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GE-FILTERS, ORDERING FILTERS AND LEFT MAPPINGS IN GE-ALGEBRAS

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Abstract. The notions of ordering filter and left mapping in a GE-algebra are introduced, and their properties are investigated. Relations between ordering filters and GE-filters are established. Conditions for an ordering filter to be a GE-filter, and vice versa, are provided. The conditions under which a left mapping becomes injective or an identity are explored. The conditions under which the GE-kernel of a self-mapping will be a GE-filter are provided. It is confirmed that the sets of all left mappings form a semigroup, and that the sets of all idempotent left mappings form a subsemigroup. The conditions under which the sets of all left mappings can be closed with respect to a binary operation are investigated.

1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [8] proved that Hilbert algebras form a variety which is locally finite. Later, several authors introduced many concepts to explore the concept of Hilbert algebras (see [5–7, 9, 10, 14–16]). Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1]). Also, Bandaru et al. introduced several concepts in GE-algebras and investigated its related properties (see [2–4, 12, 13, 17, 18]). Left mappings is very useful concept and many researchers have used it in various mathematical fields. For example, Kondo introduced the

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notion of left mapping on BCK-algebras and investigated some properties of it (see [11]). He showed that in a positive implicative BCK-algebra, if a left mapping is surjective, then it is also an injective one.

In this paper, we introduce the notion of ordering filter in a GE-algebra and provide the conditions for an ordering filter to be a GE-filter. Also, we explore the necessary condition for a GE-filter to be an ordering filter. We introduce the concept of left mapping on GE-algebras and investigate related properties. We define the GE-kernel of a left mapping of a GE-algebra and provide the conditions under which GE-kernel to be a GE-filter. We prove that the set $L(X)$ of all left mappings of a GE-algebra X is closed under the function composition $\circ$ and also a semigroup. We define the operation "$\circledast$" on $L(X)$ by $(f \circledast g)(x) = f(x) * g(x)$ for all $x \in X$ and $f, g \in L(X)$ and observe that the set $L(X)$ is not closed under $\circledast$. Finally, we investigate the conditions under which $L(X)$ be closed with respect to $\circledast$.

This study particularly focuses on ordering filters and left mappings within these algebras, offering a comprehensive exploration of their properties and interrelations. Ordering filters in GE-algebras serve as critical tools for understanding the hierarchical structure and organization within these algebraic systems. Ordering filters help identify and analyze hierarchical relationships and dependencies among elements in a GE-algebra, offering a clearer picture of the overall structure. Establishing relations between ordering filters and GE-filters not only bridges the concepts but also enhances the understanding of how different filters interact and coexist within the algebraic framework. The comprehensive study of ordering filters and left mappings in GE-algebras offers valuable contributions to the understanding of these algebraic structures. By exploring their properties, interrelations, and conditions for specific behaviors, this research paves the way for further advancements in the field of algebra and its applications in logic, computation, and beyond. The motivation lies in the quest for deeper knowledge, the development of new mathematical tools, and the potential for practical applications arising from a robust understanding of GE-algebras.

2. Preliminaries

Definition 1 ([1]). *By a GE-algebra we mean a non-empty set Y with a constant 1 and a binary operation $*$ satisfying the following axioms:*

(GE1) $\gamma_1 * \gamma_1 = 1$,
(GE2) $1 * \gamma_1 = \gamma_1$,
(GE3) $\gamma_1 * (\varpi_2 * \sigma_3) = \gamma_1 * (\varpi_2 * (\gamma_1 * \sigma_3))$

for all $\gamma_1, \varpi_2, \sigma_3 \in Y$.

In a GE-algebra Y, a binary relation "$\leq$" is defined by

$$(\forall \wp_3, \wp_4 \in Y)\,(\wp_3 \leq \wp_4 \;\Leftrightarrow\; \wp_3 * \wp_4 = 1)\,. \tag{1}$$

Definition 2 ([1,2,4]). *A GE-algebra Y is said to be*

- *transitive if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y) \left(\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4) \right). \tag{2}$$

- *commutative if it satisfies:*

$$(\forall \wp_3, \wp_4 \in Y) \left((\wp_3 * \wp_4) * \wp_4 = (\wp_4 * \wp_3) * \wp_3 \right). \tag{3}$$

- *left exchangeable if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y) \left(\wp_3 * (\wp_4 * \wp_5) = \wp_4 * (\wp_3 * \wp_5) \right). \tag{4}$$

- *belligerent if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y) \left(\wp_3 * (\wp_4 * \wp_5) = (\wp_3 * \wp_4) * (\wp_3 * \wp_5) \right). \tag{5}$$

