sufficient to provide general information about them. For detailed information on other types of integral equations such as Fredholm integral equations, we refer the reader to [25–29].
The standard form of the integral equation is

$$\psi(\eta)\phi(\eta) = \varphi(\eta) + \lambda \int_{\vartheta(\eta)}^{\mu(\eta)} \mathcal{K}(\eta,\xi)\phi(\xi)d\xi, \tag{1}$$

where $\mathcal{K}(\eta,\xi)$ is a bivariate known kernel, $\psi(\eta)$ and $\varphi(\eta)$ are known functions, $\vartheta(\eta)$, $\mu(\eta)$ are integration limits, λ is a non-zero real or complex parameter and $\phi(\eta)$ is unknown function needs to be determined. The classical form of Volterra integral equations is [30],

$$\psi(\eta)\phi(\eta) = \varphi(\eta) + \lambda \int_{a}^{\eta} \mathcal{K}(\eta,\xi)\phi(\xi)d\xi, \quad a \le \xi \le \eta \le b, \quad [a,b] \subset (-\infty,\infty). \tag{2}$$

On the other hand, when at least one of the limits of integration in an integral equation becomes infinite or when the bivariate kernel of an integral equation becomes infinite at one or more points within the range of integration, in this case, the integral equation is called as a singular integral equation. One of them is Abel integral equation and it is given as follows for $\eta > 0$:

$$\frac{1}{\Gamma(\alpha)} \int_{a}^{\eta} \frac{1}{(\eta-\xi)^{1-\alpha}}\phi(\xi)d\xi = \varphi(\eta), \quad a \le \xi \le \eta \le b, \quad [a,b] \subset (-\infty,\infty), \tag{3}$$

where $\Gamma(\cdot)$ is Gamma function defined by $\Gamma(\alpha) = \int_0^\infty \eta^{\alpha-1}e^{-\eta}d\eta$, and $\alpha \in (0,1)$ [31].

Generalized sampling operators

Butzer and his students introduced the theory of generalized sampling operators at RWTH Aachen in the late 1970s. Then it turned out that this study was very interesting both in terms of theory and practice. Before summarizing generalized sampling operators, readers who want to get detailed information on this topic can refer to the following studies [19–22, 32].

A generalized sampling operators generated by a suitable kernel function $\chi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is defined for a uniformly continuous and bounded functions $\phi \in C(\mathbb{R})$ as follows:

$$\left(\mathscr{S}_w^\chi \phi\right)(\eta) = \sum_{k=-\infty}^{\infty} \phi\left(\frac{k}{w}\right) \chi(w\eta - k), \tag{4}$$

for $\eta \in \mathbb{R}$ and $w > 0$. It is worth noting that the values $\phi\left(k/w\right)_{k=-\infty}^{\infty}$ are called sampled values taken at the nodes k/w for $k \in \mathbb{Z}$, which are a uniform grid on $\mathbb{R}$. Additionally, the generalized sampling operators are well-defined when the following conditions are held for any $s \in \mathbb{R}$:

$$\sum_{k=-\infty}^{\infty} |\chi(s-k)| < \infty,$$

the absolute convergence being uniform on compact subsets of $\mathbb{R}$, and

$$\sum_{k=-\infty}^{\infty} \chi(s-k) = 1.$$

In addition to these facts, one can say that $\mathscr{S}_w^\chi$ are linear and bounded operators mapping $C(\mathbb{R})$ into itself, having the operator norm

$$\|\mathscr{S}_w^\chi\|_{[C(\mathbb{R}),C(\mathbb{R})]} = \sup_{s\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(s-k)|,$$

and

$$\lim_{w\to\infty} \|\mathscr{S}_w^\chi \phi - \phi\|_{C(\mathbb{R})} = 0.$$

Recently, a number of progressions were observed for the development of the generalized sampling operators, focusing on certain aspects of both theory and applications. In more detail, in [33], the authors considered a new definition of generalized sampling type series utilizing an approach defined by Durrmeyer for the Bernstein polynomials. On the other hand, in [34], the authors introduced appropriate linear combinations for a multivariate version of the generalized sampling series. Both studies provide a better order of approximation theoretically proved. Along with these, in [35], the authors proposed some solutions to solve the problems encountered in real-life signal processing.

One of the most significant generalizations of the generalized sampling theorem is sampling Kantorovich operators which use the integral mean of ϕ on small intervals around the sample nodes in place of the exact value of the function at these nodes, [36–39]. In other words, the sampling Kantorovich operators can be obtained by replacing the sampled values with the Steklov mean of f on the interval $[k/w, (k+1)/w]$, which is,

$$\overline{\phi}\left(\frac{k}{w}\right) = w\int_{k/w}^{(k+1)/w} \phi(s)ds.$$

This is a point where the sampling Kantorovich operators are bounded in $L^p(\mathbb{R})$, and also, as a general idea, in Orlicz spaces, under classical singularity presumptions on the kernel χ.

