http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 4, Pages 1098–1113 (2024)
DOI:10.31801/cfsuasmas.1494749
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: June 3, 2024; Accepted: September 22, 2024

IDEAL THEORY OF (m, n)-NEAR RINGS

Fahimeh MOHAMMADI[1] and Bijan DAVVAZ[2]

[1,2]Department of Mathematical Sciences, Yazd University, Yazd, IRAN

Abstract. The aim of this research work is to define and characterize a new class of n-ary algebras that we call (m, n)-near rings. We investigate the notions of i-R-groups, i-(m, n)-near field, prime ideals, primary ideals and subtractive ideals of (m, n)-near rings. We describe the concept of homomorphisms between (m, n)-near rings that preserve the (m, n)-near ring structure, and give some results in this respect.

1. Introduction

Polyadic groups were introduced in 1928 by W. Dörnte [10]. An important role in n-group theory is the paper [12], for more details see [7,11]. Then, n-ary operations are used then in the study of (m, n)-rings [5,6,13] and (m, n)-semirings [1,3,8].

Let A be a non-empty set. A map $h : A^m \longrightarrow A$ is called an m-ary operation. A non-empty set A with an m-ary operation h is called an m-ary groupoid that is denoted by (A, h). The sequence $z_i, z_{i+1}, ..., z_m$ is denoted by z_i^m where $1 \leq i \leq m$. For all $1 \leq i \leq j \leq m$, the phrase $h(z_1, z_2, ..., z_i, k_{i+1}, ..., k_j, l_{j+1}, ..., l_m)$ is represented as $h(z_1^i, k_{i+1}^j, l_{j+1}^m)$. In this case when $k_{i+1} = k_{i+2} = ... = k_j = k$, it is expressed as $h(z_1^i, k^{(j-i)}, l_{j+1}^m)$. An m-ary groupoid (A, h) is called an m-ary semigroup if h is associative; that is,

$$h(z_1^{i-1}, h(z_i^{m+i-1}), z_{m+i}^{2m-1}) = h(z_1^{j-1}, h(z_j^{m+j-1}), z_{m+j}^{2m-1}),$$

for all $z_1, z_2, ..., z_{2m-1} \in A$ where $1 \leq i \leq j \leq m$. An m-ary semigroupoid (A, h) is named an m-ary group if for all $c_1^{i-1}, c_{i+1}^n, b \in A$ exist $z_1^n \in A$, such that $h(c_1^{i-1}, z_i, c_{i+1}^n) = b$ for every $1 \leq i \leq n$. We say f is commutative if $h(z_1, z_2, ..., z_m) = h(z_{\eta(1)}, z_{\eta(2)}, ..., z_{\eta(m)})$, for every permutation η of $\{1, 2, ..., m\}$

2020 *Mathematics Subject Classification.* 16Y30, 16Y99.
Keywords. (m, n)-near ring, (m, n)-near field, ideal, homomorphism.
1 f1991mohammadi@gmail.com; 0009-0003-1562-9231;
2 davvaz@yazd.ac.ir-Corresponding author; 0000-0003-1941-5372.

and $z_1, z_2, ..., z_m \in A$. An m-ary semigroup (A,h) is called a semi-abelian or $(1,m)$-commutative if $h(z, c^{(m-2)}, k) = h(k, c^{(m-2)}, z)$, for all $c, z, k \in A$.

2. (m,n)-Near Rings

We refer to [2, 4, 14], for details about near rings. In this section, we define the (m,n)-near ring and give examples for it and present definitions of α_1-(m,n)-near ring, α_2-(m,n)-near ring, R_0, R_c, constant near ring, i-zero divisor, $Z_{i,j}(R)$. We present some results in this respect.

Definition 1. *Assume that A is a non-empty set and h, k be r-ary and s-ary operations on A, respectively. In this case (A,h,k) is named an i-(r,s)-near ring, if the following conditions hold:*

1. *(A,h) is an r-ary group (not necessarily abelian),*
2. *(A,k) is an s-ary semigroup,*
3. *The s-ary operation k is i-distributive with respect to the r-ary operation h,*

where the definition of i-distributive condition is as follows: for every $c_1, c_2, ..., c_n$, $d_1, d_2 ..., d_m \in R$, if $i = n$, then

$$k(c_1^{n-1}, h(d_1, d_2, ..., d_m)) = h(k(c_1^{n-1}, d_1), k(c_1^{n-1}, d_2), ..., k(c_1^{n-1}, d_m)).$$

If $i = 1$ then

$$k(h(d_1, d_2, ..., d_m), c_2^n) = h(k(d_1, c_2^n), k(d_2, c_2^n), ..., k(d_m, c_2^n)).$$

If $1 < i < n$ then

$$\begin{aligned} &k(c_1^{i-1}, h(d_1, d_2, ..., d_m), c_{i+1}^n) \\ &= h(k(c_1^{i-1}, d_1, c_{i+1}^n), k(c_1^{i-1}, d_2, c_{i+1}^n), ..., k(c_1^{i-1}, d_m, c_{i+1}^n)). \end{aligned}$$

Throughout this paper, we explain i-(m,n)-near ring by (m,n)-near ring. It is clear that every (m,n)-ring [5] is an (m,n)-near ring.

Example 1. *Assume that (H,l) is an m-ary group with the identity element 0 and $N(H) = \{h : H \longrightarrow H \mid h \text{ is a function }\}$. Then $(N(H), l, \circ)$ is an $(m,2)$-near ring, where $\circ$ is the composition of functions.*

1. *We know $(N(H), l)$ is an m-ary group (not necessarily abelian).*
2. *It is clear that $(N(H), \circ)$ is a 2-ary semigroup.*
3. *The 2-ary operation $\circ$ is 1-distributive with respect to the m-ary operation f.*

We notice that in this $(m,2)$-near ring the 2-distributive law fails to retain. To consider this, let $d, d_j, c_i \in H, b_i \neq 0, 1 \leq j \leq m, 1 \leq i \leq 2$ and $h_{d_j} : H \longrightarrow H$, $h_{c_i} : H \longrightarrow H$ for all $g \in H$, by $h_{d_j}(g) = d_j$, $h_{c_i}(g) = c_i$. Now, for $i = 2$, we have

$$\begin{aligned} [h_{c_1} \circ (l(h_{d_1}, h_{d_2}, ..., h_{d_m}))](g) &= h_{c_1}(l((h_{d_1}(g), h_{d_2}(g), ..., h_{d_m}(g)) \\ &= h_{c_1}(l(d_1, d_2, ..., d_m)) = l(d_1, d_2, ..., d_m), \end{aligned}$$

and

$$\begin{aligned}[l(h_{c_1}\circ h_{d_1}, h_{c_1}\circ h_{d_2}, ..., h_{c_1}\circ h_{d_m})](g) &= l(h_{c_1}(h_{d_1}(g)), h_{c_1}(h_{d_2}(g)), ..., h_{c_1}(h_{d_1}(g)))\\ &= l(h_{c_1}(d_1), h_{c_1}(d_2), ..., h_{c_1}(d_n))\\ &= l(c_1^{(m)}).\end{aligned}$$

This shows that

$$[h_{c_1}\circ(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))](g) \neq [l(h_{c_1}\circ h_{d_1}, h_{c_1}\circ h_{d_2}, ..., h_{c_1}\circ h_{d_m})](g).$$