- *antisymmetric if the binary relation "$\leq$" is antisymmetric.*

Proposition 1 ([1]). *Every GE-algebra Y satisfies the following items.*

$$(\forall \gamma_1 \in Y) \left(\gamma_1 * 1 = 1 \right). \tag{6}$$
$$(\forall \gamma_1, \varpi_2 \in Y) \left(\gamma_1 * (\gamma_1 * \varpi_2) = \gamma_1 * \varpi_2 \right). \tag{7}$$
$$(\forall \gamma_1, \varpi_2 \in Y) \left(\gamma_1 \leq \varpi_2 * \gamma_1 \right). \tag{8}$$
$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y) \left(\gamma_1 * (\varpi_2 * \sigma_3) \leq \varpi_2 * (\gamma_1 * \sigma_3) \right). \tag{9}$$
$$(\forall \gamma_1 \in Y) \left(1 \leq \gamma_1 \;\Rightarrow\; \gamma_1 = 1 \right). \tag{10}$$
$$(\forall \gamma_1, \varpi_2 \in Y) \left(\gamma_1 \leq (\varpi_2 * \gamma_1) * \gamma_1 \right). \tag{11}$$
$$(\forall \gamma_1, \varpi_2 \in Y) \left(\gamma_1 \leq (\gamma_1 * \varpi_2) * \varpi_2 \right). \tag{12}$$
$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y) \left(\gamma_1 \leq \varpi_2 * \sigma_3 \;\Leftrightarrow\; \varpi_2 \leq \gamma_1 * \sigma_3 \right). \tag{13}$$

If Y is transitive, then

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y) \left(\gamma_1 \leq \varpi_2 \;\Rightarrow\; \sigma_3 * \gamma_1 \leq \sigma_3 * \varpi_2,\; \varpi_2 * \sigma_3 \leq \gamma_1 * \sigma_3 \right). \tag{14}$$
$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y) \left(\gamma_1 * \varpi_2 \leq (\varpi_2 * \sigma_3) * (\gamma_1 * \sigma_3) \right). \tag{15}$$
$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y) \left(\gamma_1 \leq \varpi_2,\; \varpi_2 \leq \sigma_3 \;\Rightarrow\; \gamma_1 \leq \sigma_3 \right). \tag{16}$$

Lemma 1 ([1]). *In a GE-algebra Y, the following facts are equivalent each other.*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y) \left(\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4) \right). \tag{17}$$
$$(\forall \wp_3, \wp_4, \wp_5 \in Y) \left(\wp_3 * \wp_4 \leq (\wp_4 * \wp_5) * (\wp_3 * \wp_5) \right). \tag{18}$$

Definition 3 ([1]). *A subset F of a GE-algebra Y is called a GE-filter of Y if it satisfies:*

$$1 \in F, \tag{19}$$
$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 * \wp_4 \in F,\; \wp_3 \in F \;\Rightarrow\; \wp_4 \in F). \tag{20}$$

Lemma 2 ([1]). *In a GE-algebra Y, every GE-filter F of Y satisfies:*

$$(\forall \wp_3, \wp_4 \in Y) \left(\wp_3 \leq \wp_4,\; \wp_3 \in F \;\Rightarrow\; \wp_4 \in F \right). \tag{21}$$

Definition 4 ([1])**.** *A non-empty subset F of a GE-algebra Y is called a GE-subalgebra of Y if $\wp_3 * \wp_4 \in F$ for any $\wp_3, \wp_4 \in F$.*

3. GE-filters and ordering filters

In what follows, let Y denote a GE-algebra unless otherwise specified.

Definition 5. *A subset F of Y is called an ordering filter of Y if it satisfies* (21) *and*

$$(\forall \wp_3, \wp_4 \in F)(\exists \wp_5 \in F)(\wp_5 \leq \wp_3,\ \wp_5 \leq \wp_4). \tag{22}$$

We denote by $OF(Y)$ the set of all ordering filters of Y. It is clear that $\{1\}, Y \in OF(Y)$ and every ordering filter contains the element 1.

Example 1. *We take a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5, \zeta_6\}$ with the operation table given by Table 1.*

Table 1. The binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
1	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
ρ_2	1	1	1	ϵ_4	ϵ_4	1
ι_3	1	1	1	ι_5	ι_5	ζ_6
ϵ_4	1	ρ_2	1	1	1	ζ_6
ι_5	1	1	ι_3	1	1	1
ζ_6	1	1	ι_3	ι_5	ι_5	ζ_6

Then $F_1 := \{1, \rho_2, \iota_3, \zeta_6\}$ and $F_2 := \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ are ordering filter of Y. But $F_3 := \{1, \rho_2, \iota_3, \iota_5\}$ is not an ordering filter of Y since $\iota_5 \in F_3$ and $\iota_5 \leq \epsilon_4$ but $\epsilon_4 \notin F_3$. Also, $F_4 := \{1, \rho_2, \iota_3, \epsilon_4\}$ is not an ordering filter of Y since $\rho_2, \epsilon_4 \in F_4$, $\iota_5 \leq \rho_2$ and $\iota_5 \leq \epsilon_4$ but $\iota_5 \notin F_4$.

In general, any ordering filter may not be a GE-filter as seen in the following example.

Example 2. *The ordering filter $F_2 := \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ in Example 1 is not a GE-filter of Y since $\rho_2 * \zeta_6 = 1 \in F_2$ and $\rho_2 \in F_2$, but $\zeta_6 \notin F_2$.*

We provide conditions for an ordering filter to be a GE-filter.