In recent times, the asymptotic behaviour of the generalized sampling operators has been studied, which yields precise estimates of the pointwise and uniform convergence of these operators to ϕ [33, 40, 41]. Particularly, in [41], the Voronovskya type formula for the generalized sampling operators, under appropriate singularity presumptions on the kernel function χ, has been given as follows for at least twice differentiable function ϕ at the point η,

$$\lim_{w\to\infty} w^2 \left[\left(\mathcal{S}_w^{\chi}\phi\right)(\eta) - \phi(\eta) \right] = \mathcal{A}_{\chi}\phi''(\eta), \tag{5}$$

where $\mathcal{A}_{\chi}$ is an absolute constant depending only on χ.

3 Construction of the numerical method

In this part, we construct a numerical scheme to find a numerical solution to the second and the first kind Volterra integral equations and Abel integral equations with the presented method. In line with this objective, we use the truncated type operators. In other words, whenever the operators (4) converge, for the positive integer N, f can be approximated by,

$$\left(\mathcal{S}_w^{\chi,N}\phi\right)(\eta) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\chi(w\eta - k). \tag{6}$$

Thus one can find the approximate solution of given integral equations for the arbitrary interval. Throughout this and the next sections, we take the integral equations defined in $\tilde{I} := [a, b]$ such that $-\infty < a \leq \eta \leq b < \infty$.

Numerical scheme for the second kind Volterra integral equations

So as to solve the second kind Volterra integral equations, firstly, we approximate the unknown function in (2) via (6) as follows:

$$\phi(\eta) \simeq \mathcal{S}_w^{\chi,N}(\phi(\eta)) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\chi(w\eta - k), \tag{7}$$

for properly selected kernel χ. Then substituting (7) into (2) in case of $\psi(\eta) = 1$, one readily deduces the following equation, that is to say

$$\mathcal{S}_w^{\chi,N}(\phi(\eta)) = \varphi(\eta) + \lambda \int_a^{\eta} \mathcal{K}(\eta,\xi)\mathcal{S}_w^{\chi,N}(\phi(\xi))d\xi, \quad \eta \in \tilde{I}, \tag{8}$$

which yields

$$\sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\chi(w\eta - k) = \varphi(\eta) + \lambda \int_a^{\eta} \mathcal{K}(\eta,\xi) \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\chi(w\xi - k)d\xi.$$

The point to note here is that it is possible to benefit the interchangeability properties of the integral and the sum by using the result that the generalized sampling operators are uniformly convergent, proved in [42]. Thereupon by re-composition the above equation, we deduce that

$$\varphi(\eta) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right) \left[\chi(w\eta - k) - \lambda \int_a^{\eta} \mathcal{K}(\eta, \xi)\chi(w\xi - k)d\xi\right].$$

The point to be noted here is that we ignore the endpoints of the approximation interval which compute the solution in order to avoid the singularity issue by manipulating the endpoints with any arbitrary small number ε. In addition, we need to replace η with $\eta_l = l/w + \varepsilon$, for $l = -N, \cdots, N$ before calculating the unknown coefficients $f(k/w)$. That is,

$$\varphi(\eta_l) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right) \left[\chi(w\eta_l - k) - \lambda \int_a^{\eta_l} \mathcal{K}(\eta_l, \xi)\chi(w\xi - k)d\xi\right].$$

This equation can be expressed in the matrix form as follows:

$$[\mathbf{P}][\mathbf{X}] = [\mathbf{S}],$$

where

$$[\mathbf{P}] = \left[\chi(w\eta_l - k) - \lambda \int_a^{\eta_l} \mathcal{K}(\eta_l, \xi)\chi(w\xi - k)d\xi\right]_{(2N+1)\times(2N+1)}, \quad l, k = -N, \cdots, N, \tag{9}$$

$$[\mathbf{S}] = \left[\begin{array}{ccccc} \varphi(\eta_{-N}), & \varphi(\eta_{-N+1}), & \cdots, & \varphi(\eta_{N-1}), & \varphi(\eta_N) \end{array}\right]^T_{(2N+1)\times 1}, \tag{10}$$

$$[\mathbf{X}] = \left[\begin{array}{ccccc} \phi(-N/w), & \phi((-N+1)/w), & \cdots, & \phi((N-1)/w), & \phi(N/w) \end{array}\right]^T_{(2N+1)\times 1}. \tag{11}$$

Algorithm 1: Generalized sampling operators method for solving second kind Volterra integral equations

Input: η_l, $l = -N \cdots, N$

```
for i ← −N to N do
    for k ← −N to N do
        Compute [P]_(2N+1)×(2N+1)
    end
end
Calculate [P^−1]
for k ← −N to N do
    Compute [S]_(2N+1)×(2N+1)
end
Calculate [X] = [P^−1][S].
```

Output: Compute $\sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right) \chi(w\eta - k)$, using $[\mathbf{X}]$.

The matrix equation $[\mathbf{P}][\mathbf{X}] = [\mathbf{S}]$ can be computed as long as the matrix $[\mathbf{P}]$ must be an invertible matrix. Then, to do this it is necessary to compute the matrix $[\mathbf{P}]$ and the vector $[\mathbf{S}]$ as an initial act. Then, one can easily deduce the matrix $[\mathbf{X}]$ utilizing $[\mathbf{X}] = [\mathbf{P}^{-1}][\mathbf{H}]$. Finally, the approximate solution of the second kind Volterra integral equation can be obtained by substituting the matrix [**X**] in Eq. (7). Now we summarize the algorithm of the presented method above.

