Adıyaman University Journal of Science

dergipark.gov.tr/adyusci

ADYUSCI

7(2) (2017) 133-140

Some Inequalities for Positive Linear Maps of Operators

İbrahim Halil GÜMÜŞ [1,*], Xiaohui FU [2]

[1]*Adıyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adıyaman, Türkiye, igumus@adiyaman.edu.tr*

[2]*Hainan Normal University, School of Mathematics and Statics, Haikou, P.R. China, fxh662@sina.com*

Abstract

Drawing inspiration from Lin [3], we generalize some operator inequalities due to Mond et al. [1] as follows: Let A be positive operator on a Hilbert space with $0 < m \le A \le M$. Then for $2 < p < \infty$ and every normalized positive linear map Φ,

$$\Phi^p(A^2) \le \left(\frac{\left(M^2 + m^2\right)^p}{4M^p m^p} \right)^2 \Phi(A)^{2p}.$$

Let A be positive operator on a Hilbert space with $0 < m \le A \le M$. Then for $1 \le p < \infty$ and every normalized positive linear map Φ,

$$\Phi^p(A^{-2}) \le \left(\frac{1}{4(Mm)^p} \left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p} \right)^2 \Phi(A)^{-2p}.$$

Keywords: Positive Operators, Operator Inequalities, Normalized Positive Linear Maps.

* Corresponding Author

Received: 04 October 2017 Accepted: 20 December 2017

Operatörlerin Pozitif Lineer Dönüşümleri için Bazı Eşitsizlikler

Özet

Lin'in [3] teki çalışmasından ilham alarak, Mond ve Pecaric'in [1] deki çalışmasında verilen bazı operatör eşitsizliklerinin genelleştirilmesi şu şekilde yapıldı: A, Hilbert uzayı üzerinde $0 < m \le A \le M$ şartını sağlayan bir pozitif operatör olmak üzere, $2 < p < \infty$ ve her normalize edilmiş Φ pozitif lineer dönüşümü için

$$\Phi^p(A^2) \le \left(\frac{\left(M^2 + m^2\right)^p}{4M^p m^p} \right)^2 \Phi(A)^{2p}$$

eşitsizliği geçerlidir. Yine A, Hilbert uzayı üzerinde $0 < m \le A \le M$ şartını sağlayan bir pozitif operatör olmak üzere, $1 \le p < \infty$ ve her normalize edilmiş Φ pozitif lineer dönüşümü için

$$\Phi^p(A^{-2}) \le \left(\frac{1}{4(Mm)^p} \left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p} \right)^2 \Phi(A)^{-2p}$$

eşitsizliği geçerlidir.

Anahtar Kelimeler: Pozitif Operatörler, Operatör Eşitsizlikleri, Normalize Edilmiş Pozitif Lineer Dönüşümler

1. Introduction

Let M, m be scalars and I be the identity operator. We write $A \ge 0$ to mean that the operator A is positive. If $A - B \ge 0$ $(A - B \le 0)$, then we write $A \ge B$ $(A \le B)$. A^* stands for the adjoint of A. Other capital letters are used to denote the general elements of the C^*-algebra L(*H*) of all bounded linear operators acting on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. $L_+(H)$ is the cone of positive (i.e., non-negative semi-definite) operators. Let $S(\alpha, \beta, H)$ be the totality of all self-adjoint operators on *H* whose spectral are contained in an interval (α, β). A (non-linear) transformation which maps $L_+(H)$, the set of positive operators on H, into $L_+(K)$ will be called positive. The operator norm is denoted by $\|\bullet\|$. A positive linear map Φ preserves order-relation, that is, $A \le B$ implies $\Phi(A) \le \Phi(B)$, and preserves adjoint operation, that is, $\Phi(A^*) = \Phi(A)^*$. It is said to be normalized if it transforms I_H to I_K (we use, in both cases, only I). If Φ is normalized, it maps $S(\alpha, \beta, H)$ to $S(\alpha, \beta, K)$.

It is well known that for two positive operators $A, B,$

$$A \ge B \Rightarrow A^p \ge B^p \qquad for \quad 0 \le p \le 1,$$

but

$$A \ge B \Rightarrow A^p \ge B^p \qquad for \quad p > 1.$$

Let $0 < m \le A \le M$ and Φ be normalized positive linear map. Mond and Pecaric [1] proved the following operator inequality:

$$\Phi(A^2) \le \frac{(M+m)^2}{4Mm} \Phi(A)^2. \tag{1.1}$$

Lin [3] obtained

$$\Phi(A^{-1})^2 \le \left(\frac{(M+m)^2}{4Mm} \right)^2 \Phi(A)^{-2}. \tag{1.2}$$

If we replace A by A^{-1} in (1.1), we get

$$\Phi(A^{-2}) \le \frac{(M+m)^2}{4Mm} \Phi(A^{-1})^2, \tag{1.3}$$

which is

$$\frac{4Mm}{(M+m)^2} \Phi(A^{-2}) \le \Phi(A^{-1})^2. \tag{1.4}$$

Combining (1.2) and (1.4), we have

$$\Phi(A^{-2}) \le \left(\frac{(M+m)^2}{4Mm} \right)^3 \Phi(A)^{-2}. \tag{1.5}$$

Fujii et al. [2] proved that t^2 is order preserving in the following sense.