Theorem 1. *In a transitive GE-algebra, every ordering filter is a GE-filter.*

Proof. Let F be an ordering filter of Y. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in F$ and $\wp_3 \in F$. If $\wp_3 = 1$, then $\wp_4 = 1 * \wp_4 \in F$. Suppose that $\wp_3 \neq 1$ and $\wp_4 \neq 1$. Then there exists $\wp_5 \in F$ such that $\wp_5 \leq \wp_3 * \wp_4$ and $\wp_5 \leq \wp_3$ by (22). Using (GE2), (2), (7) and (9), we have

$$1 = \wp_5 * (\wp_3 * \wp_4) \leq \wp_3 * (\wp_5 * \wp_4) \leq (\wp_5 * \wp_3) * (\wp_5 * (\wp_5 * \wp_4))$$
$$= (\wp_5 * \wp_3) * (\wp_5 * \wp_4) = 1 * (\wp_5 * \wp_4) = \wp_5 * \wp_4,$$

which implies from (10) and (16) that $1 = \wp_5 * \wp_4$, i.e., $\wp_5 \leq \wp_4$. Hence $\wp_4 \in F$ by (21), and hence F is a GE-filter of Y. □

Corollary 1. *Every ordering filter is a GE-filter in a belligerent GE-algebra.*

Proof. If Y is a belligerent GE-algebra, then

$$(\wp_3 * \wp_4) * ((\wp_5 * \wp_3) * (\wp_5 * \wp_4)) = (\wp_3 * \wp_4) * (\wp_5 * (\wp_3 * \wp_4))$$
$$= (\wp_3 * \wp_4) * (\wp_5 * ((\wp_3 * \wp_4) * (\wp_3 * \wp_4)))$$
$$= (\wp_3 * \wp_4) * (\wp_5 * 1) = (\wp_3 * \wp_4) * 1 = 1,$$

and so $\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4)$ for all $\wp_3, \wp_4, \wp_5 \in Y$. Thus Y is a transitive GE-algebra, and hence every ordering filter is a GE-filter by Theorem 1. □

In the next example, we show there exists a GE-filter that is not an ordering filter.

Example 3. *We take a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5, \zeta_6\}$ *in which the binary operation "$*$" is provided in Table 2.*

TABLE 2. The binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
1	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
ρ_2	1	1	1	ϵ_4	ϵ_4	ζ_6
ι_3	1	ρ_2	1	ι_5	ι_5	ζ_6
ϵ_4	1	1	ι_3	1	1	ζ_6
ι_5	1	1	1	1	1	ζ_6
ζ_6	1	ρ_2	ι_3	ϵ_4	ι_5	1

The set $F := \{1, \iota_3, \zeta_6\}$ *is a GE-filter of* Y*, but it is not an ordering filter of* Y *because there does not exist* $\wp_5 \in F$ *such that* $\wp_5 \leq \iota_3$ *and* $\wp_5 \leq \zeta_6$.

We would like to explore the conditions necessary for a GE-filter to be an ordering filter.

For every elements $\hbar_1$ and $\hbar_2$ of Y, we consider the set:

$$(Y; \hbar_2, \hbar_1) := \{\wp_3 \in Y \mid \hbar_2 \leq \hbar_1 * \wp_3\}. \tag{23}$$

It is clear that $1, \hbar_1, \hbar_2 \in (Y; \hbar_2, \hbar_1)$ and $(Y; 1, 1) = \{1\}$. If $(Y; \hbar_2, \hbar_1)$ has the least element, it will be denoted by $\hbar_2 \circledast \hbar_1$.

Definition 6 ([13]). *A GE-algebra Y is called an $\circledast$-GE-algebra if there exists $\hbar_1 \circledast \hbar_2$ for all $\hbar_1, \hbar_2 \in Y$.*

Lemma 3 ([13]). *If Y is an $\circledast$-GE-algebra, then*

$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 \circledast \wp_4 \leq \wp_3,\ \wp_3 \circledast \wp_4 \leq \wp_4). \tag{24}$$

Theorem 2. *Every GE-filter is an ordering filter in an $\circledast$-GE-algebra.*

Proof. Let F be a GE-filter of an $\circledast$-GE-algebra Y, and let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \in F$ and $\wp_3 \leq \wp_4$. Then $\wp_3 * \wp_4 = 1 \in F$, and thus $\wp_4 \in F$ by (20). Let $\wp_3, \wp_4 \in F$. Since $\wp_3 \leq \wp_4 * (\wp_3 \circledast \wp_4)$, we get $\wp_3 \circledast \wp_4 \in F$ by Lemma 2 and (20). Using Lemma 3, we can see that F is an ordering filter of Y. □

4. Left Mappings

Definition 7. *A self mapping $\eth$ on a GE-algebra Y is called a left mapping of Y if it satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)(\eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)). \tag{25}$$

It is clear that the identity mapping $\eth : Y \to Y$, $\wp_3 \mapsto \wp_3$, is a left mapping of Y.

Example 4. *We take a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 3.*

Table 3. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	ρ_2	1	1
ι_5	1	ρ_2	ι_3	1	1

Let $\eth$ be a self mapping on Y given as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\eth$ is a left mapping of Y.