For $i=1$, we have

$$\begin{aligned}(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))\circ h_{c_1}(g) &= (l(h_{d_1}, h_{d_2}, ..., h_{d_m}))(c_1)\\ &= l(h_{d_1}(c_1), h_{d_2}(c_1), ..., h_{d_m}(c_1))\\ &= l(d_1, d_2, ..., d_m),\end{aligned}$$

and

$$\begin{aligned}&[l(h_{d_1}\circ h_{d_1}, h_{d_2}\circ h_{c_1}, ..., h_{d_m}\circ h_{c_1})](g)\\ &\quad = l((h_{d_1}\circ h_{c_1})(g), (h_{d_2}\circ h_{c_1})(g), ..., (h_{d_m}\circ h_{c_1})(g))\\ &\quad = l((h_{d_1})(c_1), (h_{d_2})(c_1), ..., (h_{d_m})(c_1))\\ &\quad = l(d_1, d_2, ..., d_m).\end{aligned}$$

Hence,

$$[(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))\circ h_{c_2}](g) = [l((h_{d_1}\circ h_{c_1}), (h_{d_2}\circ h_{c_1}), ..., (h_{d_m}\circ h_{c_1}))](g).$$

Therefore $N(H)$ fails to satisfy the i-distributive for $i=2$.

Example 2. *Consider the additive group $\mathbb{Z}_{mn}$. Then $(\mathbb{Z}_{mn}, h)$ is a group, where $h(c_1, c_2, ..., c_m) = c_1 + c_2 + ... + c_m$. We define k on $\mathbb{Z}_{mn}$ by $k(c_1, c_2, ..., c_n) = c_1$, for all $c_1, c_2, ..., c_n \in \mathbb{Z}_{mn}$. It is easy to see $(\mathbb{Z}_{mn}, h, k)$ is an (m,n)-near ring. For $1 < i \leqslant n$, we have*

$$\begin{aligned}&k(c_1, c_2, ..., c_{i-1}, h(d_1, d_2, ..., d_m), c_{i+1}, ..., c_n) = c_1\\ &h(k(c_1, c_2, ..., c_{i-1}, d_1, c_{i+1}, ..., c_n), ..., k(c_1, c_2, ..., c_{i-1}, d_m, c_{i+1}, ..., c_n))\\ &\quad = h(c_1^{(m)}) = mc_1.\end{aligned}$$

If $mn = m-1$, then $\overline{m} = \overline{1} \in \mathbb{Z}_{mn}$. Hence, for all $1 < i \leqslant n$, $(\mathbb{Z}_{mn-1}, h, k)$ is i-distributive. For $i=1$, we have

$$\begin{aligned}&k(h(d_1, d_2, ..., d_m), c_2, ..., c_n) = h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m\\ &h(k(d_1, c_2, ..., c_n), k(d_2, d_2, ..., d_n), ..., k(d_m, c_1, ..., c_n))\\ &\quad = h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m.\end{aligned}$$

Consequently, for $i=1$, $(\mathbb{Z}_{mn-1}, h, k)$ is 1-distributive.

Assume that A is an (m,n)-near ring. The element $e \in A$ is named an identity element if $k(e^{(i-1)}, s, e^{(n-i)}) = s$ for all $s \in A$ and $1 \leqslant i \leqslant n$.

Example 3. *We know $(\mathbb{R}, +, \cdot)$ is an (m,n)-near ring with two binary operations m-addition and n-multiplication. 1 is an identity element in $(\mathbb{R}, +, \cdot)$.*

Assume that (A,h,k) is an (m,n)-near ring. $m \in A$ is named i-cancellable, if for all $1 \leq i \leq n$, $c_i, d_i \in A$ and $k(c_1^{i-1}, m, c_i^n) = k(d_1^{i-1}, m, d_i^n)$, then $c_i = d_i$ for all $1 \leq i \leq n$. $m \neq 0$ is named an i-zero divisor, if there exist nonzero elements $c_1, c_2, ..., c_n \in R$ such that $k(c_1^{i-1}, m, c_{i+1}^n) = 0$. An (m,n)-near ring (A,h,k) is called integral near ring if it has no zero divisors. An i-(m,n)-near field is a non-empty set P together with two binary operations h and k such that (P,h) is a group (not necessarily abelian), (P,k) is a group and n-ary operation k is i-distributive with respect to the m-ary operation h.

Example 4. *Set of rational numbers with two binary operations h and k so that $k(d_1, d_2, ..., d_n) = d_1$ and $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$ for $d_i \in \mathbb{Q}$, $(\mathbb{Q}, h, k)$ is an (m,n)-near field.*

Definition 2. *Let (A,h,k) be an (m,n)-near ring,*

(1) *If for every $e \in A$ exists $z \in A$ such that $e = k(z^{(n-1)}, e, z^{(n-1)})$, then A is named an α_1-(m,n)-near ring.*
(2) *If for every $e \in A-\{0\}$ exists $z \in A-\{0\}$ such that $z = k(z^{(n-1)}, e, z^{(n-1)})$, then A is named an α_2-(m,n)-near ring.*

Example 5. *$(N(H), l, \circ)$ defined in Example 1 is an α_2-(m,n)-near ring.*

Example 6. *$(\mathbb{Z}_{mn}, h, k)$ defined in Example 2 is an α_2-(m,n)-near ring.*

Definition 3. *Let (A,h,k) be an (m,n)-near ring,*

(1) *A subgroup (O,h) of an m-ary group (A,h) with the property $k(O^{(n)}) \subset M$ is named an (m,n)-subnear ring of (A,h,k), It is shown by $O \leqslant N$.*
(2) *A subnear ring O of A is named i-invariant, if $h(A^{(i-1)}, O, A^{(m-i)}) \subseteq O$.*

If O is i-invariant for all $1 \leqslant i \leqslant m$, then O is named invariant.

Example 7. *The triple $(2\mathbb{Z}, h, k)$ is an (m,n)-subnear ring of the (m,n)-near ring $(\mathbb{Z}, h, k)$, that $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$ and $k(e_1, e_2 + ..., e_n) = e_1 \cdot e_2 \cdot ... \cdot e_n$.*

Definition 4. *Let (A,h,k) be an (m,n)-near ring and 0 is the identity element of (A,h). Then, $A_0 = \{r \in A \mid k(0^{(s-1)}, r, 0^{(n-s)}) = 0,\ 1 \leq s \leq n\}$ is called the zero symmetric part of A. In addition, $A_c = \{r \in R \mid k(0^{(s-1)}, r, 0^{(n-s)}) = r,\ 1 \leq s \leq n\}$ is named a resistant part of A. An (m,n)-near ring A is named a zero symmetric near ring if $A = A_0$. An (m,n)-near ring A is named a constant (m,n)-near ring if $A = A_c$.*

Lemma 1. *A_0 and A_c are (m,n)-subnear rings of the (m,n)-near ring (A,h,k).*

Proof. We show that A_0 is a subgroup of A. If $x_1, x_2, ..., x_m \in A_0$ then

$$k(0^{(i-1)}, x_j, 0^{(n-i)}) = 0 \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq n.$$

