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On a Class of Difference Equations System of Fifth-Order

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Abstract

In the current paper, we investigate the following new class of system of difference equations

$$u_{n+1} = f^{-1}\left(g\left(v_{n-1}\right)\frac{A_1 f\left(u_{n-2}\right)+B_1 g\left(v_{n-4}\right)}{C_1 f\left(u_{n-2}\right)+D_1 g\left(v_{n-4}\right)}\right),$$
$$v_{n+1} = g^{-1}\left(f\left(u_{n-1}\right)\frac{A_2 g\left(v_{n-2}\right)+B_2 f\left(u_{n-4}\right)}{C_2 g\left(v_{n-2}\right)+D_2 f\left(u_{n-4}\right)}\right), \ n\in\mathbb{N}_0,$$

where the initial conditions u_{-p}, v_{-p}, for $p=\overline{0,4}$ are real numbers, the parameters A_r, B_r, C_r, D_r, for $r\in\{1,2\}$ are real numbers, $A_r^2+B_r^2\neq 0\neq C_r^2+D_r^2$, for $r\in\{1,2\}$, f and g are continuous and strictly monotone functions, $f(\mathbb{R})=\mathbb{R}$, $g(\mathbb{R})=\mathbb{R}$, $f(0)=0$, $g(0)=0$. In addition, we solve aforementioned general two dimensional system of difference equations of fifth-order in explicit form. Moreover, we obtain the solutions of mentioned system according to whether the parameters being zeros or not. Finally, we present an interesting application.

1. Introduction

The notation of $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, stand for the set of natural, non-negative integer, integer and real number, respectively. If $\gamma,\delta\in\mathbb{Z}$, $\gamma\leq\delta$ the notation $\beta=\overline{\gamma,\delta}$ means $\{\beta\in\mathbb{Z}:\gamma\leq\beta\leq\delta\}$.

Difference equations emerge from mathematical models of physical events, numerical solutions of differential equations or generation functions. There has been an intense interest in nonlinear difference equations. Some mathematicians are interested in nonlinear difference equations in these days in [1], [2], [3], [4], [5]. In addition, systems of difference equations are studied by some authors in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

One of the interesting difference equations is

$$w_{n+2}=\Phi w_{n+1}+\Psi w_n, \ n\in\mathbb{N}_0, \tag{1.1}$$

where the initial values w_0, w_1 and the parameters Φ and Ψ are real numbers. Equation (1.1) is solved by De Moivre in [22].

The solution of (1.1) is given by

$$w_n=\frac{\left(w_1-\lambda_2 w_0\right)\lambda_1^n-\left(w_1-\lambda_1 w_0\right)\lambda_2^n}{\lambda_1-\lambda_2}, \ n\in\mathbb{N}_0, \tag{1.2}$$

when $\Psi\neq 0$ and $\Phi^2+4\Psi\neq 0$,

$$w_n=\left(\left(w_1-\lambda_1 w_0\right)n+\lambda_1 w_0\right)\lambda_1^{n-1}, \ n\in\mathbb{N}_0, \tag{1.3}$$

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when $\Psi \neq 0$ and $\Phi^2 + 4\Psi = 0$, where λ_1 and λ_2 are the roots of the polynomial $P(\lambda) = \lambda^2 - \Phi\lambda - \Psi = 0$. Also, the roots of characteristic equation are $\lambda_{1,2} = \frac{\Phi \pm \sqrt{\Phi^2 + 4\Psi}}{2}$.

Another well-known difference equation, that is Riccati difference equation, is given by

$$w_{n+1} = \frac{\alpha w_n + \beta}{\gamma w_n + \delta},\ n \in \mathbb{N}_0, \tag{1.4}$$

for $\gamma \neq 0$, $\alpha\delta \neq \beta\gamma$, where the initial condition w_0 and the parameters α, β, γ, δ are real numbers. Equation (1.4) is reduced to equation (1.1) by using the convenient transformation.

There are general forms of the difference equations reduced to equation (1.4) by changing variables in literature. For example, the following difference equation

$$w_{n+1} = \alpha w_{n-k} + \frac{\delta w_{n-k} w_{n-k-l}}{\beta x_{n-k-l} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{1.5}$$

where k, l are fixed natural numbers, the parameters α, β, γ, δ and the initial conditions w_{-i}, $i = \overline{1, k+l}$ are real numbers and $\beta^2 + \gamma^2 \neq 0$, is solved in [23].

Some authors solved special cases of equation (1.5) in [24], [25], [26], [27], [28]. A different form of equation (1.5) continued to be studied in the literature [29], [30], [31].

In an earlier paper, Elsayed et al., deal with the following difference equation

$$u_{n+1} = \gamma_0 u_{n-1} + \frac{\gamma_1 u_{n-1} u_{n-4}}{\gamma_2 u_{n-4} + \gamma_3 u_{n-2}},\ n \in \mathbb{N}_0, \tag{1.6}$$

where the initial values u_{-p}, for $p = \overline{0,4}$ are arbitrary positive real numbers and the coefficients γ_l, for $l = \overline{0,3}$ are real numbers in [32].

Recently, Stević et al., investigate the following difference equations

$$x_{n+1} = \Phi^{-1}\left(\Phi(x_{n-1}) \frac{\alpha\Phi(x_{n-2}) + \beta\Phi(x_{n-4})}{\gamma\Phi(x_{n-2}) + \delta\Phi(x_{n-4})}\right),\ n \in \mathbb{N}_0, \tag{1.7}$$

where the initial values x_{-p}, for $p = \overline{0,4}$ and the parameters α, β, γ and δ are real numbers in [33]. Note that, the different form of equation (1.6) is equation (1.7).

Equations (1.7) can be expand in various ways. For instance, increasing order, adding periodic coefficients, expanding the dimensional, etc.