Numerical scheme for the first kind Volterra integral equations

Now we use (2) to obtain the first kind Volterra integral equations in case of $\psi(\eta) = 0$. Then if we approximate the unknown function $\phi_w(\eta)$ with (7), we obtain the following equality,

$$\varphi(\eta) = \int_a^\eta \mathcal{K}(\eta,\xi)\mathcal{S}_w^{\chi,N}(\phi(\xi))d\xi, \quad \eta \in \tilde{I}. \tag{12}$$

By following the similar steps in the previous subsection, a method for numerical solution of the first kind Volterra integral equations via generalized sampling operators can be developed. The matrix equation obtained here in this circumstance is,

$$[\mathbf{R}][\mathbf{X}] = [\mathbf{S}],$$

where

$$[\mathbf{R}] = \left[\int_a^{\eta_l} \mathcal{K}(\eta_l,\xi)\chi(w\xi-k)d\xi\right]_{(2N+1)\times(2N+1)}, \quad l,k = -N,\cdots,N, \tag{13}$$

and the vectors [**S**] and [**X**] given in Eq. (10) and Eq. (11), respectively. Similarly, we need to compute the matrix [**R**] and the vector [**S**] as a beginning. Then, one can smoothly deduce the matrix [**X**] with the help of $[\mathbf{X}] = [\mathbf{R}^{-1}][\mathbf{H}]$. In the end, the approximate solution of the first kind of Volterra integral equation can be deduced by substituting the matrix [**X**] in Eq. (7).

Numerical scheme for the Abel's integral equations

In this subsection, we provide a numerical scheme for the numerical solution of Abel's integral equation with the proposed method. For this purpose, we approximate the unknown function in (3) via (6), which yields

$$\frac{1}{\Gamma(\alpha)}\int_a^\eta \frac{1}{(\eta-\xi)^{1-\alpha}}\mathcal{S}_w^{\chi,N}(\phi(\xi))d\xi = \varphi(\eta), \quad \eta \in \tilde{I}.$$

Then this equality gives us the following equation,

$$\varphi(\eta_l) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\left[\frac{1}{\Gamma(\alpha)}\int_a^{\eta_l}\frac{1}{(\eta_l-\xi)^{1-\alpha}}\chi(w\xi-k)d\xi\right],$$

where $l = -N,\dots,N$. Ultimately, this equation can be converted to a matrix equation, that is

$$[\mathbf{K}][\mathbf{X}] = [\mathbf{S}],$$

where

$$[\mathbf{K}] = \left[\frac{1}{\Gamma(\alpha)}\int_a^{\eta_l}\frac{1}{(\eta_l-\xi)^{1-\alpha}}\chi(w\xi-k)d\xi\right]_{(2N+1)\times(2N+1)}, \quad l,k = -N,\dots,N, \tag{14}$$

and the vectors $[\mathbf{S}]$ and $[\mathbf{X}]$ given in Eq. (10) and Eq. (11), respectively.

Remark 1 *It is possible to write an algorithm similar to Algorithm 1, where only the content of the matrix $[\boldsymbol{P}]$ will change and the matrices $[\mathbf{S}]$ and $[\boldsymbol{K}]$ will be replaced by it for the first kind Volterra and Abel's integral equations, respectively.*

It is noted that we can show $\phi(k/w)$, $k = -N, \ldots, N$ by $\phi_w(k/w)$, $k = -N, \ldots, N$ that are our solution in nodes k/w, $k = -N, \ldots, N$ and by substituting them in Eq. (6), we can find $\mathscr{S}_w^{\chi,N}(\phi_w(\eta_k))$, $k = -N, \ldots, N$ that is the proposed method solution for the integral equation.

4 Numerical examples

In this section of this paper, three numerical examples are provided and tested to demonstrate the practicability and accuracy of the proposed method. The first example is related to the second kind Volterra integral equations, the second example is related to the first kind Volterra integral equations, and the last one related to Abel's integral equation. In all examples the package of *MATLAB 2020a* has been used to implement the algorithm to calculate numerical solution of the test equations considered in this study. The error is reported on the following grid points

$$\rho = \{\eta_{-N}, \ldots, \eta_N\}, \quad \eta_l = l/w \quad l = -N, \ldots, N.$$

In addition to these, we set the following notations to analyze the error of the proposed method:

$$E_w(\eta) = |\phi(\eta) - \mathscr{S}_w^{\chi,N}(\phi_w(\eta))|,$$

and

$$\|E_w\|_\infty = \max\{E_w(\eta_l), l = -N, \ldots, N\},$$

where $\phi(\eta)$ and $\mathscr{S}_w^{\chi,N}(\phi_w(\eta))$ are exact solution and approximate solution of the test integral equations respectively and η_l are the uniform grids on $\tilde{I}$. Moreover, we summarize the root mean square error as follows i.e.

$$RMSE = \sqrt{\frac{\sum\limits_{l=-N}^{N} [\phi(\eta_l) - \mathscr{S}_w^{\chi,N}(\phi_w(\eta_l))]^2}{2N+1}}.$$

Time represents the CPU time consumed in each numerical examples. Moreover, we summarize the root mean square error with RMSE.