Now, we have

$$\begin{aligned} &k(0^{(i-1)}, h(x_1, x_2, ..., x_m), 0^{(n-i)}) \\ &\quad = h(k(0^{(i-1)}, x_1, 0^{(n-i)}), k(0^{(i-1)}, x_2, 0^{(n-i)}), ..., k(0^{(i-1)}, x_m, 0^{(n-i)})) = 0. \end{aligned}$$

Therefore, $h(x_1, x_2, ...x_m) \in A_0$, and so (A_0, h) is a subgroup of (A, h, k). Next, if we take $y_1, y_2, ..., y_n \in A_0$, then for all $1 \leq i \leq n$ and $1 \leq j \leq n$, we have $k(0^{(i-1)}, y_j, 0^{(n-i)}) = 0$. Then, we obtain

$$\begin{aligned}
&k(0^{(n-1)}, k(y_1, y_2, ..., y_n)) = k(k(0^{(n-1)}, y_1), y_2, ..., y_n) = k(0, y_2, ..., y_n)\\
&= k(k(0^{(n)}), y_2, .., y_n) = k(0, k(0^{(n-1)}, y_2), y_3, ..., y_n) = k(0, 0, y_3, ..., y_n)\\
&= ... = k(0^{(n-1)}, y_n) = 0.
\end{aligned}$$

Therefore, $k(y_1, y_2, ..., y_n) \in A_0$, and so $k(A_0^{(n)}) \subset A_0$. This shows that (A_0, h, k) is an (m, n)-subnear ring of (m, n)-near ring (A, h, k). We show that A_c is a subgroup of A. Let $x_1, x_2, ..., x_m \in A_0$. Then, we have $k(0^{(i-1)}, x_j, 0^{(n-i)}) = x_j$ for $1 \leq j \leq m$ and $1 \leq i \leq n$. Now, we obtain

$$\begin{aligned}
&k(0^{(i-1)}, h(x_1, x_2, ...x_m), 0^{(n-i)})\\
&\quad = h(k(0^{(i-1)}, x_1, 0^{(n-i)}), k(0^{(i-1)}, x_2, 0^{(n-i)}), ..., k(0^{(i-1)}, x_m, 0^{(n-i)}))\\
&\quad = h(x_1, x_2, ..., x_m).
\end{aligned}$$

This yields that $h(x_1, x_2, ...x_m) \in A_c$. Hence, (A_c, h) is a subgroup of (A, h, k). Next, if $y_1, ..., y_n \in A_c$, then $k(0^{(i-1)}, y_j, 0^{(n-i)}) = y_j$, for all $1 \leq i \leq n$, $1 \leq j \leq n$. This gives that $k(0^{(n-1)}, k(y_1, y_2, ..., y_n)) = k(k(0^{(n-1)}, y_1), y_2, ..., y_n) = k(y_1, y_2, ..., y_n)$. Therefore $k(y_1, y_2, ..., y_n) \in A_c$ and $k(A_c^{(n)}) \subset A_c$. Hence, (A_c, h, k) is an (m, n)-subnear ring of (m, n)-near ring (A, h, k). □

Theorem 1. *Let (A, h, k) be an (m, n)-near ring. If $r \in A_0$ is i-cancellable, then r is not an i-zero divisor.*

Proof. Suppose that $r \in A_0$ is i-cancellable and also r is an i-zero divisor, so there exist nonzero elements $d_1, d_2, ..., d_n \in A$ such that $k(d_1^{i-1}, r, d_{i+1}^n) = 0$. Since $r \in A_0$, it follows that $k(d_1^{i-1}, r, d_{i+1}^n) = 0 = k(0^{(i-1)}, r, 0^{(n-i)})$. Again, since r is i-cancellable, it follows that for all $1 \leq i \leq n$, $d_i = 0$, that it is a contradiction. □

Let (A, h, k) be an (m, n)-near ring. The center, $Z_{i,j}(A)$, is the subset of elements in A that (i, j)-commute with element of A. In the symbol, we can write:

$$\begin{aligned}
Z_{i,j}(A) \quad &= \{b \in A \mid a_1, ..., a_n \in A \text{ and for } j > i,\\
&\quad k(a_1^{i-1}, b, a_i^n) = k(a_1^{i-1}, a_j, a_{i+1}, ..., a_{j-1}, b, a_{j+1}^n)\}.
\end{aligned}$$

Example 8. *In Example 2, for all $i, j \in 2, 3, ..., n$, we have $Z_{i,j}(A) = A$.*

Suppose that (A, h, k) is an (m, n)-near ring. If (A, k) is commutative, then A is named a commutative near ring. An element $r \in A$ is named idempotent element if $k(r^{(n)}) = r$. An element $r \in A$ is named nilpotent element if $k(r^{(n)}) = 0$.

Example 9. *In Example 2, for all $r \in \mathbb{Z}_{mn}$, we have $k(r^n) = r$, and so all elements are idempotent. Moreover, $\mathbb{Z}_{mn}$ has only one nilpotent element that is* 0*.*

Suppose that (A, h, k) is an (m, n)-near ring. A subset S of A is named nilpotent if $k(S^{(n)}) = 0$. A subset S of A is named nill if every element of S is a nilpotent element.

Theorem 2. *Assume that S is a subset of A. If S is nilpotent, then S is nill.*

Proof. Assume that S is nilpotent. Then $k(S^{(n)}) = 0$. This gives that $k(s^{(n)}) = 0$ for all $s \in S$. Hence, S is a nilpotent for all $s \in S$, then S is nill. $\square$

Definition 5. *Assume that (A, h, k) is an (m, n)-near ring and (W, h) be an m-group with identity element 0 of (A, h). W is named an i-A-group if there exists a mapping $l : \underbrace{W\times, ..., \times W}_{i-1} \times A \times \underbrace{W \times ... \times W}_{n-i} \to W$ the image of*

$$(r^{(i-1)}, s, r^{(n-i)}) \in \underbrace{W\times, ..., \times W}_{i-1} \times A \times \underbrace{W \times ... \times W}_{n-i} \to W,$$

for $s \in A$ and $r \in W$, is denoted by $l(r^{(i-1)}, s, r^{(n-i)}) = k(r^{(i-1)}, s, r^{(n-i)})$, satisfying the following conditions:

(1) $k(s_1^{i-1}, h(r_1, r_2, ..., r_m), s_{i+1}^n)$
$= h(k(s_1^{i-1}, r_1, s_{i+1}^n), k(s_1^{i-1}, r_2, s_{i+1}^n), ..., k(s_1^{i-1}, r_n, s_{i+1}^n))$.

(2) $k(t_1^{i-1}, k(z_1, z_2, ..., z_n), t_{i+1}^n) = k(t_1^{i-l-1}, k(t_{i-l}^{i-1}, z_1^{n-l}), z_{n-l+1}^n, t_{i+1}^n)$
$= k(t_1^{i-1}, z_1^s, k(z_{s+1}^n, t_{i+1}^{i+s}), t_{i+s+1}^n)$ *for all* $1 \le l \le i-1$ *and* $1 \le s \le n-i$,

for all $s_j, t_i \in W$ that $1 \le i, j \le n$. For all $r_i, z_t \in A$ that $1 \le i \le m$ and $1 \le t \le n$, we denote this i-A-group by $\underbrace{AA...A}_{i-1} W \underbrace{AA...A}_{n-i}$.