Proposition 2. *Given a left mapping $\eth$ of Y, we have*

(i) $\eth(1) = 1$,
(ii) $(\forall \wp_3 \in Y)\ (\wp_3 \leq \eth(\wp_3))$,
(iii) $(\forall \wp_3 \in Y)\ (\eth(\wp_3 * 1) = 1)$,
(iv) $(\forall \wp_3, \wp_4 \in Y)\ (\wp_3 \leq \wp_4 \ \Rightarrow\ \wp_3 \leq \eth(\wp_4))$.

Proof. (i) Using (GE1), (6) and (25), we get $\eth(1) = \eth(\eth(1) * 1) = \eth(1) * \eth(1) = 1$.

(ii) Using (GE1) and (i) and (25) induces $1 = \eth(1) = \eth(\wp_3 * \wp_3) = \wp_3 * \eth(\wp_3)$, that is, $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$.

(iii) Using (6) and (i) induces $\eth(\wp_3 * 1) = \eth(1) = 1$ for all $\wp_3 \in Y$.

(iv) Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \leq \wp_4$. Then $1 = \eth(1) = \eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)$ by (25), and so $\wp_3 \leq \eth(\wp_4)$. □

Definition 8. *The GE-kernel of a left mapping* $\eth$ *of* Y *is defined to be the set:*

$$ker(\eth) := \{\wp_3 \in Y \mid \eth(\wp_3) = 1\}. \tag{26}$$

Theorem 3. *If a left mapping* $\eth$ *of* Y *is injective, then* $ker(\eth) = \{1\}$.

Proof. Suppose $\eth$ is an injective left mapping of Y and let $\wp_3 \in ker(\eth)$. Then $\eth(\wp_3) = 1 = \eth(1)$ by Proposition 2(i), and so $\wp_3 = 1$ since $\eth$ is injective. Hence $ker(\eth) = \{1\}$. □

The following example shows that the converse of Theorem 3 is not true, that is, any left mapping $\eth$ of Y with $ker(\eth) = \{1\}$ may not be injective.

Example 5. *Consider a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 4.*

TABLE 4. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	ρ_2	1	1
ι_5	1	ρ_2	ρ_2	1	1

Define a self mapping $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 = 1, \\ \rho_2 & \text{if } \wp_3 \in \{\rho_2, \iota_3\}, \\ \epsilon_4 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ *is a left mapping of* Y *with* $ker(\eth) = \{1\}$. *But it is not injective since* $\eth(\rho_2) = \rho_2 = \eth(\iota_3)$ *but* $\rho_2 \neq \iota_3$.

Theorem 4. *If a GE-algebra* Y *is antisymmetric and transitive, then every left mapping* $\eth$ *of* Y *with* $ker(\eth) = \{1\}$ *is injective.*

Proof. Let $\eth$ be a self mapping of a transitive and antisymmetric GE-algebra Y and $ker(\eth) = \{1\}$. Let's take $\wp_3, \wp_4 \in Y$ which satisfies $\eth(\wp_3) = \eth(y)$. Then

$\eth(\wp_3) * \eth(\wp_4) = 1$ by (GE1), and so $\eth(\eth(\wp_3) * \wp_4) = 1$ by (25), that is, $\eth(\wp_3) * \wp_4 \in ker(\eth) = \{1\}$. Hence $\eth(\wp_3) \leq \wp_4$. It follows from Proposition 2(ii) that $\wp_3 \leq \eth(\wp_3) \leq \wp_4$. Similarly, we can induce $\wp_4 \leq \wp_3$ for all $\wp_3, \wp_4 \in Y$. Hence $\wp_3 = \wp_4$, and $\eth$ is injective. □

Theorem 5. *In an antisymmetric GE-algebra, every injective left mapping is the identity mapping.*

Proof. Let $\eth$ be an injective left mapping of an antisymmetric GE-algebra Y. Then $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii). Using (GE1), (25) and Proposition 2(i) induces $\eth(1) = 1 = \eth(\wp_3) * \eth(\wp_3) = \eth(\eth(\wp_3) * \wp_3)$ for all $\wp_3 \in Y$. Since $\eth$ is injective, we have $\eth(\wp_3) * \wp_3 = 1$, i.e., $\eth(\wp_3) \leq \wp_3$. Thus $\eth(\wp_3) = \wp_3$ for all $\wp_3 \in Y$ since Y is antisymmetric. Therefore $\eth$ is the identity mapping. □

In the next example, we claim that if Y is not antisymmetric, then any injective left mapping may not be the identity mapping.

Example 6. *Consider a set* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 5.*

Table 5. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	ι_3	ϵ_4	ι_5
ι_3	1	1	1	1	ι_5
ϵ_4	1	1	1	1	ι_5
ι_5	1	ρ_2	1	1	1

Then Y is a GE-algebra which is not antisymmetric. Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 = 1, \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \iota_3, \\ \iota_3 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ is an injective mapping of Y which is not an identity mapping of Y.