In this paper, we are interested in the following general two dimensional form of equation (1.7)

$$\begin{aligned} u_{n+1} &= f^{-1}\left(g(v_{n-1}) \frac{A_1 f(u_{n-2}) + B_1 g(v_{n-4})}{C_1 f(u_{n-2}) + D_1 g(v_{n-4})}\right), \\ v_{n+1} &= g^{-1}\left(f(u_{n-1}) \frac{A_2 g(v_{n-2}) + B_2 f(u_{n-4})}{C_2 g(v_{n-2}) + D_2 f(u_{n-4})}\right),\ n \in \mathbb{N}_0, \end{aligned} \tag{1.8}$$

where the initial conditions u_{-p}, v_{-p}, for $p = \overline{0,4}$ are real numbers, the parameters A_r, B_r, C_r, D_r, for $r \in \{1,2\}$, are real numbers, $A_r^2 + B_r^2 \neq 0 \neq C_r^2 + D_r^2$, for $r \in \{1,2\}$, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. We obtain the solutions of system (1.8) in explicit form according to states of parameters by changing the variable. In addition, we present an application, which indicates that some conclusions in [32] are not correct.

2. Explicit-form solutions of system (1.8)

In this section, we investigate the solutions of system (1.8) in explicit-form.

Theorem 2.1. *Assume that A_r, B_r, C_r, $D_r \in \mathbb{R}$, for $r \in \{1,2\}$, $A_1^2 + B_1^2 \neq 0 \neq C_1^2 + D_1^2$, $A_2^2 + B_2^2 \neq 0 \neq C_2^2 + D_2^2$, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. So, the general system (1.8) is solvable in explicit-form.*

Proof. If at least one of the initial values $u_{-p}=0$ or $v_{-p}=0$, for $p=\overline{0,4}$, then the solution of system (1.8) is not defined. Moreover, assume that $u_{n_0}=0$ for some $n_0\in\mathbb{N}_0$. Then from system (1.8) we have $v_{n_0+2}=0$. These facts along with (1.8) imply that v_{n_0+5} is not defined. Similarly, suppose that $v_{n_0}=0$ for some $n_0\in\mathbb{N}_0$. Then from system (1.8) we have $u_{n_0+2}=0$. These facts along with (1.8) imply that u_{n_0+5} is not defined. Hence, for every well-defined solution of system (1.8), we have

$$u_nv_n\neq 0,\ n\geq -4. \tag{2.1}$$

From (2.1) and the conditions of the theorem we have

$$f(u_n)\neq 0,\ g(v_n)\neq 0,\ n\geq -4.$$

Now, we examine the solutions of system (1.8) for two cases:

Case 1**:** First, suppose that $A_1D_1-B_1C_1\neq 0$, $A_2D_2-B_2C_2\neq 0$ and $C_1\neq 0\neq C_2$. Let

$$x_n=\frac{f(u_n)}{g(v_{n-2})},\ y_n=\frac{g(v_n)}{f(u_{n-2})},\ n\geq -2. \tag{2.2}$$

From (1.8) and monotonicity of f and g, we obtain

$$\begin{aligned} f(u_{n+1}) &= g(v_{n-1})\frac{A_1f(u_{n-2})+B_1g(v_{n-4})}{C_1f(u_{n-2})+D_1g(v_{n-4})},\\ g(v_{n+1}) &= f(u_{n-1})\frac{A_2g(v_{n-2})+B_2f(u_{n-4})}{C_2g(v_{n-2})+D_2f(u_{n-4})},\ n\in\mathbb{N}_0. \end{aligned} \tag{2.3}$$

By using the change of variables (2.2) in (2.3) we get

$$x_{n+1}=\frac{A_1x_{n-2}+B_1}{C_1x_{n-2}+D_1},\ y_{n+1}=\frac{A_2y_{n-2}+B_2}{C_2y_{n-2}+D_2},\ n\in\mathbb{N}_0. \tag{2.4}$$

Let

$$k_m^{(j)}=x_{3m+j},\ l_m^{(j)}=y_{3m+j},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.5}$$

Then from (2.4) and (2.5) we obtain

$$k_{m+1}^{(j)}=\frac{A_1k_m^{(j)}+B_1}{C_1k_m^{(j)}+D_1},\ l_{m+1}^{(j)}=\frac{A_2l_m^{(j)}+B_2}{C_2l_m^{(j)}+D_2}, \tag{2.6}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. The equations in (2.6) are named Riccati type difference equations in literature.

Let

$$k_m^{(j)}=\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+p_j,\ l_m^{(j)}=\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+h_j,\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}, \tag{2.7}$$

for some $p_j,h_j\in\mathbb{R}$, $j\in\{-2,-1,0\}$.

From (2.6) and (2.7) we have

$$\left(\frac{z_{m+2}^{(j)}}{z_{m+1}^{(j)}}+p_j\right)\left(C_1\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+C_1p_j+D_1\right)-\left(A_1\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+A_1p_j+B_1\right)=0,$$

$$\left(\frac{t_{m+2}^{(j)}}{t_{m+1}^{(j)}}+h_j\right)\left(C_2\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+C_2h_j+D_2\right)-\left(A_2\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+A_2h_j+B_2\right)=0,$$

for some $m \in \mathbb{N}_0$, $j \in \{-2,-1,0\}$.

Let

$$p_j = -\frac{D_1}{C_1},\ h_j = -\frac{D_2}{C_2},\ j \in \{-2,-1,0\}.$$

Then, we get

$$\begin{aligned} C_1^2 z_{m+2}^{(j)} - C_1 (A_1 + D_1) z_{m+1}^{(j)} + (A_1 D_1 - B_1 C_1) z_m^{(j)} &= 0, \\ C_2^2 t_{m+2}^{(j)} - C_2 (A_2 + D_2) t_{m+1}^{(j)} + (A_2 D_2 - B_2 C_2) t_m^{(j)} &= 0, \end{aligned} \tag{2.8}$$

for $m \in \mathbb{N}_0$, $j \in \{-2,-1,0\}$.