Example 10. *If we consider $W = \mathbb{Z}$ in Example 2, then W is an 1-$\mathbb{Z}_{mn}$-group. By taking $i = 1$ in Definition 5, the conditions of the definition are satisfied,*

$$k(h(r_1, r_2, ..., r_m), s_2^n) = h(k(r_1, s_2^n), k(r_2, s_2^n), ..., k(r_m, s_2^n)) = h(r_1, r_2, ..., r_m),$$
$$k(k(s_1, s_2, ..., s_n), t_2^n) = k(s_1^l, k(s_{l+1}^n, t_2^{1+l}), t_{2+l}^n) = s_1.$$

In Definition 5, if $k(r^{(i-1)}, g, r^{(n-i)}) = 0$ for all $g \in W$ yields $r = 0$, then W is a faithful i-A-group.

Example 11. *In Example 2, $\mathbb{Z}_{mn}$ operates faithfully on $\mathbb{Z}$.*

Assume that (A, h, k) is an (m, n)-near ring. A subgroup H of an i-A-group W is named an i-A-subgroup (written as $H \le_A W$), if it is closed under the operation of A and $k(r^{(i-1)}, h, r^{(n-i)}) \in H$ for all $r \in A$, $h \in H$. Suppose that W_1 and W_2 are two A-groups, $s : W_1 \to W_2$ is named i-A-homomorphism, if for all $l, l_1, ..., l_n \in W_1$ and for all $r \in A$, $s(h(l_1, l_2, ..., l_m)) = h(s(l_1), s(l_2), ..., s(l_m))$ and $s(k(r^{(i-1)}, l, r^{(n-i)})) = k(r^{(i-1)}, s(l), r^{(n-i)})$. If H is the kernel of an i-A-homomorphism, then it is named an i-A-normal subgroup and we write $H \trianglelefteq_A W$.

Example 12. *If $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$, $k(d_1, d_2, ..., d_n) = d_1 \cdot d_2 \cdot ... \cdot d_n$, then $(\mathbb{R}, h, k)$ is an (m, n)-near ring and $\mathbb{Q}$ (the set of rationales) is a i-$\mathbb{R}$-subgroup of $\mathbb{R}$.*

Assume that W is an i-A-group. W is named a unitary i-A-group if A be a near ring with unity 1 so that $k(1^{(i-1)}, x, 1^{(n-i)}) = x$ for all $x \in W$.

Example 13. *If in Example 4, $d_j = 1$ for $j \in \{1, 2..., i-1, i+1, ..., n\}$, then* $k(1^{(i-1)}, x, 1^{(n-i)}) = \underbrace{1 \cdot 1 \cdot ... \cdot 1}_{i-1} \cdot x \cdot \underbrace{1 \cdot 1 \cdot ... \cdot 1}_{n-i} = x$.

Theorem 3. *In an α_1-(m,n)-near ring for every $a \in A$ exist some $s \in A$ if $n = 2i+1$, then*

(1) $k(s^{(i)}, a^{(i+1)}) = k(a^{(i+1)}, s^{(i)})$,
(2) $a = k(s^{(i)}, k(s^{(i)}, ..., k(s^{(i)}, a, s^{(i)}), ..., s^{(i)}), s^{(i)})$.

Proof. (1) Suppose that A is an α_1-(m,n)-near ring and $a \in A$. So there exists $s \in R$ such that $a = k(s^{(i-1)}, a, s^{(n-i)})$. This implies that

$$\begin{aligned} &k(s^{(i)}, a^{(i+1)}) = k(s^{(i)}, a, a^{(i)}) = k(s^{(i)}, a, k(s^{(i)}, a, s^{(i)}), a^{(i-1)}) \\ &= k(k(s^{(i)}, a, s^{(i)}), a, s^{(i)}, a^{i-1}) = k(a, a, s^{(i)}, a^{(i-1)}) \\ &= k(a, a, s^{(i)}, a, k(s^{(i)}, a, s^{(i)}, a^{(i-3)})) = k(a, a, k(s^{(i)}, a, s^{(i)}), a, s^{(i)}, a^{(i-3)}) \\ &= k(a, a, a, a, s^{(i)}, a^{(i-3)}) = ... = k(a^{(i+1)}, s^{(i)}). \end{aligned}$$

(2) We have

$$\begin{aligned} &k(s^{(i)}, k(s^{(i)}, ..., k(s^{(i)}, a, s^{(i)}), ..., s^{(i)}), s^{(i)}) \\ &= k(s^{(i)}, k(s^{(i)}, a, s^{(i)}), s^{(i)}) = a. \end{aligned}$$

□

A subnear ring M of a (m,n)-near ring A is named an α_2-subnear ring if for every $a \in M$ exists an $s \in M$ so that $n = 2i+1$, $k(s^{(i)}, a, s^{(i)}) = s$.

Theorem 4. *Suppose that A is an α_2-(m,n)-near ring. In this case*

(1) *Every invariant subgroup W of A is an α_2-subnear ring.*
(2) *Every ideal I of a zero symmetric α_2-near ring A is an α_2-subnear ring.*

Proof. (1) Take $a \in W - \{0\}$. Since A is an α_2-near ring there exists $s \in A$ such that $k(s^{(i)}, a, s^{(i)}) = s$. Now W is an invariant subgroup of A implies that $k(s^{(i)}, a, s^{(i)}) \in W$. Then $s \in W$. Consequently W is an α_2-subnear ring.

(2) Assume that I is an ideal of the zero symmetric α_2-near ring A. Let $a \in I - \{0\}$. Since A is an α_2-near ring, so there exists $s \in A - \{0\}$ so that $k(s^{(i)}, a, s^{(i)}) = s$. Now, we have $k(s^{(i)}, a, s^{(i)}) \in k((A - \{0\})^{(i)}, I - \{0\}, (A - \{0\})^{(i)}) \subseteq I - \{0\}$. The desired result now follows. □

3. Ideals and Homomorphisms of (m,n)-Near Rings

We define the notions of i-ideal, zero near ring, prime ideal, semi-symmetric, $A(S)$, k-ideal, i-N-primary and i-P-primary in the (m,n)-near rings and assert a few related theorems.

Assume that I is a non-empty subgroup of an (m,n)-near ring (A, h, k). Then I is named a normal subgroup of A if for all $a_i \in A$ and $s_1^{i-1}, s_{i+1}^m \in A$, $1 \le i, j \le m$, there is $b_j \in I$ that $h(s_1^{i-1}, a_i, s_{i+1}^m) = h(s_1^{j-1}, b_j, s_{j+1}^m)$.

Definition 6. *Suppose that I is a non-empty subset of an (m,n)-near ring (A,h,k). In this case I is named an ideal of A if*

(1) *I is a normal subgroup of m-ary group (A,h), (I,h) is an m-ary group,*
(2) *for every $a_1,a_2,...,a_n \in A$, $k(a_1^{i-1},I,a_{i+1}^n) \subseteq I$,*
(3) *for all $r_1,...,r_{j-1},r_{j+1},...,r_m,s_1,...,s_{j-1},s_{j+1},...,s_n \in A$ and $1 \le k \le n$, $d \in I$, there exists $l \in I$ that*

$$k(s_1^{j-1},h(r_1^{k-1},d,r_{k+1}^m),s_{j+1}^n)$$
$$= h(k(s_1^{j-1},r_1,s_{j+1}^n),k(s_1^{j-1},r_2,s_{j+1}^n),...,k(s_1^{j-1},r_{k-1},s_{j+1}^n),l,$$
$$(s_1^{j-1},r_{k+1},s_{j+1}^n),...,k(s_1^{j-1},r_n,s_{j+1}^n)).$$

I is named an i-ideal of A if it satisfies (1) and (2) and I is named a j-ideal of A for $j \ne i$ if it satisfies (1) and (3).