Theorem 6. *If $\eth$ is a left mapping of Y, then $ker(\eth)$ and $Im(\eth)$ are GE-subalgebras of Y.*

Proof. Let $\wp_3, \wp_4 \in ker(\eth)$. Then $\eth(\wp_3) = 1 = \eth(\wp_4)$. Hence $\eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4) = \wp_3 * 1 = 1$ by (6) and (25), i.e., $\wp_3 * \wp_4 \in ker(\eth)$. Thus ker($\eth$) is a GE-subalgebra of Y.

Let $\hbar_1, \hbar_2 \in Im(\eth)$. Then there exist $\hbar_3, \hbar_4 \in Y$ such that $\eth(\hbar_3) = \hbar_1$ and $\eth(\hbar_4) = \hbar_2$. Now $\hbar_3 \in Y$ implies that $\eth(c) \in Y$, and so $\hbar_1 * \hbar_2 = \eth(\hbar_3) * \eth(\hbar_4) = \eth(\eth(\hbar_3) * \hbar_4) \in Im(\eth)$. Hence $Im(\eth)$ is a GE-subalgebra of Y. □

In the following example, we can see that $Im(\eth)$ is neither ordering filter nor GE-filter.

Example 7. *Let $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ be the GE-algebra in Example 5. Define a self mapping $\eth$ on Y as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\} \\ \rho_2 & \text{if } \wp_3 \in \{\rho_2, \iota_3\}. \end{cases}$$

Then $\eth$ is a left mapping of Y with $Im(\eth) = \{1, \rho_2\}$. But $Im(\eth)$ is neither an ordering filter of Y nor a GE-filter of Y since $\rho_2 \leq \iota_3$ and $\rho_2 \in Im(\eth)$ but $\iota_3 \notin Im(\eth)$.

Question 9. *If $\eth$ is a left mapping of Y, is $ker(\eth)$ a GE-filter of Y or an ordering filter of Y?*

The next example shows that the answer to Question 9 is negative.

Example 8. *1. Consider a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 6.*

TABLE 6. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	1	1	1
ι_5	1	ρ_2	ι_3	1	1

Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \iota_3\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ is a left mapping of Y and its kernel is $ker(\eth) = \{1, \iota_3\}$ which is not a GE-filter of Y since $\iota_3 * \rho_2 = 1 \in ker(\eth)$ and $\iota_3 \in ker(\eth)$, but $\rho_2 \notin ker(\eth)$.

2. Consider a set $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 7.

Then Y is a GE-algebra. Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \epsilon_4, \iota_5\} \\ \iota_3 & \text{if } \wp_3 = \iota_3. \end{cases}$$

TABLE 7. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	1
ι_3	1	ρ_2	1	ϵ_4	ρ_2
ϵ_4	1	ι_5	ι_3	1	ι_5
ι_5	1	1	1	ϵ_4	1

Then $\eth$ is a left mapping of Y with $ker(\eth) = \{1, \rho_2, \epsilon_4, \iota_5\}$. But $ker(\eth)$ is not an ordering filter of Y since $\rho_2 \leq \iota_3$ and $\rho_2 \in ker(\eth)$ but $\iota_3 \notin ker(\eth)$.

We explore the conditions under which a positive answer to Question 9 may come out.

Theorem 7. *If a self mapping $\eth$ on Y is an endomorphism, i.e., $\eth(\wp_3 * \wp_4) = \eth(\wp_3) * \eth(\wp_4)$ for all $\wp_3, \wp_4 \in Y$, then $ker(\eth)$ is a GE-filter of Y.*

Proof. Assume that $\eth : Y \to Y$ is an endomorphism. Then $\eth(1) = \eth(\wp_3 * \wp_3) = \eth(\wp_3) * \eth(\wp_3) = 1$ for all $\wp_3 \in Y$, that is, $1 \in ker(\eth)$. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in ker(\eth)$ and $\wp_3 \in ker(\eth)$. Since $\eth$ is an endomorphism, it follows that

$$1 = \eth(\wp_3 * \wp_4) = \eth(\wp_3) * \eth(\wp_4) = 1 * \eth(\wp_4) = \eth(\wp_4),$$

that is $\wp_4 \in ker(\eth)$. Therefore $ker(\eth)$ is a GE-filter of Y. □

Corollary 2. *Let $\eth$ be a left mapping of Y. If $\eth$ is an endomorphism, then $ker(\eth)$ is a GE-filter of Y.*

Theorem 8. *Let $\eth$ be a left mapping of Y which is idempotent, that is, $\eth(\eth(\wp_3)) = \eth(\wp_3)$ for all $\wp_3 \in Y$. If Y is commutative, then $ker(\eth)$ is a GE-filter of Y.*

Proof. We first show the following assertion.