Example 1

For the following second kind Volterra integral equation, we take the following equation,

$$\phi(\eta) = e^{-\eta^2} - \frac{1}{2}\left(\frac{1}{e} - e^{-\eta^2}\right)\eta + \int_{-1}^{\eta} \eta\xi\phi(\xi)d\xi, \text{ on } \tilde{I},$$

with the exact solution $\phi(\eta) = e^{-\eta^2}$ on $\tilde{I} = [-1, 1]$. In this example, $\varphi(\eta) = e^{-\eta^2} - \frac{1}{2}\left(\frac{1}{e} - e^{-\eta^2}\right)\eta$, $\mathscr{K}(\eta, \xi) = \eta\xi$ and $\lambda = 1$.

Additionally, for this experiment we use the univariate Fejer kernel defined by

$$\mathcal{X}(\eta) = \frac{1}{2}\operatorname{sinc}^2\left(\frac{\eta}{2}\right),$$

for $\eta \in \mathbb{R}$, where the sinc function is given by

$$\operatorname{sinc}(\eta) := \begin{cases} \dfrac{\sin(\pi\eta)}{\pi\eta}, & \text{if } \eta \in \mathbb{R} - \{0\}, \\ 1, & \text{if } \eta = 0. \end{cases}$$

In Figure 1, the Fejer kernel can be shown.

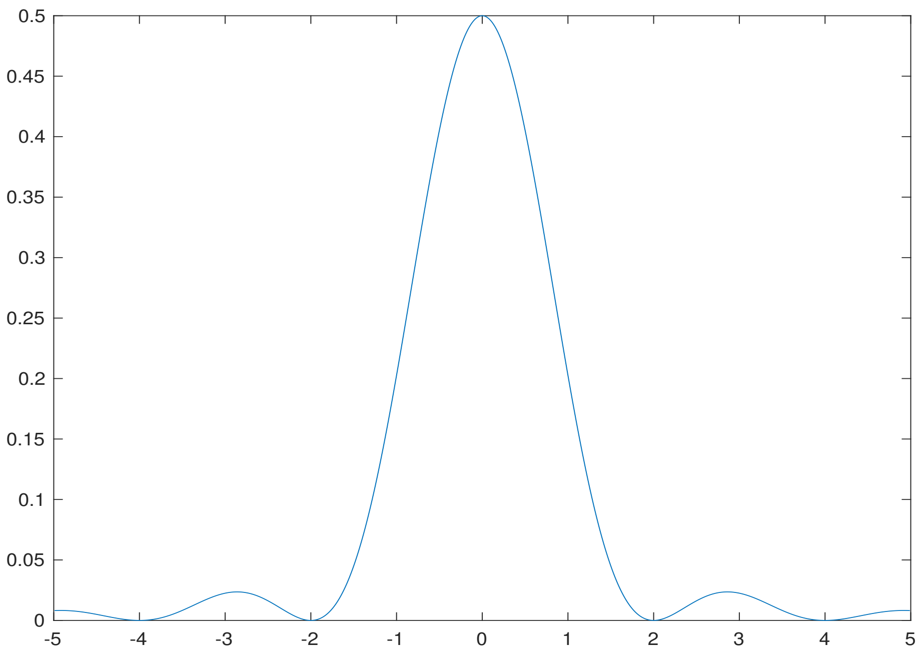

Figure 1. The univariate Fejer kernel $\mathcal{X}(\eta)$

Thus, we have the following generalized sampling operator by substituting univariate Fejer kernel to (6), that is to say

$$\left(\mathcal{S}_w^{\mathcal{X},N}\phi\right)(\eta) = \frac{1}{2}\sum_{k=-N}^{N}\phi\left(\frac{k}{w}\right)\operatorname{sinc}^2\left(\frac{w\eta - k}{2}\right).$$

In Table 1, numerical results of solution of the second kind Volterra integral equation which obtained by the proposed technique are presented. These results confirm that the proposed method is an approximation process for the second kind Volterra integral equation. In addition to this, in Figure 2, computational solution and exact solution of test problem have been provided. This graph shows the convergence properties of the presented method as well.