If for every $1 \le i \le n$, I is an i-ideal, then I is named an ideal of A.

Example 14. *Let $\mathbb{Z}$ and $\mathbb{Q}$ be the set of integers and the set of rational numbers, respectively. Consider two (m,n)-near rings $(\mathbb{Z},h,k)$ and $(\mathbb{Q},h,k)$, where $h(d_1,d_2,,...,d_m) = d_1+d_2+...+d_m$ and $k(d_1,d_2,...,d_n) = d_1 \cdot d_2 \cdot ... \cdot d_{n-1} \cdot d_n$. Then $\mathbb{Z}$ is an (m,n)-subnear ring of $\mathbb{Q}$, but $\mathbb{Z}$ is not an ideal of the near ring $\mathbb{Q}$.*

Remark 1. *If $J_1,J_2,...,J_n$ and $I_1,I_2,I_2,...,I_m$ are ideals of a near ring A, then*

(1) *$h(I_1,I_2,...,I_m)$ is an ideal of A,*
(2) *$J_1 \cap J_2 \cap ... \cap J_n$ is an ideal of A,*
(3) *$k(J_1,J_2,...,J_n)$ is an ideal of A.*

Assume that (A,h,k) is an (m,n)-near ring and I is an ideal. (A,h) is a group and I is a normal subgroup. The quotient group $(A/I,H,K)$ is defined. An m-ary operation h on the cosets is defined by the m-ary operation h as follows:

$$H(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),...,h(d_{m_1},d_{m_2},...,d_{m_{m-1}},I))$$
$$= h(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...$$
$$h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}}h(d_{m_1},d_{m_2},...,d_{m_{m-1}},I)....)).$$

An n-ary operation k on cosets is defined by the n-ary operation k as follows:

$$K(h(d_{1_1},d_{1_2},...,d_{1_{n-1}},I),...,h(d_{n_1},d_{n_2},...,d_{n_{n-1}},I))$$
$$= h(k(h(d_{1_1},d_{1_2},...,d_{1_{n-1}},I),...,h(d_{(i-1)_1},d_{(i-1)_2},...,d_{(i-1)_{(n-1)}},I),d_{i_1},$$
$$h(d_{(i+1)_1},d_{(i+1)_2},...,d_{(i+1)_{n-1}},I))...,h(d_{n_1},d_{n_2},...,d_{n_{m-1}},I)),...,$$
$$k(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),...,h(d_{(i-1)_1},d_{(i-1)_2},...,d_{(i-1)_{m-1}},I),d_{i_{m-1}},$$
$$h(d_{(i+1)_1},d_{(i+1)_2}...,d_{(i+1)_{m-1}},I))...,h(d_{n_1},d_{n_2},...,d_{n_{m-1}},I)),I).$$

Theorem 5. *If I is an ideal in an (m,n)-near ring (A,h,k), then $(A/I,H,K)$, where the operations H and K are defined as above, has the structure of an (m,n)-near ring.*

Proof. We prove that H is well defined. Assume that

$$h(d_{i_1},d_{i_2},...,d_{i_{m-1}},I) = h(e_{i_1},e_{i_2},...,e_{i_{m-1}},I),$$

for $1 \le i \le m$. Then

$H(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),...,h(d_{m_1},d_{m_2},...,d_{m_{m-1}},I))$
$= h(h(d_{1_1},d_{1_2}...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...,$
$h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}}h(d_{m_1},...,d_{m_{m-1}},I)....))$
$= h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...,$
$h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= h(d_{1_1},d_{2_2},...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...$
$h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}},h(I,e_{m_1},e_{m_2},...,e_{m_{m-1}})...))$
$= h(d_{1_1},d_{1_2},...,h(d_{1_{(m-1)}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...,$
$h(h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}},I),h(e_{m_1},e_{m_2},...,e_{m_{m-1}})...))$
$= h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},,h(d_{3_1},d_{3_2},...,d_{3_{m-1}},...,$
$h(h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},I),e_{m_1},e_{m_2},...,e_{m_{(m-1)}})...))$
$= ... = h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(h(d_{2_1},d_{2_2},...,d_{2_{m-1}},I),e_{3_1},e_{3_2},...,e_{3_{m-1}}),...,$
$h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(h(e_{2_1},e_{2_2},...,e_{2_{m-1}},I),e_{3_1},e_{3_2},...,e_{3_{m-1}}),...,$
$h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(h(I,e_{2_1},e_{2_2},...,e_{2_{m-1}}),e_{3_1},...,e_{3_{m-1}}),...,$
$h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= h(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),e_{2_1},e_{2_2},...,e_{2_{m-1}}),h(e_{3_1},e_{3_2},...,e_{3_{m-1}},...$
$h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= h(e_{1_1},e_{1_2},...,e_{1_{m-1}},h(e_{2_1},e_{2_2},...,e_{2_{m-1}},h(e_{3_1},e_{3_2},...,e_{3_{m-1}},...,$
$h(e_{(m-1)_1},e_{(m-1)_2},...,e_{(m-1)_{m-1}},h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)...))$
$= H(h(e_{1_1},e_{1_2},...,e_{1_{m-1}},I),...,h(e_{m_1},e_{m_2},...,e_{m_{m-1}},I)).$

Since I is an ideal, then the operator k is well defined and since (A,h) is an m-ary group so $(A/I,H)$ is an m-ary group. Furthermore, since (A,k) is an n-ary semigroup, it follows that $(A/I,K)$ is an n-ary semigroup. The n-ary operation k is i-distributive with respect to the m-ary operation h. Thus, the n-ary operation k is i-distributive with respect to the m-ary operation H. □

An (m,n)- near ring (A,h,k) is named simple if A does not have non-trivial ideals. A proper ideal I of (A,h,k) is named maximal if $I \subseteq J \subseteq A$ and J is an ideal of A implies that either $I = J$ or $J = A$. A proper ideal I of an (m,n)-near ring (A,h,k) is named prime, if for every ideals $A_1,A_2,...,A_n$ of A, $k(A_1,A_2,...,A_n) \subseteq I$ implies $A_1 \subseteq I$ or $A_2 \subseteq I$ or ... or $A_n \subseteq I$. A proper ideal I of an (m,n)-near ring (A,h,k) is named weakly prime, if for any ideals $A_1,A_2,...,A_n$ of A, $\{0\} \neq k(A_1,A_2,...,A_n) \subseteq I$ implies $A_1 \subseteq I$ or $A_2 \subseteq I$ or ... or $A_n \subseteq I$. Clearly, every prime ideal is weakly prime and (0) is always weekly prime ideal of (A,h,k). An ideal I of an (m,n)-near ring (A,h,k) is named semi-symmetric if $k(\underbrace{z,z,...,z}_{n}) \in I$, implies $k(\underbrace{\langle z\rangle,\langle z\rangle,...,\langle z\rangle}_{n}) \subseteq I$.