Assume that $\Delta_1 := (A_1 + D_1)^2 - 4(A_1 D_1 - B_1 C_1) \neq 0$, $\Delta_2 := (A_2 + D_2)^2 - 4(A_2 D_2 - B_2 C_2) \neq 0$. Then by employing formula (1.2), we have

$$\begin{aligned} z_m^{(j)} &= \frac{\left(z_1^{(j)} - \lambda_2 z_0^{(j)}\right)\lambda_1^m - \left(z_1^{(j)} - \lambda_1 z_0^{(j)}\right)\lambda_2^m}{\lambda_1 - \lambda_2}, \\ t_m^{(j)} &= \frac{\left(t_1^{(j)} - \widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^m - \left(t_1^{(j)} - \widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^m}{\widehat{\lambda}_1 - \widehat{\lambda}_2}, \end{aligned} \tag{2.9}$$

for $m \in \mathbb{N}_0$, $j \in \{-2,-1,0\}$, where $\lambda_{1,2} = \frac{(A_1+D_1)\pm\sqrt{\Delta_1}}{2C_1}$, $\widehat{\lambda}_{1,2} = \frac{(A_2+D_2)\pm\sqrt{\Delta_2}}{2C_2}$. Equations in (2.9) are the general solutions to equations in (2.8).

By using (2.9) in (2.7), we obtain

$$\begin{aligned} k_m^{(j)} &= \frac{\left(z_1^{(j)} - \lambda_2 z_0^{(j)}\right)\lambda_1^{m+1} - \left(z_1^{(j)} - \lambda_1 z_0^{(j)}\right)\lambda_2^{m+1}}{\left(z_1^{(j)} - \lambda_2 z_0^{(j)}\right)\lambda_1^m - \left(z_1^{(j)} - \lambda_1 z_0^{(j)}\right)\lambda_2^m} - \frac{D_1}{C_1} \\ &= \frac{\left(k_0^{(j)} + \frac{D_1}{C_1} - \lambda_2\right)\lambda_1^{m+1} - \left(k_0^{(j)} + \frac{D_1}{C_1} - \lambda_1\right)\lambda_2^{m+1}}{\left(k_0^{(j)} + \frac{D_1}{C_1} - \lambda_2\right)\lambda_1^m - \left(k_0^{(j)} + \frac{D_1}{C_1} - \lambda_1\right)\lambda_2^m} - \frac{D_1}{C_1}, \\ l_m^{(j)} &= \frac{\left(t_1^{(j)} - \widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^{m+1} - \left(t_1^{(j)} - \widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^{m+1}}{\left(t_1^{(j)} - \widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^m - \left(t_1^{(j)} - \widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^m} - \frac{D_2}{C_2} \\ &= \frac{\left(l_0^{(j)} + \frac{D_2}{C_2} - \widehat{\lambda}_2\right)\widehat{\lambda}_1^{m+1} - \left(l_0^{(j)} + \frac{D_2}{C_2} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(l_0^{(j)} + \frac{D_2}{C_2} - \widehat{\lambda}_2\right)\widehat{\lambda}_1^m - \left(l_0^{(j)} + \frac{D_2}{C_2} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^m} - \frac{D_2}{C_2}, \end{aligned}$$

for $m \in \mathbb{N}_0$, $j \in \{-2,-1,0\}$, from the last equalities with (2.5) we have

$$\begin{aligned} x_{3m+j} &= \frac{\left(x_j + \frac{D_1}{C_1} - \lambda_2\right)\lambda_1^{m+1} - \left(x_j + \frac{D_1}{C_1} - \lambda_1\right)\lambda_2^{m+1}}{\left(x_j + \frac{D_1}{C_1} - \lambda_2\right)\lambda_1^m - \left(x_j + \frac{D_1}{C_1} - \lambda_1\right)\lambda_2^m} - \frac{D_1}{C_1}, \\ y_{3m+j} &= \frac{\left(y_j + \frac{D_2}{C_2} - \widehat{\lambda}_2\right)\widehat{\lambda}_1^{m+1} - \left(y_j + \frac{D_2}{C_2} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(y_j + \frac{D_2}{C_2} - \widehat{\lambda}_2\right)\widehat{\lambda}_1^m - \left(y_j + \frac{D_2}{C_2} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^m} - \frac{D_2}{C_2}, \end{aligned} \tag{2.10}$$

for $m \in \mathbb{N}_0$, $j \in \{-2,-1,0\}$.

From (2.2), we get

$$\begin{aligned} f\left(u_n\right) &= x_n g\left(v_{n-2}\right)=x_n y_{n-2} f\left(u_{n-4}\right)=x_n y_{n-2} x_{n-4} g\left(v_{n-6}\right)=x_n y_{n-2} x_{n-4} y_{n-6} f\left(u_{n-8}\right) \\ &= x_n y_{n-2} x_{n-4} y_{n-6} x_{n-8} g\left(v_{n-10}\right)=x_n y_{n-2} x_{n-4} y_{n-6} x_{n-8} y_{n-10} f\left(u_{n-12}\right), \ n \geq 8, \\ g\left(v_n\right) &= y_n f\left(u_{n-2}\right)=y_n x_{n-2} g\left(v_{n-4}\right)=y_n x_{n-2} y_{n-4} f\left(u_{n-6}\right)=y_n x_{n-2} y_{n-4} x_{n-6} g\left(v_{n-8}\right) \\ &= y_n x_{n-2} y_{n-4} x_{n-6} y_{n-8} f\left(u_{n-10}\right)=y_n x_{n-2} y_{n-4} x_{n-6} y_{n-8} x_{n-10} g\left(v_{n-12}\right), n \geq 8. \end{aligned} \tag{2.11}$$