$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 \in ker(\eth),\ \wp_3 \leq \wp_4 \ \Rightarrow\ \wp_4 \in ker(\eth)). \tag{27}$$

Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \in ker(\eth)$ and $\wp_3 \leq \wp_4$. Then $\wp_4 = (\wp_4 * \wp_3) * \wp_3$ since Y is commutative. Hence

$$\eth(\wp_4) = \eth((\wp_4 * \wp_3) * \wp_3) = (\wp_4 * \wp_3) * \eth(\wp_3) = (\wp_4 * \wp_3) * 1 = 1,$$

and so $\wp_4 \in ker(\eth)$. It is clear that $1 \in ker(\eth)$ by Proposition 2(i). Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in ker(\eth)$ and $\wp_3 \in ker(\eth)$. Then $1 = \eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)$, and so $\wp_3 \leq \eth(\wp_4)$. It follows from (27) that $\eth(\wp_4) \in ker(\eth)$. Thus $1 = \eth(\eth(\wp_4)) = \eth(\wp_4)$ by the idempotency of $\eth$ which shows that $\wp_4 \in ker(\eth)$. Therefore $ker(\eth)$ is a GE-filter of Y. □

In Theorem 8, if Y is not commutative, then $ker(\eth)$ is not a GE-filter of Y as shown in the following example.

Example 9. *Consider a set* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 8.*

TABLE 8. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ι_5	ι_5
ι_3	1	1	1	ϵ_4	ι_5
ϵ_4	1	1	1	1	1
ι_5	1	ρ_2	ι_3	1	1

Then Y *is a GE-algebra, and it is not commutative since* $(\rho_2 * \iota_3) * \iota_3 = 1 * \iota_3 = \iota_3 \neq \rho_2 = 1 * \rho_2 = (\iota_3 * \rho_2) * \rho_2$. *Define a self mapping* $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{otherwise.} \end{cases}$$

Then $\eth$ *is the idempotent left mapping of* Y, *and its kernel is* $ker(\eth) = \{1, \epsilon_4, \iota_5\}$ *which is not a GE-filter of* Y *since* $\epsilon_4 * \rho_2 = 1 \in ker(\eth)$ *and* $\epsilon_4 \in ker(\eth)$ *but* $\rho_2 \notin ker(\eth)$.

The next example shows that any left mapping may not be idempotent.

Example 10. *Consider a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 9.*

TABLE 9. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	1	1
ι_3	1	1	1	1	1
ϵ_4	1	ρ_2	ι_3	1	1
ι_5	1	ρ_2	ι_3	1	1

Define a self mapping $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{if } \wp_3 = \iota_3. \end{cases}$$

Then $\eth$ *is a left mapping of* Y. *But it is not idempotent since* $\eth(\eth(\iota_3)) = \eth(\rho_2) = 1 \neq \rho_2 = \eth(\iota_3)$.

Theorem 9. *Let* $\eth$ *be a left mapping of* Y*. If* $\eth$ *is idempotent, then*

$$(\forall \wp_3 \in Y)(\eth(\wp_3) = \wp_3 \;\Leftrightarrow\; \wp_3 \in Im(\eth)). \tag{28}$$

$$ker(\eth) \cap Im(\eth) = \{1\}. \tag{29}$$

Proof. Let $\eth$ be an idempotent left mapping of Y. It is clear that if $\eth(\wp_3) = \wp_3$, then $\wp_3 \in Im(\eth)$. Let $\wp_3 \in Im(\eth)$. Then there exists $\wp_4 \in Y$ such that $\eth(\wp_4) = \wp_3$. Hence $\eth(\wp_3) = \eth(\eth(\wp_4)) = \eth(\wp_4) = \wp_3$, and thus (28) is valid. If $\wp_3 \in ker(\eth) \cap Im(\eth)$, then $\eth(\wp_3) = 1$ and $\eth(\wp_4) = \wp_3$ for some $\wp_4 \in Y$. Hence $1 = \eth(\wp_3) = \eth(\eth(\wp_4)) = \eth(\wp_4) = \wp_3$, and so $ker(\eth) \cap Im(\eth) = \{1\}$. □

Lemma 4. *Every commutative GE-algebra* Y *satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)\left(\wp_3 \le \wp_4 \;\Rightarrow\; (\exists \hbar_1 \in Y)(\wp_4 = \hbar_1 * \wp_3)\right). \tag{30}$$

Proof. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \le \wp_4$. Then $\wp_3 * \wp_4 = 1$ and so

$$\wp_4 = 1 * \wp_4 = (\wp_3 * \wp_4) * \wp_4 = (\wp_4 * \wp_3) * \wp_3 = \hbar_1 * \wp_3$$

where $\hbar_1 = \wp_4 * \wp_3$. □

Lemma 5. *Every GE-algebra* Y *satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)\left((\exists \hbar_1 \in Y)(\wp_4 = \hbar_1 * \wp_3) \Rightarrow \wp_3 \le \wp_4\right). \tag{31}$$

Proof. Suppose that $\wp_4 = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Then

$$\wp_3 * \wp_4 = \wp_3 * (\hbar_1 * \wp_3) = \wp_3 * (\hbar_1 * (\wp_3 * \wp_3)) = \wp_3 * (\hbar_1 * 1) = \wp_3 * 1 = 1$$

by (GE1), (GE3) and (6). Hence $\wp_3 \le \wp_4$. □

Proposition 3. *Let* Y *be a commutative GE-algebra which satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)((((\wp_3 * \wp_4) * \wp_4) * \wp_4) = \wp_3 * \wp_4). \tag{32}$$