Theorem 6. *For an ideal P of an (m,n)-near ring (A,h,k), the following statements are equivalent:*

(1) *P is prime.*

(2) *If* $d_i \notin P$ *and* $1 \leqslant i \leqslant n$*, then* $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \nsubseteq P$.

Proof. To prove (1) $\Rightarrow$ (2) assume P is a prime ideal and $d_i \notin P$ for $1 \leqslant i \leqslant n$. Then $\langle d_i\rangle \nsubseteq P$. If $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \subseteq P$, P is a prime ideal, then $\langle d_1\rangle \subseteq P$ or $\langle d_2\rangle \subseteq P$ or ... or $\langle d_n\rangle \subseteq P$. This is a contradiction. Hence, $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \nsubseteq P$. So (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (1), suppose that $I_1, I_2, ..., I_n$ are ideals of R such that $k(I_1, I_2, ..., I_n) \subseteq P$. Assume that $I_1, I_2, ..., I_n \nsubseteq P$, Then by (2), we have $k(I_1, I_2, ..., I_n) \nsubseteq P$, that is a contradiction. Hence , $I_1 \subseteq P$ or $I_2 \subseteq P$ or ... or $I_n \subseteq P$. So, P is a prime ideal. The proof of (2) $\Rightarrow$ (1) is completed. □

An (m, n)-near ring (A, h, k) is named a zero near ring if $k(\underbrace{A, A, ..., A}_{n}) = 0$. Assume that A is an (m, n)-near ring. The intersection of all prime ideals of A is named the prime radical of A and is denoted by (A). For any proper ideal I of A, the intersection of all prime ideals of A containing I is named the prime radical of I and is denoted by $P(I)$.

Lemma 2. *Every integral* (m, n)*-near ring is prime.*

Proof. Assume that (A, h, k) is an integral (m, n)-near ring. It is enough to show (0) is a prime ideal. Let $I_1, I_2, ..., I_n$ be ideals of A such that $k(I_1, ..., I_n) \subset (0)$. If either $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$, then there is nothing to prove. If possible, suppose that $I_1 \neq (0)$ or $I_2 \neq (0)$ or ... or $I_n \neq (0)$, then we can choose $0 \neq a_1 \in I_1$, $0 \neq a_2 \in I_2$, ..., $0 \neq a_n \in I_n$ such that $k(a_1, a_2, ..., a_n) = 0$, which is in contrast to the fact that A is integral. Therefore, either $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$. Thus, we proved that (0) is a prime ideal of A. Hence, A is a prime (m, n)-near ring. □

Theorem 7. *If the* (m, n)*-near ring* (A, h, k) *is simple, then either* A *is prime or* A *is a zero* (m, n)*-near ring.*

Proof. Assume that A is not a zero (m, n)-near ring. Then $k(A^{(n)}) \neq (0)$. We prove that (0) is a prime ideal of A. Assume that $I_1, I_2, ..., I_n$ are ideals of A such that $k(I_1, I_2, ..., I_n) \subseteq (0)$. Since $I_1, I_2, ..., I_n$ are ideal of A and A is simple, so $I_1, I_2, ..., I_n \in \{(0), A\}$. Then $k(A^{(n)}) \subseteq k(I_1, I_2, ..., I_n) \subseteq (0)$. It is a contradiction. Hence, $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$. Thus, (0) is a prime ideal of A. This yields that A is a prime (m, n)-near ring. □

Theorem 8. *If* I *is a semi-symmetric ideal of an* (m, n)*-near ring* (A, h, k)*, then* $P(I)$ *is completely semiprime.*

Proof. Suppose that $k(a^{(n)}) \in P(I)$. So, $k(k(a^{(n)})^{(n)}) \in I$. Because I is semi-symmetric, $\langle k(k(a^{(n)})^{(n)})\rangle \subseteq I \subseteq P(I)$, thus $a \in P(I)$. This implies that $P(I)$ is completely semiprime. □

If I is a semi-symmetric ideal of a (m, n)-near ring (A, h, k), then

$$P(I) = \{x \in A \mid k(x^{(n)}) \in I\}.$$

An (m, n)-near ring A is named semi-symmetric if $\langle 0 \rangle$ is a semi-symmetric ideal of A.

For any subset S of an (m, n)-near ring (A, h, k),

$$A(S) = \{x \in S \mid k(A^{(i-1)}, x, A^{(n-i)}) = \{0\}\}.$$

Clearly, $A(S)$ is an i-ideal of A. An ideal I of an (m, n)-near ring (A, h, k) is named subtractive or k-ideal, if $h(d_1, d_2, ..., d_m) \in I$ for any elements $d_1, d_2, ..., d_{m-1} \in I$ and $d_m \in A$, then $d_m \in I$.

Theorem 9. *Let I be a k-ideal of an (m, n)-near ring (S, h, k) with $1 \neq 0$. The following statements are equivalent:*

(1) *I is a weakly prime ideal.*
(2) *If $B_1, B_2, ..., B_n$ are ideals of S such that $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq I$, then $B_i \subseteq I$ for some $1 \leq i \leq n$.*

Proof. It is straightforward. □

Theorem 10. *Every ideal of (m, n)-near ring (S, h, k) is weakly prime if and only if for any ideals $B_1, B_2, ..., B_n$ of S, $k(B_1, B_2, ..., B_n) = B_1$ or $k(B_1, B_2, ..., B_n) = B_2$ or .... or $k(B_1, B_2, ..., B_n) = B_n$ or $k(B_1, B_2, ..., B_n) = 0$.*

Proof. Assume that every ideal of S is weakly prime. Let $B_1, B_2, ..., B_n$ be ideals of S and $k(B_1, B_2, ..., B_n) \neq S$, so $k(B_1, B_2, ..., B_n)$ is weakly prime. If $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq k(B_1, B_2, ..., B_n)$, then we have $B_1 \subseteq k(B_1, B_2, ..., B_n)$ or $B_2 \subseteq k(B_1, B_2, ..., B_n)$ or ... or $B_n \subseteq k(B_1, B_2, ..., B_n)$ (since $k(B_1, B_2, ..., B_n)$ is weakly prime ideal of S), that is, $B_1 = k(B_1, B_2, ..., B_n)$ or $B_2 = k(B_1, B_2, ..., B_n)$ or ... or $B_n = k(B_1, B_2, ..., B_n)$. If $k(B_1, B_2, ..., B_n) = S$, then $B_1 = B_2 = ... = B_n = S$ whence $S^n = S$.