From (2.11), we have

$$\begin{aligned} f\left(u_{12m+i}\right) &= x_{12m+i} y_{12m+i-2} x_{12m+i-4} y_{12m+i-6} x_{12m+i-8} y_{12m+i-10} f\left(u_{12(m-1)+i}\right), \\ g\left(v_{12m+i}\right) &= y_{12m+i} x_{12m+i-2} y_{12m+i-4} x_{12m+i-6} y_{12m+i-8} x_{12m+i-10} g\left(v_{12(m-1)+i}\right), \end{aligned} \tag{2.12}$$

for $m \in \mathbb{N}_0$, $i = \overline{8,19}$. Multiplying the equalities which are obtained from (2.12), from 0 to m, it follows that

$$\begin{aligned} f\left(u_{12m+3s+p}\right) &= f\left(u_{3s+p-12}\right) \prod_{r=0}^{m}\left(x_{12r+3s+p} y_{12r+3s+p-2} x_{12r+3s+p-4} y_{12r+3s+p-6} x_{12r+3s+p-8} y_{12r+3s+p-10}\right), \\ g\left(v_{12m+3s+p}\right) &= g\left(v_{3s+p-12}\right) \prod_{r=0}^{m}\left(y_{12r+3s+p} x_{12r+3s+p-2} y_{12r+3s+p-4} x_{12r+3s+p-6} y_{12r+3s+p-8} x_{12r+3s+p-10}\right), \end{aligned} \tag{2.13}$$

for $m \in \mathbb{N}_0$, $s = \overline{3,6}$, $p = \overline{-1,1}$. From (2.13), we obtain

$$\begin{aligned} f\left(u_{12m+3s+p}\right) &= f\left(u_{3s+p-12}\right) \prod_{r=0}^{m}\Big(x_{3\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+p-3\lfloor\frac{p+2}{3}\rfloor} y_{3\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+p-2-3\lfloor\frac{p}{3}\rfloor} \\ &\quad \times x_{3\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+p+2+3\lfloor\frac{p-2}{3}\rfloor} y_{3\left(4r+s-1+\lfloor\frac{p-1}{3}\rfloor\right)+p-3-3\lfloor\frac{p-1}{3}\rfloor} \\ &\quad \times x_{3\left(4r+s-1+\lfloor\frac{p-3}{3}\rfloor\right)+p-5-3\lfloor\frac{p-3}{3}\rfloor} y_{3\left(4r+s-1+\lfloor\frac{p-5}{3}\rfloor\right)+p-7-3\lfloor\frac{p-5}{3}\rfloor}\Big), \\ g\left(v_{12m+3s+p}\right) &= g\left(v_{3s+p-12}\right) \prod_{r=0}^{m}\Big(y_{3\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+p-3\lfloor\frac{p+2}{3}\rfloor} x_{3\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+p-2-3\lfloor\frac{p}{3}\rfloor} \\ &\quad \times y_{3\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+p+2+3\lfloor\frac{p-2}{3}\rfloor} x_{3\left(4r+s-1+\lfloor\frac{p-1}{3}\rfloor\right)+p-3-3\lfloor\frac{p-1}{3}\rfloor} \\ &\quad \times y_{3\left(4r+s-1+\lfloor\frac{p-3}{3}\rfloor\right)+p-5-3\lfloor\frac{p-3}{3}\rfloor} x_{3\left(4r+s-1+\lfloor\frac{p-5}{3}\rfloor\right)+p-7-3\lfloor\frac{p-5}{3}\rfloor}\Big), \end{aligned} \tag{2.14}$$

for $m \in \mathbb{N}_0$, $s = \overline{3,6}$, $p = \overline{-1,1}$. By substituting the equations in (2.10) into (2.14) and by using equations in (2.2), we have

$$\begin{aligned} & u_{12m+3s+p} \\ = \ & f^{-1}\Bigg[f\left(u_{3s+p-12}\right) \\ & \times \prod_{r=0}^{m}\left(\left(\frac{\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-\frac{D_1}{C_1}\right)\right. \\ & \times \left(\frac{\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-\frac{D_2}{C_2}\right) \end{aligned}$$

$$\times\left(\frac{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-\frac{D_1}{C_1}\right) \tag{2.15}$$

$$\times\left(\frac{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)$$

$$\times\left(\frac{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)$$

$$\times\left(\frac{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)\Bigg)\Bigg],$$

$$v_{12m+3s+p}=g^{-1}\Bigg[g\left(v_{3s+p-12}\right)$$

$$\times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-\frac{D_2}{C_2}\right)$$

$$\times\left(\frac{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-\frac{D_1}{C_1}\right)$$

$$\times\left(\frac{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-\frac{D_2}{C_2}\right) \tag{2.16}$$

$$\times\left(\frac{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)$$

$$\times\left(\frac{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)$$

$$\times\left(\frac{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)\bigg)\bigg],$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. The formulas in (2.15) and (2.16) are the solutions of system (1.8) if $\Delta_1\neq 0\neq\Delta_2$. Assume that $\Delta_1=(A_1+D_1)^2-4(A_1D_1-B_1C_1)=0$ and $\Delta_2=(A_2+D_2)^2-4(A_2D_2-B_2C_2)=0$. So, by employing formula (1.3), we obtain

$$\begin{aligned}
z_m^{(j)} &= \left(\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)m+\lambda_1 z_0^{(j)}\right)\lambda_1^{m-1},\\
t_m^{(j)} &= \left(\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)m+\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_1^{m-1},
\end{aligned}\tag{2.17}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$, where

$$\lambda_1=\frac{A_1+D_1}{2C_1}\neq 0,\ \widehat{\lambda}_1=\frac{A_2+D_2}{2C_2}\neq 0.$$