Table 1. $\|E_w\|_\infty$, $RMSE$ and $Time$ for the numerical solution of the second kind integral equation, with $\varepsilon = 0.01$, on equally spaced grid on $\tilde{I}$

N	Proposed method		
	$\|E_n\|_\infty$	RMSE	Time
5	1.295605e-03	6.074132e-04	< 1
10	1.189609e-03	4.919707e-04	< 1
25	6.622351e-04	2.514840e-04	< 1
50	3.686320e-04	1.357361e-04	< 1
100	1.944992e-04	7.046801e-05	3.483948
150	1.320756e-04	4.759612e-05	6.669202
200	9.986052e-05	3.592332e-05	11.948382

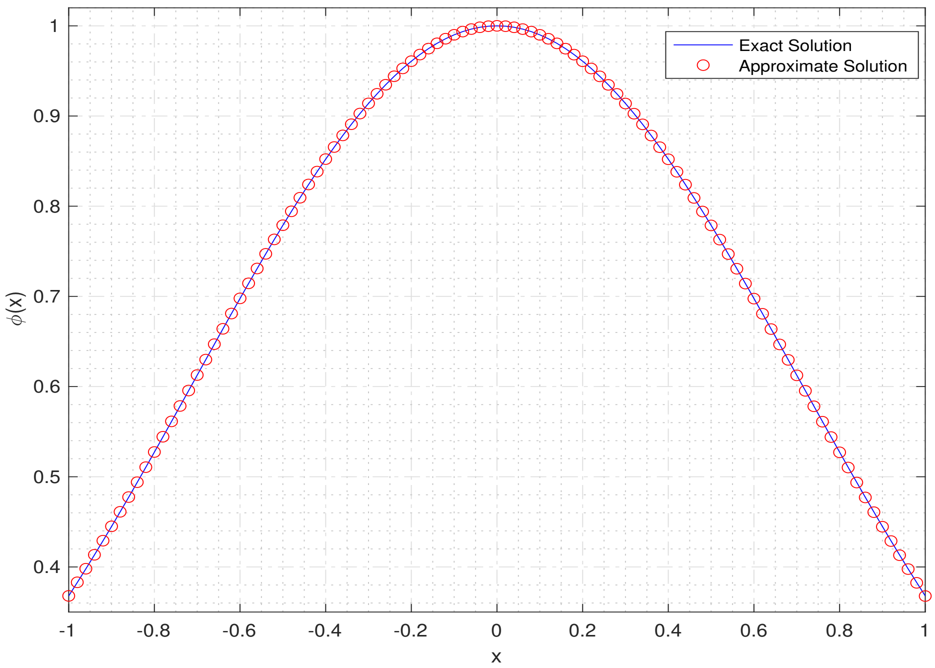

Figure 2. While the blue line represents the exact solution, the red squares represent the proposed method. The figure illustrates the accuracy of the proposed method

Example 2

In this example, we solve the first kind Volterra integral equation numerically. For that, we take the following equation,

$$-\sin(\eta) - \cos(\eta) + e^{\eta+2}(\cos(2) - \sin(2)) = \int_{-2}^{\eta} 2e^{\eta-\xi}\phi(\xi)d\xi, \text{ on } \tilde{I},$$

with the exact solution $\phi(\eta) = \sin(\eta)$ on $\tilde{I} = [-2, 2]$. In this example, $\varphi(\eta) = -\sin(\eta) - \cos(\eta) + e^{\eta+2}(\cos(2) - \sin(2))$ and $\mathcal{K}(\eta, \xi) = 2e^{\eta-\xi}$. Moreover, for this experiment we use the univariate Blackman-Harris kernel defined by

$$\mathcal{X}(\eta) = \frac{1}{2}\,\text{sinc}\,(\eta) + \frac{9}{32}\,(\text{sinc}(\eta+1) + \text{sinc}(\eta-1)) - \frac{1}{32}\,(\text{sinc}(\eta+3) + \text{sinc}(\eta-3))\,,$$

for $\eta \in \mathbb{R}$, where the sinc function defined above. In Figure 3, the Blackman-Harris kernel can be seen.

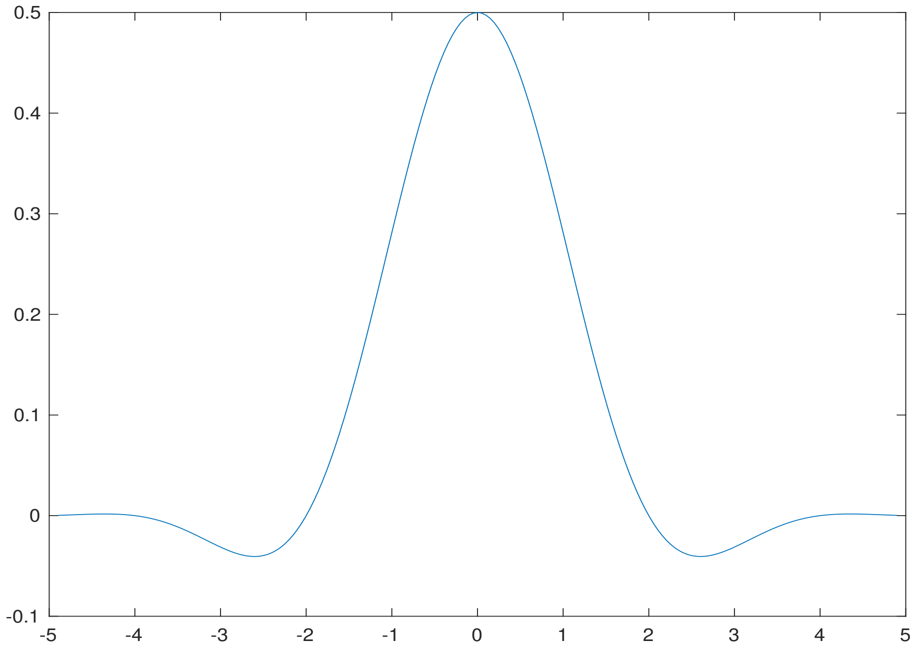

Figure 3. The univariate Blackman-Harris kernel $\mathcal{X}(\eta)$

Thus, we have the following generalized sampling operator by substituting univariate Blackman-Harris kernel to (6), that is to say