Conversely, let I be any proper ideal of S and let $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq I$ for ideals $B_1, B_2, ..., B_n$ of S. Then, either $B_1 = k(B_1, B_2, ..., B_n) \subseteq I$ or $B_2 = k(B_1, B_2, ..., B_n) \subseteq I$ or ... or $B_n = k(B_1, B_2, ..., B_n) \subseteq I$. □

Lemma 3. *If P be a subtractive ideal of i-(m, n)-near ring (S, h, k) such that $2 \leq i \leq n$, then P is a weakly prime ideal but it is not a prime ideal of (m, n)-near ring S. Moreover, $k(d_1, d_2, ..., d_n) = 0$ for some $d_1, d_2, ..., d_n \notin P$, then we have $k(d_{i-1}, P^{(n-1)}) = \{0\}$.*

Proof. If $i = 2$, assume that $k(d_1, p_1^{n-1}) \neq 0$, for some $c_1, c_2, ..., c_{n-1} \in P$. Then

$$0 \neq k(d_1, h(k(1, d_2, d_3, ..., d_n), (k(1, c_1, c_2, ..., c_{n-1}))^{(m-1)}), 1^{(n-2)}) \in P.$$

Since P is a weakly prime ideal of S, it follows that $d_1 \in P$ or

$$h(k(1, d_2, d_3, ..., d_n), (k(1, c_1, c_2, ..., c_{n-1}))^{(m-1)}) \in P,$$

that is, $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$. It is a contradiction. Therefore $k(d_1, P^{(n-1)}) = \{0\}$. Similarly, we can show that $k(P, d_2, P^{(n-2)}) = \{0\}$.

If $3 \leq i \leq n$, suppose that $k(d_{i-1}, c_1^{n-1}) \neq 0$, for some $c_1, c_2, ..., c_{n-1} \in P$. Then, we have

$$0 \neq k(1^{i-2}, d_{i-1}, h((k(c_1^{i-2}, 1, c_i, ..., c_{n-1}))^{i-2}, k(d_1^{i-2}, 1, d_i^n),$$
$$(k(c_1^{i-2}, 1, c_i^{n-1}))^{(m-i+1)}), 1^{n-i}) \in P.$$

Since P is a weakly prime ideal of S, it follows that $d_{i-1} \in P$ or

$$h((k(c_1^{i-2}, 1, c_i^{n-1}))^{(i-2)}, k(d_1^{i-2}, 1, d_i^n), (k(c_1^{i-2}, 1, c_i^{n-1}))^{(m-i+1)}) \in P,$$

that is, $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$. It is a contradiction. Therefore, we derive that $k(d_{i-1}, P^{(n-1)}) = \{0\}$. □

Theorem 11. *Suppose that P is a k-ideal in an i-(m, n)-near ring (S, h, k). If P is weakly prime ideal but not prime, then $P^n = \{0\}$.*

Proof. Assume that $k(c_1, c_2, ..., c_n) \neq 0$ for some $c_1, c_2, ..., c_n \in P$ and $k(d_1, d_2, ..., d_n) = 0$ for some $d_1, d_2, ..., d_n \notin P$, where P is not a prime ideal of S. Hence

$$0 \neq k(d_1^{i-2}, h(d_n, p_i^{m-1}), d_{i+1}^n)$$
$$= h(k(d_1, d_2, ..., d_{i-1}, d_i, d_{i+1}, ..., d_n), (k(d_1^{i-1}, p_i, d_{i+1}^n))^{(m-1)}) \in P.$$

Hence either $d_1 \in P$ or ... or $d_{i-1} \in P$ or $d_{i+1} \in P$ or ... or $d_n \in P$ or $h(d_i, c_i^{m-1}) \in P$, thus either $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$, that it is a contradiction. Hence $P^n = \{0\}$. □

Corollary 1. *Assume that P is a weakly prime ideal of (m, n)-near ring (S, h, k). If P is not a prime ideal of S, then $P \subseteq Nil\ S$, where $Nil\ S$ denotes the set of all nilpotent element of S.*

A k-ideal in a commutative (m, n)-near ring (S, h, k) satisfying that $P^n = \{0\}$.

Lemma 4. *Assume that l is a homomorphism of (m, n)-near ring (S_1, h, k) onto (m, n)-near ring (S_2, h', k'). Then each of the following statements is true:*

1. *If Y is an ideal (k-ideal) in S_1, then $l(Y)$ is an ideal (k-ideal) in S_2.*
2. *If W is an ideal (k-ideal) in S_2, then $l^{-1}(W)$ is an ideal (k-ideal) in S_1.*

Proof. It is straightforward. □

Theorem 12. *If $l : S_1 \longrightarrow S_2$ is a homomorphism of (m, n)-near rings and P is a prime ideal in S_2, then $l^{-1}(P)$ is a prime ideal in S_1.*

Proof. By Lemma 4, $l^{-1}(P)$ is an ideal of (S_1, h, k). If $k(d_1, d_2, ..., d_n) \in l^{-1}(P)$, then $l(k(d_1, d_2, ..., d_n)) \in P$ implies $k'(l(d_1), l(d_2), ..., l(d_n)) \in P$. Hence P is a prime ideal of S_2 therefore it follows that either $l(d_1) \in P$ or $l(d_2) \in P$ or ... or $l(d_n) \in P$ and thus either $d_1 \in l^{-1}(P)$ or $d_2 \in l^{-1}(P)$ or ... or $d_n \in l^{-1}(P)$. Thus $l^{-1}(P)$ is a prime ideal of S_1. □

Theorem 13. *If (S,h,k) be an (m,n)-near ring such that $S=\langle d_1,d_2,...,d_k\rangle$ for $k=\max\{n,m\}$, is a finitely generated ideal of S, Then each proper k-ideal A of S is included in a maximal k-ideal of S.*

Proof. Assume that β is the set of all k-ideals B of S satisfying $A\subseteq B\subset S$, that is partially ordered by inclusion. Take a chain $\{B_i \mid i\in I\}$ in β. Then $B=\bigcup B_i$ is a k-ideal of S, because if $d_1,d_2,...,d_{n-1},h(d_1,d_2,...,d_n)\in B$ then by the definition of B, there is $i_1,i_2,...,i_{n-1},j\in I$ such that $d_1\in B_{i_1},d_2\in B_{i_2},...,d_{n-1}\in B_{i_{n-1}},h(d_1,d_2,...,d_n)\in B_j$, as B_i partially ordered by inclusion, then $B_j\subseteq B_{i_1}$ or $B_{i_1}\subseteq B_j$. Without reduce totality of problem assuming $B_{i_1},B_{i_2},...,B_{i_{n-1}}\subseteq B_j$. So $d_1,d_2,...,d_{n-1},h(d_1,d_2,...,d_n)\in B_j$ because B_j is a k-ideal. Thus $d_n\in B_j$ and $B_j\subseteq B$ then $d_n\in B$ so B is a k-ideal and $S=\langle d_1,d_2,...,d_k\rangle$ implies $B\neq S$ and hence $B\in\beta$. So by Zorn's lemma, β has a maximal element. □

Corollary 2. *Let (S,h,k) be an (m,n)-near ring with identity 1. Then each proper j-ideal of S is included in a maximal j-ideal of S.*

Proof. The proof is immediate by taking $S=\langle 1\rangle$. □

Lemma 5. *If C,D be two j-ideals of an (m,n)-near ring (S,h,k), then $C\cap D$ is a j-ideal.*

Proof. Let C,D be two j-ideals of S, then by definition j-ideal, C and D are subgroups of m-ary group (S,h). so $C\cap D$ is a subgroup of m-ary group (S,h). It is enough to prove for every $d_1,d_2,...,d_n\in S$, $k(d_1^{k-1},C\cap D,d_{k+1}^n)\subseteq C\cap D$. because C is a j-ideal, $k(d_1^{k-1},C\cap D,d_{k+1}^n)\subseteq k(d_1^{k-1},C,d_{k+1}^n)\subseteq C$ and because D is a j-ideal, $k(d_1^{k-1},C\cap D,d_{k+1}^n)\subseteq k(d_1^{k-1},D,d_{k+1}^n)\subseteq D$. therefore $k(d_1^{i-1},C\cap D,d_{i+1}^n)\subseteq C\cap D$. □

Definition 7. *An equivalence relation ρ on an (m,n)-near ring (S,f,g) is called a congruence on S if for any $a_1,a_2,...,a_m,b_1,b_2,...,b_n\in S$ such that $a\rho b$, then for all $1\leq i\leq n$ and $1\leq j\leq m$:*

(1) $f(a_1^{j-1},a,a_{j+1}^m)\rho f(a_1^{j-1},b,a_{j+1}^m)$;
(2) $g(b_1^{i-1},a,b_{i+1}^n)\rho g(b_1^{i-1},b,b_{i+1}^n)$.