Note that equations in (2.17) are the solutions to the system (2.8) if $\Delta_1=0=\Delta_2$. From (2.7) and (2.17), we get

$$\begin{aligned}
k_m^{(j)} &= \frac{\left(\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)(m+1)+\lambda_1 z_0^{(j)}\right)\lambda_1}{\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)m+\lambda_1 z_0^{(j)}}-\frac{D_1}{C_1}\\
&= \frac{\left(\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)m+\lambda_1}-\frac{D_1}{C_1},\\
l_m^{(j)} &= \frac{\left(\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)(m+1)+\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_1}{\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)m+\widehat{\lambda}_1 t_0^{(j)}}-\frac{D_2}{C_2}\\
&= \frac{\left(\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(m+1)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)m+\widehat{\lambda}_1}-\frac{D_2}{C_2},
\end{aligned}\tag{2.18}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. By using (2.5) in (2.18), we obtain

$$\begin{aligned}
x_{3m+j} &= \frac{\left(\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)m+\lambda_1}-\frac{D_1}{C_1},\\
y_{3m+j} &= \frac{\left(\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(m+1)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)m+\widehat{\lambda}_1}-\frac{D_2}{C_2},
\end{aligned}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. From (2.14), we have

$$u_{12m+3s+p} = f^{-1}\Bigg[f(u_{3s+p-12})\times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor+1\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\lambda_1}-\frac{D_1}{C_1}\right)$$
$$\times\left(\frac{\left(\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(4r+s+\lfloor\frac{p}{3}\rfloor+1)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(4r+s+\lfloor\frac{p}{3}\rfloor)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)$$
$$\times\left(\frac{\left(\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor+1\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\lambda_1}-\frac{D_1}{C_1}\right) \tag{2.19}$$
$$\times\left(\frac{\left(\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)$$
$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)$$
$$\times\left(\frac{\left(\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)\Bigg)\Bigg],$$

$$v_{12m+3s+p} = g^{-1}\Bigg[g(v_{3s+p-12})$$
$$\times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor+1\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)$$
$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)(4r+s+\lfloor\frac{p}{3}\rfloor+1)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)(4r+s+\lfloor\frac{p}{3}\rfloor)+\lambda_1}-\frac{D_1}{C_1}\right)$$

$$\times\left(\frac{\left(\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor+1\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right) \tag{2.20}$$
$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)$$
$$\times\left(\frac{\left(\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)$$
$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)\Bigg)\Bigg],$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$, if $\Delta_1=0=\Delta_2$.
Now assume that $C_1=0=C_2$, $D_1\neq 0\neq D_2$. In this case, equations in (2.4) turn into

$$x_{n+1}=\frac{A_1}{D_1}x_{n-2}+\frac{B_1}{D_1},\ y_{n+1}=\frac{A_2}{D_2}y_{n-2}+\frac{B_2}{D_2},\ n\in\mathbb{N}_0.$$

Thus,

$$k_{m+1}^{(j)}=\frac{A_1}{D_1}k_m^{(j)}+\frac{B_1}{D_1},\ l_{m+1}^{(j)}=\frac{A_2}{D_2}l_m^{(j)}+\frac{B_2}{D_2},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.21}$$

If $A_1=D_1$ and $A_2=D_2$ then from (2.21), we have

$$k_m^{(j)}=\frac{B_1}{D_1}m+k_0^{(j)},\ l_m^{(j)}=\frac{B_2}{D_2}m+l_0^{(j)},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\},$$

so

$$x_{3m+j}=\frac{B_1}{D_1}m+x_j,\ y_{3m+j}=\frac{B_2}{D_2}m+y_j,\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.22}$$

From (2.2), (2.14) and (2.22), we get

$$u_{12m+3s+p} = f^{-1}\Bigg[f\left(u_{3s+p-12}\right) \tag{2.23}$$
$$\times\prod_{r=0}^{m}\Bigg(\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}\right)$$
$$\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}\right)$$

$$\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}\right)\Bigg)\Bigg],$$

$$\begin{aligned} v_{12m+3s+p} &= g^{-1}\Bigg[g\left(v_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}\right)\\ &\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}\right)\\ &\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}\right)\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}\right)\\ &\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}\right)\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}\right)\Bigg)\Bigg], \end{aligned} \tag{2.24}$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. Hence, the formulas in (2.24) and (2.24) are solutions of system (1.8) in this case. Suppose that $A_1\neq D_1$ and $A_2\neq D_2$. By using (2.21), we get

$$\begin{aligned} k_m^{(j)} &= \left(\frac{A_1}{D_1}\right)^m k_0^{(j)}+\frac{B_1}{A_1-D_1}\left(\left(\frac{A_1}{D_1}\right)^m-1\right),\\ l_m^{(j)} &= \left(\frac{A_2}{D_2}\right)^m l_0^{(j)}+\frac{B_2}{A_2-D_2}\left(\left(\frac{A_2}{D_2}\right)^m-1\right), \end{aligned}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. That is,

$$\begin{aligned} x_{3m+j} &= \left(\frac{A_1}{D_1}\right)^m\left(x_j+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1},\\ y_{3m+j} &= \left(\frac{A_2}{D_2}\right)^m\left(y_j+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}, \end{aligned} \tag{2.25}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. From (2.2), (2.14) and (2.25), we get

$$\begin{aligned} u_{12m+3s+p} &= f^{-1}\Bigg[f\left(u_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p+2}{3}\rfloor}\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p}{3}\rfloor}\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-2}{3}\rfloor}\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\Bigg)\Bigg], \end{aligned} \tag{2.26}$$

$$\begin{aligned} v_{12m+3s+p} &= g^{-1}\Bigg[g(v_{3s+p-12})\times\prod_{r=0}^{m}\Bigg(\Bigg(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p+2}{3}\rfloor}\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\Bigg) \\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p}{3}\rfloor}\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right) \\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-2}{3}\rfloor}\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right) \\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right) \\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right) \\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\Bigg)\Bigg], \end{aligned} \tag{2.27}$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. Then, the solutions of system (1.8) are given by the equations in (2.26) and (2.27) in this case.
Case 2**:** Assume that $A_1D_1=B_1C_1$, $A_2D_2=B_2C_2$. If $A_1=0$ and $B_1\neq 0$. Then $C_1=0$ and $D_1\neq 0$. If $A_2=0$ and $B_2\neq 0$. Then $C_2=0$ and $D_2\neq 0$. From system (1.8), we have

$$u_{n+1}=f^{-1}\left(\frac{B_1}{D_1}g(v_{n-1})\right),\ v_{n+1}=g^{-1}\left(\frac{B_2}{D_2}f(u_{n-1})\right),\ n\in\mathbb{N}_0. \tag{2.28}$$