$$\left(\mathscr{S}_w^{\mathcal{X},N}\phi\right)(\eta) = \sum_{k=-N}^{N} \phi\left(\frac{k}{w}\right)\left[\frac{1}{2}\operatorname{sinc}(w\eta - k) + \frac{9}{32}\left(\operatorname{sinc}(w\eta - k + 1) + \operatorname{sinc}(w\eta - k - 1)\right)\right.$$
$$\left. - \frac{1}{32}\left(\operatorname{sinc}(w\eta - k + 3) + \operatorname{sinc}(w\eta - k - 3)\right)\right].$$

In Table 2, numerical results of the solution of the first kind Volterra integral equation which computed by the proposed method are presented. These results confirm the approximation properties of the presented method. In addition to this, in Figure 4, computational solution and exact solution of test problem have been provided. This graph also shows the convergence properties of the presented method.

Table 2. $\|E_w\|_\infty$, *RMSE* and *Time* for the numerical solution of the second kind integral equation, on equally spaced grid on $\tilde{I}$

N	Proposed method		
	$\|E_n\|_\infty$	RMSE	Time
5	5.619397e-02	1.270206e-02	< 1
10	3.012573e-02	5.103643e-03	< 1
15	1.384453e-02	2.111580e-03	< 1
20	4.580176e-03	8.216078e-04	< 1

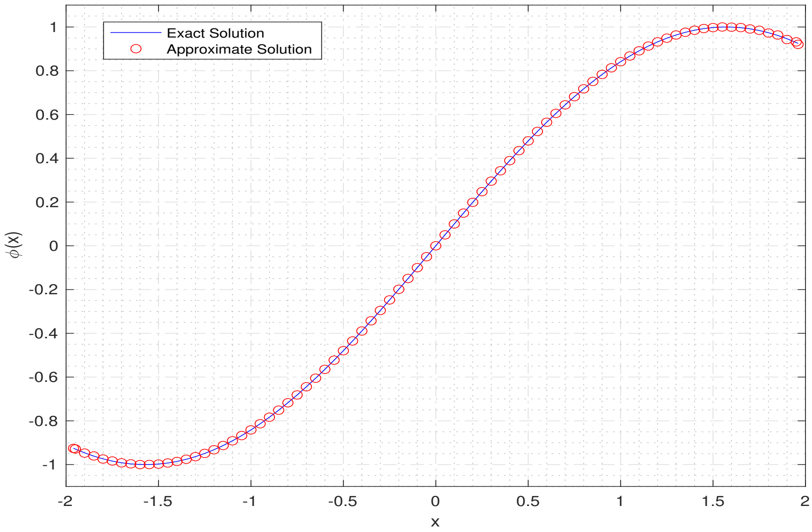

Figure 4. Numerical solution of the first kind Volterra integral equation via generalized sampling operators method. While the blue line represents the exact solution, the red squares represent the proposed method. The figure illustrates the accuracy of the proposed method

Example 3

Finally, we present a numerical example for the Abel's integral equation which is

$$\frac{4}{15}\sqrt{\eta+1}\left(4\eta^2-2\eta+9\right)=\int_{-1}^{\eta}\frac{1}{\sqrt{\eta-\xi}}\phi(\xi)d\xi,\ \text{on}\ \tilde{I},$$

with the exact solution $\phi(\eta)=\eta^2+1$ on $\tilde{I}=[-1,1]$. In this example, $\varphi(\eta)=\frac{4}{15}\sqrt{\eta+1}\left(4\eta^2-2\eta+9\right)$. Moreover, we use the univariate Blackman-Harris kernel for this example.

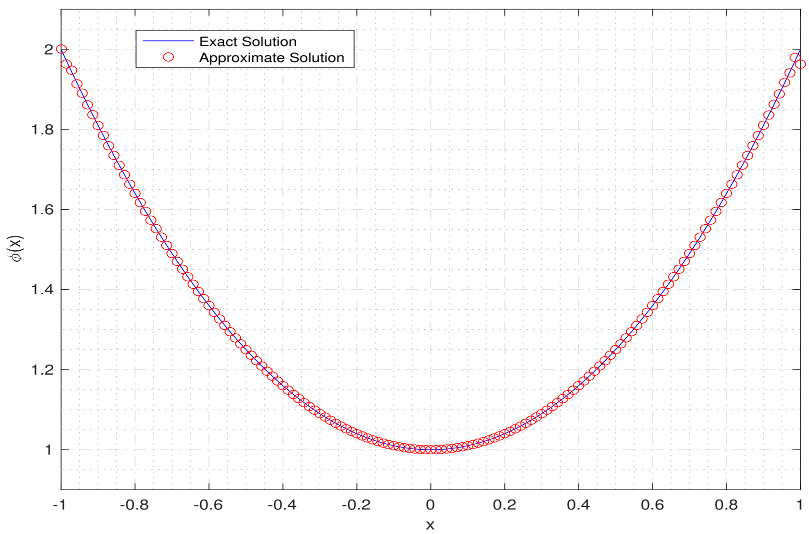

Figure 5. While the blue line represents the exact solution, the red squares represent the proposed method. The figure illustrates the accuracy of the proposed method