Proposition 1.1 *Let* $0 < A \le B$ *and* $0 < m \le A \le M$ *. Then the following inequality holds:*

$$A^2 \le \frac{(M+m)^2}{4Mm} B^2.$$

A quick use of the above proposition and (1.1) give the following preliminary result

Proposition 1.2 *Let* $0 < m \le A \le M$ *. Then for normalized positive linear map* Φ*:*

$$\Phi(A^2)^2 \le \frac{(M^2+m^2)^2}{4M^2m^2}\left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi(A)^4. \tag{1.6}$$

It is interesting to ask whether t^p $(p \ge 1)$ for the inequalities (1.1) and (1.5) is order preserving. This is a main motivation for the present paper.

In this paper, we give p-power $(p > 2)$ of inequality (1.1) and present an operator inequality which is refinement of (1.5). Furthermore, we achieve a generalization of the refinement inequality.

2. Main Results

We give some lemmas before we give the main theorems of this paper:

Lemma 2.1 [6] *Let* A *and* B *be positive operators. Then for* $1 \le r < \infty$

$$\left\| A^r + B^r \right\| \le \left\| (A+B)^r \right\|. \tag{2.1}$$

Lemma 2.2 [5] *Let* $A, B > 0$*.Then the following norm inequality holds:*

$$\|AB\| \le \frac{1}{4}\|A+B\|^2. \tag{2.2}$$

Lemma 2.3 [4, p. 41] *Let* $A > 0$ *and* Φ *be normalized positive linear map. Then*

$$\Phi(A)^{-1} \le \Phi(A^{-1}). \tag{2.3}$$

Lemma 2.4 *Let* $0 < m \le A \le M$ *. Then for normalized positive linear map* Φ*:*

$$\Phi(A^{-2})^{\frac{1}{2}} \le \Phi(A^{-1}) + \frac{(M-m)^2}{4Mm(M+m)}. \tag{2.4}$$

Proof : In [1, (14)], we replace A by A^{-1} and have the result.

Now we prove the first main result in the following theorem.

Theorem 2.5 *Let* $0 < m \le A \le M$. *Then for every normalized positive linear map* Φ,

$$\Phi(A^2)^p \le \left(\frac{(M^2+m^2)^p}{4M^p m^p}\right)^2 \Phi(A)^{2p}, \qquad 2 < p < \infty. \tag{2.5}$$

Proof : The operator inequality (2.5) is equivalent to

$$\left\|\Phi(A^2)^{\frac{p}{2}}\Phi^{-p}(A)\right\| \le \frac{(M^2+m^2)^p}{4M^p m^p}. \tag{2.6}$$

Compute

$$\left\|\Phi(A^2)^{\frac{p}{2}}(Mm)^p\Phi^{-p}(A)\right\| \le \frac{1}{4}\left\|\Phi(A^2)^{\frac{p}{2}} + (M^2m^2\Phi(A)^{-2})^{\frac{p}{2}}\right\|^2 \qquad (by\ (2.2))$$

$$\le \frac{1}{4}\left\|(\Phi(A^2) + M^2m^2\Phi(A)^{-2})^{\frac{p}{2}}\right\|^2 \qquad (by\ (2.1))$$

$$= \frac{1}{4}\left\|\Phi(A^2) + M^2m^2\Phi(A)^{-2}\right\|^p$$

$$\le \frac{1}{4}\left\|(M+m)\Phi(A) - mMI + M^2m^2\Phi(A)^{-2}\right\|^p. \quad (by\ [1, (10)])$$

Note that

$$(M-\Phi(A))(m-\Phi(A))\Phi(A)^{-2} \le 0,$$

then

$$Mm\Phi(A)^{-2} + I \le (M+m)\Phi(A)^{-1}. \tag{2.7}$$

Thus

$$\left\|\Phi(A^2)^{\frac{p}{2}}(Mm)^p\Phi(A)^{-p}\right\| \le \frac{1}{4}\left\|(M+m)\Phi(A)-mMI+M^2m^2\Phi(A)^{-2}\right\|^p$$

$$\le \frac{1}{4}\left\|(M+m)\Phi(A)-mMI+Mm((M+m)\Phi(A)^{-1}-I)\right\|^p \qquad (by\ (2.7))$$

$$= \frac{1}{4}\left\|(M+m)(\Phi(A)+Mm\Phi(A)^{-1})-2mMI\right\|^p$$

$$\le \frac{1}{4}\left\|(M+m)(M+m)I-2mMI\right\|^p \qquad (by\ (2.3)\ and\ [3,(2.3)])$$

$$= \frac{1}{4}\left(M^2+m^2\right)^p.$$

That is

$$\left\|\Phi(A^2)^{\frac{p}{2}}\Phi(A)^{-p}\right\| \le \frac{\left(M^2+m^2\right)^p}{4M^pm^p}.$$

Thus (2.5) holds.