From (2.28) we easily get

$$u_n=f^{-1}\left(\frac{B_1B_2}{D_1D_2}f(u_{n-4})\right),\ v_n=g^{-1}\left(\frac{B_1B_2}{D_1D_2}g(v_{n-4})\right),\ n\geq 3. \tag{2.29}$$

By using (2.29), we obtain

$$u_{4m+i}=f^{-1}\left(\left(\frac{B_1B_2}{D_1D_2}\right)^{m+1}f(u_{i-4})\right),\ v_{4m+i}=g^{-1}\left(\left(\frac{B_1B_2}{D_1D_2}\right)^{m+1}g(v_{i-4})\right), \tag{2.30}$$

$m\in\mathbb{N}_0$, $i=\overline{3,6}$.
If $A_1\neq 0$ and $B_1=0$. Then $D_1=0$ from which it follows that $C_1\neq 0$. If $A_2\neq 0$ and $B_2=0$. Then $D_2=0$ from which it follows that $C_2\neq 0$. From system (1.8), we get

$$u_{n+1}=f^{-1}\left(\frac{A_1}{C_1}g(v_{n-1})\right),\ v_{n+1}=g^{-1}\left(\frac{A_2}{C_2}f(u_{n-1})\right),\ n\in\mathbb{N}_0. \tag{2.31}$$

From (2.31) we easily get

$$u_n=f^{-1}\left(\frac{A_1A_2}{C_1C_2}f(u_{n-4})\right),\ v_n=g^{-1}\left(\frac{A_1A_2}{C_1C_2}g(v_{n-4})\right), n\geq 1. \tag{2.32}$$

By using (2.32), we obtain

$$u_{4m+i}=f^{-1}\left(\left(\frac{A_1A_2}{C_1C_2}\right)^{m+1}f(u_{i-4})\right),\ v_{4m+i}=g^{-1}\left(\left(\frac{A_1A_2}{C_1C_2}\right)^{m+1}g(v_{i-4})\right), \tag{2.33}$$

$m \in \mathbb{N}_0$, $i = \overline{3,6}$.
If $D_1 = 0$ so $C_1 \neq 0$. This means $B_1 = 0$, $A_1 \neq 0$. If $D_2 = 0$ so $C_2 \neq 0$. This means $B_2 = 0$, $A_2 \neq 0$. Then we have system (2.31). Moreover, the equalities in (2.33) are solutions of system (2.31).
Assume that $C_1 = 0$ so $D_1 \neq 0$. This means $A_1 = 0$, $B_1 \neq 0$. Suppose that $C_2 = 0$ so $D_2 \neq 0$. This means $A_2 = 0$, $B_2 \neq 0$. So, we obtain system (2.28). In addition, equalities in (2.30) are solutions of system (2.28).
Suppose that $A_1B_1C_1D_1 \neq 0$ and $A_2B_2C_2D_2 \neq 0$. It means $A_1 = \frac{B_1C_1}{D_1}$ and $A_2 = \frac{B_2C_2}{D_2}$. Moreover, we have system (2.28). Similarly, it means $B_1 = \frac{A_1D_1}{C_1}$ and $B_2 = \frac{A_2D_2}{C_2}$. □

3. An application

In this section, we give an application for system (1.8).

Remark 3.1. *If $f = g$, $A_1 = A_2$, $B_1 = B_2$, $C_1 = C_2$, $D_1 = D_2$, $u_{-p} = v_{-p}$, $p = \overline{0,4}$, then, the system (1.8) turns into the following equation*

$$u_{n+1} = f^{-1}\left(f(u_{n-1})\frac{A_1 f(u_{n-2}) + B_1 f(u_{n-4})}{C_1 f(u_{n-2}) + D_1 f(u_{n-4})}\right),\ n \in \mathbb{N}_0. \tag{3.1}$$

Behavior of solutions to equation (1.6) is mentioned in [32]. But somethings are not correct in [32].
Equation (1.6) can be expressed as

$$u_{n+1} = u_{n-1}\frac{\gamma_0\gamma_3 u_{n-2} + (\gamma_0\gamma_2 + \gamma_1)u_{n-4}}{\gamma_2 u_{n-4} + \gamma_3 u_{n-2}},\ n \in \mathbb{N}_0. \tag{3.2}$$

Firstly, the authors of [32] studied to obtain the equilibrium point of equation (1.6). Then, using a great deal calculations, they found $\overline{u} = 0$. If

$$(1-\gamma_0)(\gamma_2+\gamma_3) \neq \gamma_1,$$

an unique equilibrium point of equation (1.6) is $\overline{u} = 0$.
Suppose that an equilibrium point of equation (1.6) is $\overline{u}$. So we get the following equation

$$\overline{u} = \gamma_0\overline{u} + \frac{\gamma_1\overline{u}^2}{(\gamma_2+\gamma_3)\overline{u}}. \tag{3.3}$$

From (3.3), we see that it must be

$$(\gamma_2+\gamma_3) \neq 0 \text{ and } \overline{u} \neq 0.$$

This exterminates the probability $\overline{u} = 0$.
Suppose that $\overline{u} \neq 0$. Moreover, equation (3.3) means

$$\overline{u}\left(1-\gamma_0-\frac{\gamma_1}{\gamma_2+\gamma_3}\right) = 0,$$

so we have

$$1-\gamma_0-\frac{\gamma_1}{\gamma_2+\gamma_3} = 0. \tag{3.4}$$

From equation (3.4), the equilibrium point of the difference equation is $\overline{u} \neq 0$. It implies that the idea in [32] Theorem 3, under the condition, zero equilibrium point of equation (1.6) is local asymptotic stable is not corect, because it is not an equilibrium point at all.
In addition, Theorem 4 in [32] is expressed as:

Theorem 3.2. *If $\gamma_2(1-\gamma_0) \neq \gamma_1$, then the unique equilibrium point of Equation (1.6) is globally asymptotically stable.*

The particular case of equation (3.1) is equation (3.2) with

$$f(x) = x,\ A_1 = \gamma_0\gamma_3,\ B_1 = \gamma_0\gamma_2 + \gamma_1,\ C_1 = \gamma_2,\ D_1 = \gamma_3.$$

Example 3.3. *Keep in mind the equation (1.6) with*

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 1,$$

and then we get the following equation

$$u_{n+1} = u_{n-1}\frac{u_{n-2} + 2u_{n-4}}{u_{n-2} + u_{n-4}},\ n \in \mathbb{N}_0. \tag{3.5}$$

Equation (3.5) is derived from equation (3.1) with $f(x) = x$ and $x \in \mathbb{R}$,

$$A_1 = C_1 = D_1 = 1,\ B_1 = 2. \tag{3.6}$$

By using (3.6) the first equation in (2.8), we get

$$p_1(\lambda) = \lambda^2 - 2\lambda - 1,$$

and its roots are

$$\lambda_1 = 1 + \sqrt{2} \text{ and } \lambda_2 = 1 - \sqrt{2}.$$

Then, we obtain

$$\gamma_2(1 - \gamma_0) - \gamma_1 = -1 \neq 0,$$

the restriction $\gamma_2(1 - \gamma_0) \neq \gamma_1$ in Theorem 3.2 is valid.
By using the parameters A_1, B_1, C_1, D_1 are as in (3.6) and (2.15)-(2.16), where $f(x) = x$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
u_{12m+3s+p} &= u_{3s+p-12} \\
&\times \prod_{r=0}^{m} \Bigg(\left(\frac{\left(\frac{u_{p-3\lfloor \frac{p+2}{3} \rfloor}}{u_{p-3\lfloor \frac{p+2}{3} \rfloor-2}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p+2}{3} \rfloor+1} - \left(\frac{u_{p-3\lfloor \frac{p+2}{3} \rfloor}}{u_{p-3\lfloor \frac{p+2}{3} \rfloor-2}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p+2}{3} \rfloor+1}}{\left(\frac{u_{p-3\lfloor \frac{p+2}{3} \rfloor}}{u_{p-3\lfloor \frac{p+2}{3} \rfloor-2}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p+2}{3} \rfloor} - \left(\frac{u_{p-3\lfloor \frac{p+2}{3} \rfloor}}{u_{p-3\lfloor \frac{p+2}{3} \rfloor-2}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p+2}{3} \rfloor}} - 1 \right) \\
&\times \left(\frac{\left(\frac{u_{p-2-3\lfloor \frac{p}{3} \rfloor}}{u_{p-4-3\lfloor \frac{p}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p}{3} \rfloor+1} - \left(\frac{u_{p-2-3\lfloor \frac{p}{3} \rfloor}}{u_{p-4-3\lfloor \frac{p}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p}{3} \rfloor+1}}{\left(\frac{u_{p-2-3\lfloor \frac{p}{3} \rfloor}}{u_{p-4-3\lfloor \frac{p}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p}{3} \rfloor} - \left(\frac{u_{p-2-3\lfloor \frac{p}{3} \rfloor}}{u_{p-4-3\lfloor \frac{p}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p}{3} \rfloor}} - 1 \right) \\
&\times \left(\frac{\left(\frac{u_{p+2+3\lfloor \frac{p-2}{3} \rfloor}}{u_{p+3\lfloor \frac{p-2}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-2}{3} \rfloor+1} - \left(\frac{u_{p+2+3\lfloor \frac{p-2}{3} \rfloor}}{u_{p+3\lfloor \frac{p-2}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-2}{3} \rfloor+1}}{\left(\frac{u_{p+2+3\lfloor \frac{p-2}{3} \rfloor}}{u_{p+3\lfloor \frac{p-2}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-2}{3} \rfloor} - \left(\frac{u_{p+2+3\lfloor \frac{p-2}{3} \rfloor}}{u_{p+3\lfloor \frac{p-2}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-2}{3} \rfloor}} - 1 \right) \\
&\times \left(\frac{\left(\frac{u_{p-3-3\lfloor \frac{p-1}{3} \rfloor}}{u_{p-5-3\lfloor \frac{p-1}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-1}{3} \rfloor} - \left(\frac{u_{p-3-3\lfloor \frac{p-1}{3} \rfloor}}{u_{p-5-3\lfloor \frac{p-1}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-1}{3} \rfloor}}{\left(\frac{u_{p-3-3\lfloor \frac{p-1}{3} \rfloor}}{u_{p-5-3\lfloor \frac{p-1}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-1}{3} \rfloor-1} - \left(\frac{u_{p-3-3\lfloor \frac{p-1}{3} \rfloor}}{u_{p-5-3\lfloor \frac{p-1}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-1}{3} \rfloor-1}} - 1 \right) \\
&\times \left(\frac{\left(\frac{u_{p-5-3\lfloor \frac{p-3}{3} \rfloor}}{u_{p-7-3\lfloor \frac{p-3}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-3}{3} \rfloor} - \left(\frac{u_{p-5-3\lfloor \frac{p-3}{3} \rfloor}}{u_{p-7-3\lfloor \frac{p-3}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-3}{3} \rfloor}}{\left(\frac{u_{p-5-3\lfloor \frac{p-3}{3} \rfloor}}{u_{p-7-3\lfloor \frac{p-3}{3} \rfloor}} + 1 - \lambda_2\right)\lambda_1^{4r+s+\lfloor \frac{p-3}{3} \rfloor-1} - \left(\frac{u_{p-5-3\lfloor \frac{p-3}{3} \rfloor}}{u_{p-7-3\lfloor \frac{p-3}{3} \rfloor}} + 1 - \lambda_1\right)\lambda_2^{4r+s+\lfloor \frac{p-3}{3} \rfloor-1}} - 1 \right)
\end{aligned} \tag{3.7}$$