In Figure 5, we can observe how the proposed method converges the exact solution of Abel's integral equations.
The three numerical examples given above show that the generalized sampling operators method can be an alternative to other computational methods for numerical solutions of integral equations.

5 Concluding remarks

In this paper, we have proposed and tested a numerical scheme to solve integral equations utilizing generalized sampling operators. For this, firstly, we construct the numerical scheme for the solution. Then we provide the convergence analysis of the proposed method with the aid of Voronovskaya type formula for the generalized sampling operators. Finally, in order to validate our theoretical result, we present some numerical experiments with different kernels.

Declarations

Use of AI tools

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

All data generated or analyzed during this study are included in this article.

Ethical approval (optional)

The author states that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The author declares that he has no conflict of interest.

Funding

No funding

Author's contributions

The author has read and agreed to the published version of the manuscript.

Acknowledgements

Not applicable

References

[1] Archibald, T. and Tazzioli, R. Integral equations between theory and practice: the cases of Italy and France to 1920. *Archive for History of Exact Sciences*, 68, 547-597, (2014). [CrossRef]

[2] Abel, N.H. Aufloesung einer mechanischen Aufgabe. *Journal für die Reine und Angewandte Mathematik (Crelle)*, 1, 153-157, (1826). [CrossRef]

[3] Gorenflo, R. and Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*. Springer: Vienna, (1997).

[4] Moiseiwitsch, B.L. *Integral Equations*. Longman: London and New York, (1977).

[5] Babolian, E. and Davari, A. Numerical implementation of Adomian decomposition method for linear Volterra integral equations of the second kind. *Applied Mathematics and Computation*, 165, 223-227, (2005). [CrossRef]

[6] Izadian, J., Salahshour, S. and Soheil, S. A numerical method for solving Volterra and Fredholm integral equations using homotopy analysis method. *AWER Procedia Information Technology & Computer Science*, 1, 406-411, (2012).

[7] Rashidinia, J. and Zarebnia, M. Solution of a Volterra integral equation by the Sinc-collocation method. *Journal of Computational and Applied Mathematics*, 206(2), 801-813, (2007). [CrossRef]

[8] Atkinson, K.E. *The Numerical Solution of Integral Equations of the Second Kind* (Vol. 4). Cambridge University Press: England, (1997).

[9] Maleknejad, K. and Aghazadeh, N. Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method. *Applied Mathematics and Computation*, 161(3), 915–922, (2005). [CrossRef]

[10] Maleknejad, K., Tavassoli Kajani, M. and Mahmoudi, Y. Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets. *Kybernetes*, 32(9/10), 1530–1539, (2003). [CrossRef]

[11] Maleknejad, K., Mollapourasl, R. and Alizadeh, M. Numerical solution of Volterra type integral equation of the first kind with wavelet basis. *Applied Mathematics and Computation*, 194(2), 400-405, (2007). [CrossRef]

[12] Shoukralla, E.S. and Markos, M.A. The economized monic Chebyshev polynomials for solving weakly singular Fredholm integral equations of the first kind. *Asian-European Journal of Mathematics*, 13(01), 2050030, (2020). [CrossRef]

[13] Wang, W. A mechanical algorithm for solving the Volterra integral equation. *Applied Mathematics and Computation*, 172(2), 1323-1341, (2006). [CrossRef]

[14] Maleknejad, K., Hashemizadeh, E. and Ezzati, R. A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation. *Communications in Nonlinear Science and Numerical Simulation*, 16(2), 647–655, (2011). [CrossRef]

[15] Usta, F., İlkhan, M. and Kara, E.E. Numerical solution of Volterra integral equations via Szász-Mirakyan approximation method. *Mathematical Methods in the Applied Sciences*, 44(9), 7491-7500, (2021). [CrossRef]

[16] Usta, F. Numerical analysis of fractional Volterra integral equations via Bernstein approximation method. *Journal of Computational and Applied Mathematics*, 384, 113198, (2021). [CrossRef]

[17] Usta, F. Bernstein Approximation technique for numerical solution of Volterra integral equations of the third kind. *Computational and Applied Mathematics*, 40, 161, (2021). [CrossRef]

[18] Acar, T., Cappelletti Montano, M., Garrancho, P. and Leonessa, V. On sequence of J. P. King type operators. *Journal of Function Spaces*, 2019, 2329060, (2019). [CrossRef]