If $\eth$ *is a left mapping of* Y*, then*

$$(\forall \wp_3 \in Y)(\exists (\wp_4, \wp_5) \in ker(\eth) \times Im(\eth))(\wp_5 = \wp_4 * \wp_3). \tag{33}$$

Proof. Since $\wp_3 \le \eth(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii), it follows from Lemma 4 that $\eth(\wp_3) = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Hence

$$(\eth(\wp_3) * \wp_3) * \wp_3 = ((\hbar_1 * \wp_3) * \wp_3) * \wp_3 = \hbar_1 * \wp_3 = \eth(\wp_3)$$

by (32). If we take $\wp_5 := \eth(\wp_3)$ and $\wp_4 := \eth(\wp_3) * \wp_3$, then $(\wp_4, \wp_5) \in ker(\eth) \times Im(\eth)$ and $\wp_5 = \wp_4 * \wp_3$. □

Proposition 4. *Let* $\eth$ *be a left mapping of* Y*. If* $\eth$ *is idempotent, then*

$$(\forall \wp_3 \in Y)(\exists (\wp_4, \wp_5) \in ker(\eth) \times Im(\eth))(\wp_4 = \wp_5 * \wp_3). \tag{34}$$

Proof. Suppose that $\eth$ is an idempotent left mapping of Y. Then $\eth(\eth(\wp_3)) = \eth(\wp_3)$ for all $\wp_3 \in Y$, and so

$$\eth(\eth(\eth(\wp_3)) * \wp_3) = \eth(\eth(\wp_3)) * \eth(\wp_3) = 1.$$

Hence $\eth(\wp_3) * \wp_3 = \eth(\eth(\wp_3)) * \wp_3 \in ker(\eth)$. It follows that $\wp_5 * \wp_3 = \wp_4$ for some $\wp_4 \in ker(\eth)$ and $\wp_5 := \eth(\wp_3) \in Im(\eth)$. □

Proposition 5. *Every left mapping $\eth$ of a commutative GE-algebra satisfies the condition* (34).

Proof. Let $\eth$ be a left mapping of a commutative GE-algebra Y. Since $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii), it follows from Lemma 4 that $\eth(\wp_3) = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Hence

$$\eth(\eth(\wp_3)) = \eth(\hbar_1 * \wp_3) = \hbar_1 * \eth(\wp_3) = \hbar_1 * (\hbar_1 * \wp_3) = \hbar_1 * \wp_3 = \eth(\wp_3)$$

for all $\wp_3 \in Y$ by (7). Hence $\eth$ is idempotent. Using Proposition 4, we know that (34) is valid. □

Denote by $L(Y)$ and $IL(Y)$ the set of all left mappings of Y and the set of all idempotent left mappings of Y, respectively. Define an operation "$\circledast$" on $L(Y)$ by $(\eth \circledast \xi)(\wp_3) = \eth(\wp_3) * \xi(\wp_3)$ for all $\wp_3 \in Y$ and $\eth, \xi \in L(Y)$.

Proposition 6. *$L(Y)$ is closed under the function composition $\circ$, that is, if $\eth$ and ξ are left mappings of Y, then $\eth \circ \xi$ is also a left mapping of Y.*

Proof. Let $\eth, \xi \in L(Y)$ and $\wp_3, \wp_4 \in Y$. Then

$$(\eth \circ \xi)(\wp_3 * \wp_4) = \eth(\xi(\wp_3 * \wp_4)) = \eth(\wp_3 * \xi(\wp_4)) = \wp_3 * \eth(\xi(\wp_4)) = \wp_3 * (\eth \circ \xi)(\wp_4),$$

and so $\eth \circ \xi$ is a left mapping of Y. □

Theorem 10. *$(L(Y), \circ)$ is a semigroup and $IL(Y)$ is a subsemigroup of $L(Y)$.*

Proof. Straightforward. □

The following example shows that $L(Y)$ is not closed under the operation "$\circledast$", that is, there are two left mappings $\eth$ and ξ of Y such that $\eth \circledast \xi$ is not a left mapping of Y.

Example 11. *Consider a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 10.*
Define self mappings $\eth$ and ξ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \iota_3. \end{cases}$$

TABLE 10. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	ι_5	1	ι_5
ι_3	1	ρ_2	1	1	1
ϵ_4	1	ρ_2	1	1	1
ι_5	1	ρ_2	1	1	1

$$\xi : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \iota_3 & \text{if } \wp_3 = \iota_3, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ and ξ are left mappings of Y and $\eth \circledast \xi$ is given as follows:

$$\eth \circledast \xi : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \iota_3, \epsilon_4\} \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

We can observe that $\eth \circledast \xi$ is not a left mapping of Y since

$$(\eth\circledast\xi)(\rho_2*\iota_3) = (\eth\circledast\xi)(\iota_5) = \iota_5 \neq 1 = \rho_2*(\epsilon_4*\iota_3) = \rho_2*(\eth(\iota_3)*(\xi(3)) = \rho_2*(\eth\circledast\xi)(\iota_3).$$

We investigate the conditions under which $L(Y)$ can be closed with respect to the operation "$\circledast$".