$$\times\left(\frac{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-1\right)\Bigg),$$

for $m\in\mathbb{N}_0,\ s=\overline{3,6},\ p=\overline{-1,1}.$
Note that

$$\begin{aligned}
&\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-1\right)\\
&=\lambda_1-1=\sqrt{2}>1,
\end{aligned}$$

when

$$\begin{aligned}
&\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}},\\
&\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}},p=\overline{-1,1}.\\
&\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}},
\end{aligned}\tag{3.8}$$

By selecting positive initial conditions providing (3.8) and using equations in (3.7), we obtain

$$\lim_{m\to\infty}u_m=\infty.$$

Now, we give numerical example to support the last equation.

Example 3.4. *Consider the equation (3.5) with the initial values $u_{-4} = 0.195$, $u_{-3} = 0.1$, $u_{-2} = 2.4$, $u_{-1} = 3$, $u_0 = 7.62$, the solution is given as in Figure (1).*

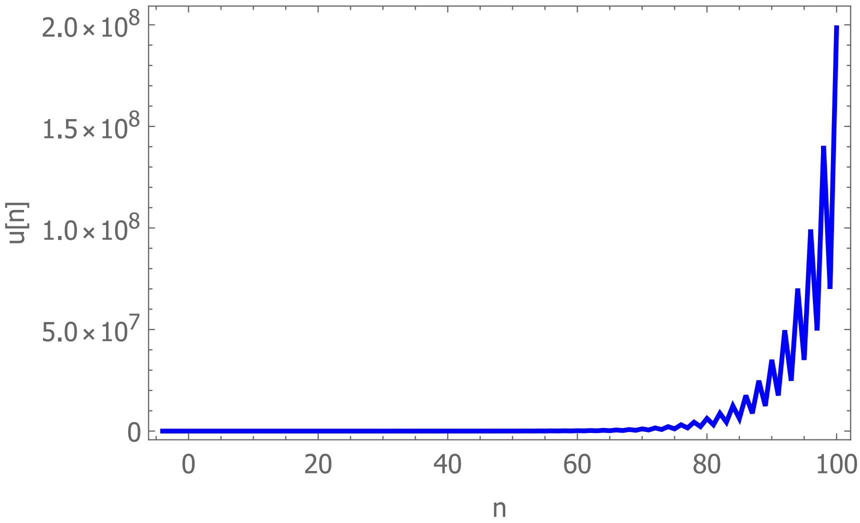

Figure 1: Plots of u_n

Then, the solution is not convergent. It is a counterexample to the claim in Theorem 3.2 (Theorem 4 in [32]).

4. Conclusion

In this study, we have solved the following general two dimensional system of difference equations

$$u_{n+1} = f^{-1}\left(g(v_{n-1})\frac{A_1 f(u_{n-2}) + B_1 g(v_{n-4})}{C_1 f(u_{n-2}) + D_1 g(v_{n-4})}\right), v_{n+1} = g^{-1}\left(f(u_{n-1})\frac{A_2 g(v_{n-2}) + B_2 f(u_{n-4})}{C_2 g(v_{n-2}) + D_2 f(u_{n-4})}\right),\ n \in \mathbb{N}_0,$$

where the parameters A_j, B_j, C_j, D_j, for $j \in \{1,2\}$ are real numbers, the initial values u_{-k}, v_{-k}, for $k = \overline{0,4}$ are real numbers, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. The following particular cases are considered:

1. if $A_1D_1 \neq B_1C_1$ and $A_2D_2 \neq B_2C_2$
   (a) if $C_1 \neq 0$, $C_2 \neq 0$,
      i. if $(A_1 + D_1)^2 - 4(A_1D_1 - B_1C_1) \neq 0$, $(A_2 + D_2)^2 - 4(A_2D_2 - B_2C_2) \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.15) and (2.16).
      ii. if $(A_1 + D_1)^2 - 4(A_1D_1 - B_1C_1) = 0$, $(A_2 + D_2)^2 - 4(A_2D_2 - B_2C_2) = 0$, then the general solutions of system (1.8) is given by formulas in (2.19) and (2.20).
   (b) if $C_1 = 0$, $C_2 = 0$,
      i. if $A_1 = D_1$, $A_2 = D_2$, then the general solutions of system (1.8) is given by formulas in (2.24) and (2.24).
      ii. if $A_1 \neq D_1$, $A_2 \neq D_2$, then the general solutions of system (1.8) is given by formulas in (2.26) and (2.27).
2. if $A_1D_1 = B_1C_1$, $A_2D_2 = B_2C_2$,
   (a) if $A_1 = 0$, $A_2 = 0$, then the general solutions of system (1.8) is given by formulas in (2.30).
   (b) if $A_1 \neq 0$, $A_2 \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.33).
   (c) if $D_1 = 0$, $D_2 = 0$, then the general solutions of system (1.8) is given by formulas in (2.33).
   (d) if $D_1 \neq 0$, $D_2 \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.30).

(e) if $A_1B_1C_1D_1 \neq 0, A_2B_2C_2D_2 \neq 0.$

i. if $A_1 = \frac{B_1C_1}{D_1}, A_2 = \frac{B_2C_2}{D_2}$, then the general solutions of system (1.8) is given by formulas in (2.30).

ii. if $B_1 = \frac{A_1D_1}{C_1}, B_2 = \frac{A_2D_2}{C_2}$, then the general solutions of system (1.8) is given by formulas in (2.33).

In addition, an application is given.

Declarations

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