[19] Butzer, P.L. and Stens, R.L. Sampling theory for not necessarily band-limited functions: a historical overview. *SIAM Review*, 34(1), 40-53, (1992). [CrossRef]

[20] Butzer, P.L., Fischer, A. and Stens, R.L. Generalized sampling approximation of multivariate signals: general theory. *Atti Sem. Mat. Fis. Univ. Modena*, 41(1), 17-37, (1993). [CrossRef]

[21] Butzer, P.L., Ries, S. and Stens, R.L. Approximation of continuous and discontinuous functions by generalized sampling series. *Journal of Approximation Theory*, 50, 25-39, (1987). [CrossRef]

[22] Butzer, P.L. and Stens, R.L. Prediction of non-bandlimited signals from past samples in terms of splines of low degree.*Mathematische Nachrichten*, 132(1), 115-130, (1987). [CrossRef]

[23] Atkinson, K.E. *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press: Cambridge, (1997).

[24] Paez-Rueda, C.I. and Bustamante-Miller, R. Novel computational approach to solve convolutional integral equations: method of sampling for one dimension, *Pontificia Universidad Javeriana: Engineering for Development*, 23(2), 1-32, (2019). [CrossRef]

[25] Wazwaz, A.M. *A First Course in Integral Equations*, World Scientific: Singapore, (1997).

[26] Ansari, K.J., Sessa, S. and Alam, A. A class of relational functional contractions with applications to nonlinear integral equations. *Mathematics*, 11(15), 3408, (2023). [CrossRef]

[27] Say, F. Asymptotics of singularly perturbed Volterra type integro-differential equation. *Konuralp Journal of Mathematics*, 8(2), 365-369, (2020).

[28] Wang, Q. and Zhou, H. Two-grid iterative method for a class of Fredholm functional integral equations based on the radial basis function interpolation. *Fundamental Journal of Mathematics and Applications*, 2(2), 117-122, (2019). [CrossRef]

[29] Zhou, H., Wang, Q. The Nyström method and convergence analysis for system of Fredholm integral equations. *Fundamental Journal of Mathematics and Applications*, 2(1), 28-32, (2019). [CrossRef]

[30] Shestopalov, Y.V. and Smirnov, Y.G. Integral Equations. *Karlstad University*, 2002.

[31] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications*, Amsterdam: Gordon and Breach Science Publishers, 1993.

[32] P.L. Butzer, W. Splettstößer, R.L. Stens, *The sampling theorem and linear prediction*, Jahresberichte Dt. Math.-Verein, **90** (1988), 1-70.

[33] Bardaro, C. and Mantellini, I. Asymptotic expansion of generalized Durrmeyer sampling type series. *Jaen Journal on Approximation*, 6(2), 143-165, (2014).

[34] Bardaro, C. and Mantellini, I. On linear combinations of multivariate generalized sampling type series. *Mediterranean Journal of Mathematics*, 10, 1833-1852, (2013). [CrossRef]

[35] Bardaro, C., Butzer, P.L., Stens, R.L. and Vinti, G. Prediction by samples from the past with error estimates covering discontinuous signals. *IEEE Transactions on Information Theory*, 56(1), 614-633, (2010). [CrossRef]

[36] Bardaro, C., Vinti, G., Butzer, P.L. and Stens, R. Kantorovich-type generalized sampling series in the setting of Orlicz spaces. *Sampling Theory in Signal and Image Processing*, 6, 29-52, (2007). [CrossRef]

[37] Costarelli, D. and Vinti, G. Approximation by multivariate generalized sampling Kantorovich operators in the setting of Orlicz spaces. *Bollettino dell'Unione Matematica Italiana*, 4(3), 445-468, (2011).

[38] Costarelli, D. and Vinti, G. Approximation by nonlinear multivariate sampling Kantorovich type operators and applications to image processing. *Numerical Functional Analysis and Optimization*, 34(8), 819-844, (2013). [CrossRef]

[39] Costarelli, D. and Vinti, G. Degree of approximation for nonlinear multivariate sampling Kantorovich operators on some function spaces. *Numerical Functional Analysis and Optimization*, 36(8), 964- 990, (2015). [CrossRef]

[40] Bardaro, C. and Mantellini, I. Asymptotic formulae for linear combinations of generalized

sampling operators. *Zeitschrift für Analysis und ihre Anwendunge*, 32, 279-298, (2013). [CrossRef]

[41] Bardaro, C. and Mantellini, I. A quantitative Voronvoskaja formula for generalized sampling operators. *East Journal on Approximations*, 15(4), 459-471, (2009).

[42] Bardaro, C. and Vinti, G. Uniform convergence and rate of approximation for a nonlinear version of the generalized sampling operator. *Results in Mathematics*, 34, 224-240, (1998). [CrossRef]

Mathematical Modelling and Numerical Simulation with Applications (MMNSA)
(https://dergipark.org.tr/en/pub/mmnsa)

How to cite this article: Usta, F. (2024). Approximate solution of integral equations based on generalized sampling operators. *Mathematical Modelling and Numerical Simulation with Applications*, 4(2), 149-164. https://doi.org/10.53391/mmnsa.1487545