Theorem 11. *Let Y be a belligerent GE-algebra. For every $\eth, \xi \in L(Y)$, we have*

(i) *$\eth \circledast \xi \in L(Y)$.*

(ii) *If $\eth \circ \xi = \xi \circ \eth$ and ξ is idempotent, then $\eth \circledast \xi \in IL(Y)$.*

Proof. (i) For every $\wp_3, \wp_4 \in Y$, we get

$$\begin{aligned}(\eth \circledast \xi)(\wp_3 * \wp_4) &= \eth(\wp_3 * \wp_4) * \xi(\wp_3 * \wp_4) = (\wp_3 * \eth(\wp_4)) * (\wp_3 * \xi(\wp_4)) \\ &= \wp_3 * (\eth(\wp_4) * \xi(\wp_4)) = \wp_3 * (\eth \circledast \xi)(\wp_4).\end{aligned}$$

Hence $\eth \circledast \xi \in L(Y)$.

(ii) For every $\wp_3 \in Y$, we have

$$\begin{aligned}&((\eth \circledast \xi) \circ (\eth \circledast \xi))(\wp_3) = (\eth \circledast \xi)((\eth \circledast \xi)(\wp_3)) = (\eth \circledast \xi)(\eth(\wp_3) * \xi(\wp_3)) \\ &= \eth(\eth(\wp_3) * \xi(\wp_3)) * \xi(\eth(\wp_3) * \xi(\wp_3)) = (\eth(\wp_3) * \eth(\xi(\wp_3))) * (\eth(\wp_3) * \xi(\xi(\wp_3))) \\ &= (\eth(\wp_3) * \xi(\eth(\wp_3))) * (\eth(\wp_3) * \xi(\wp_3)) = \xi(\eth(\wp_3) * \eth(\wp_3)) * (\eth(\wp_3) * \xi(\wp_3)) \\ &= \xi(1) * (\eth(\wp_3) * \xi(\wp_3)) = 1 * (\eth(\wp_3) * \xi(\wp_3)) = (\eth(\wp_3) * \xi(\wp_3)) = (\eth \circledast \xi)(\wp_3).\end{aligned}$$

and thus $\eth \circledast \xi \in IL(Y)$. □

Proposition 7. *Let $\eth, \xi \in L(Y)$ satisfy $(\xi \circledast \eth)(\wp_3) = 1$ for all $\wp_3 \in Y$. If Y is antisymmetric and $\eth$ is idempotent, then $Im(\eth) \subseteq Im(\xi)$.*

Proof. If $\wp_4 \in Im(\eth)$, then $\eth(\wp_4) = \wp_4$ by (28) and hence

$$\xi(\wp_4) * \wp_4 = \xi(\wp_4) * \eth(\wp_4) = (\xi \circledast \eth)(\wp_4) = 1,$$

that is, $\xi(\wp_4) \leq \wp_4$. Since $\wp_4 \leq \xi(\wp_4)$ by Proposition 2(ii) and Y is antisymmetric, we have $\wp_4 = \xi(\wp_4) \in Im(\xi)$. Thus $Im(\eth) \subseteq Im(\xi)$. □

Theorem 12. *For every $\eth, \xi \in L(Y)$, we have*

(i) *If $\eth \circ \xi = \xi \circ \eth$, $Im(\eth) \subseteq Im(\xi)$ and ξ is idempotent, then $\xi \circledast \eth$ is constant on Y with the value 1.*
(ii) *If $\eth$ is idempotent, then $ker(\xi) \cap Im(\eth) \subseteq Im(\xi \circledast \eth)$.*

Proof. (i) Assume that $\eth \circ \xi = \xi \circ \eth$, $Im(\eth) \subseteq Im(\xi)$ and ξ is idempotent. Then Theorem 9 yields $(\xi \circ \eth)(\wp_3) = \eth(\wp_3)$ for all $\wp_3 \in Y$. Hence

$$\begin{aligned}(\xi \circledast \eth)(\wp_3) &= \xi(\wp_3) * \eth(\wp_3) = \xi(\wp_3) * (\xi \circ \eth)(\wp_3) \\ &= \xi(\wp_3) * (\eth \circ \xi)(\wp_3) = \eth(\xi(\wp_3) * \xi(\wp_3)) \\ &= \eth(1) = 1\end{aligned}$$

for all $\wp_3 \in Y$.

(ii) Suppose that $\eth$ is idempotent and let $\wp_4 \in ker(\xi) \cap Im(\eth)$. Then $\xi(\wp_4) = 1$ and $\eth(\wp_3) = \wp_4$ for some $\wp_3 \in Y$. It follows that

$$\wp_4 = \eth(\wp_3) = 1 * \eth(\eth(\wp_3)) = \xi(\wp_4) * \eth(\wp_4) = (\xi \circledast \eth)(\wp_4) \in Im(\xi \circledast \eth).$$

Thus $ker(\xi) \cap Im(\eth) \subseteq Im(\xi \circledast \eth)$.

□

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