Remark 2.6 We cann't get the inequality (1.6) when $p=2$, but we obtain the relation between $\Phi(A^2)^p$ and $\Phi(A)^{2p}$ for $p>2$ and moreover the form of the inequality (2.5) is simple.

Theorem 2.7 *Let* $0<m\le A\le M$. *Then for every normalized positive linear map* Φ,

$$\Phi(A^{-2}) \le \frac{1}{4^2M^2m^2}\left(M+m+\frac{(M-m)^2}{4(M+m)}\right)^4\Phi(A)^{-2}. \qquad (2.8)$$

Proof : The inequality (2.8) is equivalent to

$$\left\|\Phi(A^{-2})^{\frac{1}{2}}\Phi(A)\right\| \le \frac{1}{4Mm}\left(M+m+\frac{(M-m)^2}{4(M+m)}\right)^2.$$

Compute

$$\left\|Mm\Phi(A^{-2})^{\frac{1}{2}}\Phi(A)\right\| \le \frac{1}{4}\left\|Mm\Phi(A^{-2})^{\frac{1}{2}}+\Phi(A)\right\|^2$$

$$\le \frac{1}{4}\left\| Mm\Phi(A^{-1}) + \frac{(M-m)^2}{4(M+m)} + \Phi(A) \right\|^2 \qquad (by \text{ Lemma 2.4})$$

$$\le \frac{1}{4}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^2 . \qquad (by\ [3, (2.3)])$$

That is

$$\left\| \Phi(A^{-2})^{\frac{1}{2}} \Phi(A) \right\| \le \frac{1}{4Mm}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^2 .$$

Thus (2.8) holds.

Remark 2.8 It is easy to compute that $\frac{1}{4^2 M^2 m^2}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^4$ is smaller than $\left(\frac{(M+m)^2}{4Mm} \right)^3$ in the right side of (1.5). Thus (2.8) is a refinement of (1.5).

In the next theorem, we give a generalization of (2.8).

Theorem 2.9 *Let* $0 < m \le A \le M$. *Then for every normalized positive linear map* Φ *and* $1 \le p < \infty$,

$$\Phi(A^{-2})^p \le \left(\frac{1}{4(Mm)^p}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p} \right)^2 \Phi(A)^{-2p}. \qquad (2.9)$$

Proof : The operator inequality (2.9) is equivalent to

$$\left\| \Phi(A^{-2})^{\frac{p}{2}} \Phi(A)^p \right\| \le \frac{1}{4(Mm)^p}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p} . \qquad (2.10)$$

Compute

$$\left\| (Mm)^p \Phi(A^{-2})^{\frac{p}{2}} \Phi(A)^p \right\| \le \frac{1}{4}\left\| (mM\Phi(A^{-2})^{\frac{1}{2}})^p + \Phi(A)^p \right\|^2 \qquad (by\ (2.2))$$

$$\le \frac{1}{4}\left\| Mm\Phi(A^{-2})^{\frac{1}{2}} + \Phi(A) \right\|^{2p} \qquad (by\ (2.2))$$

$$= \frac{1}{4}\left\| Mm\Phi(A^{-1}) + \frac{(M-m)^2}{4(M+m)} + \Phi(A) \right\|^{2p} \quad (by \text{ Lemma 2.4})$$

$$\leq \frac{1}{4}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p}. \qquad (by\,[3, (2.3)])$$

That is

$$\left\| \Phi(A^{-2})^{\frac{p}{2}} \Phi(A)^p \right\| \leq \frac{1}{4(Mm)^p}\left(M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p}.$$

Thus (2.9) holds.

Remark 2.10 When $p = 1$, the inequality (2.9) is (2.8). Thus the inequality (2.9) is a generalization of (2.8).

Acknowledgements

The work was supported by the natural science foundation of Hainan Province (No: 114007).

References

[1] Mond, B., Pecaric, J. E., *Converses of Jensen's inequality for linear maps of operators*, An. Univ. Vest Timis. Ser. Mat. Inform. 31(2), 223-228, 1993.

[2] Fujii, M., Izumino, S., Nakamoto, R., Seo, Y., *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J. 8, 117-122, 1997.

[3] Lin, M., *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. 402, 127-132, 2013.

[4] Bhatia, R., *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[5] Bhatia, R., Kittaneh, F., *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl. 308, 203-211, 2000.

[6] Ando, T., Zhan, X., *Norm inequalities related to operator monotone functions*, Math. Ann., 315, 771-780, 1999.