Let ρ be a congruence on an (m,n)-near ring (S,f,g). Then, the congruence class of x, S is denoted by $x\rho$ and is defined by $x\rho=\{y\in S\mid (x,y)\in\rho\}$. The set of all congruence classes of S is denoted by S/ρ.

Theorem 14. *Let (S,h,k) be an (m,n)-near ring, then $(S/\rho,h,k)$ is an (m,n)-near ring under the operations*

$$h(d_1\rho,d_2\rho,...,d_m\rho)=h(d_1,d_2,...,d_m)\rho,$$
$$k(d_1\rho,d_2\rho,...,d_n\rho)=k(d_1,d_2,...,d_n)\rho,$$

where $d_1,d_2,...,d_m\in S$ is called quotient near ring.

Proof. Let $d_1\rho, d_2\rho, ..., d_{2m-1}\rho, e_1\rho, e_2\rho, ..., e_m\rho$ be elements of S/ρ. Then for each $1 \leq i \leq j \leq m$,

$$h(d_1\rho, d_2\rho, ..., d_{i-1}\rho, h(d_i\rho, d_{i+1}\rho, ..., d_{m+i-1}\rho), d_{m+i}\rho, d_{m+i+1}\rho, d_{2m-1}\rho)$$
$$= h(d_1\rho, d_2\rho, ..., d_{j-1}\rho, h(d_j\rho, d_{j+1}\rho, ..., d_{m+j-1}\rho), d_{m+j}\rho, d_{m+j+1}\rho, ..., d_{2m-1}\rho).$$

So, the addition is associative on S/ρ. Similarly, the multiplication is associative, too. Finally, in order to show that the right i-distributivity, we have

$$k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, h(e_1\rho, e_2\rho, ..., e_m\rho), d_{i+1}\rho, d_{i+2}\rho, ..., d_n\rho)$$
$$= h(k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_1\rho, d_{i+1}\rho, ..., d_n\rho),$$
$$k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_2\rho, d_{i+1}\rho, ..., d_n\rho),$$
$$..., k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_m\rho, d_{i+1}\rho, ..., d_n\rho)).$$

Therefore, we derive that S/ρ is an (m, n)-near ring. □

Lemma 6. *If (A, h, k) be an (m, n)-near ring with $1 \neq 0$. Then A has at least one j-maximal ideal.*

Proof. Since $\{0\}$ is a proper j-ideal of A, the set Δ of all proper j-ideals of A is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by using Zorn's lemma to this partially ordered set, a maximal j-ideal of A is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. □

Now, we define the concept of a homomorphism between (m, n)-near rings and assert some theorems in this respect.

Definition 8. *A mapping η from the (m, n)-near ring (A, h, k) into the (m, n)-near ring (A', h', k') will be named a homomorphism if for each $d_1, d_2, ..., d_m \in R$*

(1) $(k(d_1, d_2, ..., d_n))\eta = k'((d_1)\eta, (d_2)\eta, ..., (d_n)\eta)$,
(2) $(h(d_1, d_2, ..., d_m))\eta = h'((d_1)\eta, (d_2)\eta, ..., (d_m)\eta)$.

A homomorphism η from the (m, n)-near ring (A, h, k) onto the (m, n)-near ring (A', h', k') is named maximal if for each $d \in A'$ there exists $c_d \in \eta^{-1}(\{d\})$ such that $h(y, ker(\eta)^{(m-1)}) \subset h(c_d, ker(\eta)^{(m-1)})$ for each $y \in \eta^{-1}(\{d\})$ and $ker(\eta) = \{y \in A \mid y\eta = 0\}$.

Lemma 7. *Suppose that η is a homomorphism from the (m, n)-near ring (A, h, k) onto the (m, n)-near ring (A', h', k'). If η be maximal, then $ker(\eta)$ is a Q-ideal, where $Q = \{c_d\}_{d \in A'}$.*

Proof. It is clear that $\bigcup_{d \in A} h(c_d, ker(\eta)^{(m-1)}) = A$. Let c_d and c_b be different elements in Q and $d \neq b$. Let $h(c_d, ker(\eta)^{(m-1)}) \cap h(c_b, ker(\eta)^{(m-1)}) \neq \emptyset$. Thus, there exist $k_1, k_2, ..., k_{m-1}, k'_1, k'_2, ..., k'_{m-1} \in ker(\eta)$ such that $h(c_d, k_1^{m-1}) = h(c_b, {k'}_1^{m-1})$. Thus,

$$\begin{aligned} d &= h'(c_d\eta, k_1\eta, ..., k_{m-1}\eta) = (h(c_d, k_1^{m-1}))\eta = (h(c_b, {k'}_1^{m-1}))\eta \\ &= h'(c_b\eta, k'_1\eta, ..., k'_{m-1}\eta) = b. \end{aligned}$$

This is a contradiction. Hence, we derive that $ker(\eta)$ is a Q-ideal. □

Lemma 8. *Let A, A', η and Q be stated in Lemma 7, and $c_{d_1}, c_{d_2}, ..., c_{d_m}, c_{d_{m+1}}$ be elements in Q.*

(1) *If* $h(h(c_{d_1}, c_{d_2}, ..., c_{d_m}), ker(\eta)^{(m-1)}) \subset h(c_{d_{m+1}}, ker(\eta)^{(m-1)})$, *then* $h'(d_1, d_2, ..., d_m) = d_{m+1}$.

(2) *If* $h(k(c_{d_1}, c_{d_2}, ..., c_{d_n}), ker(\eta)^{(m-1)}) \subset h(c_{d_{n+1}}, ker(\eta)^{(m-1)})$, *then* $k'(d_1, d_2, ..., d_n) = d_{n+1}$.

Proof. (1) Since

$$\begin{aligned} h(c_{d_1}, c_{d_2}, ..., c_{d_m}) \quad & \in h(h(c_{d_1}, c_{d_2}, ..., c_{d_m}), ker(\eta)^{(m-1)}) \\ & \subset h(c_{d_{m+1}}, ker(\eta)^{(m-1)}), \end{aligned}$$

it conforms that there are $k_1, k_2, ..., k_{m-1} \in ker(\eta)$ such that $h(c_{d_1}, c_{d_2}, ..., c_{d_m}) = h(c_{d_{m+1}}, k_1^{m-1})$. Thus, we get

$$\begin{aligned} h'(d_1, d_2, ..., d_m) \quad & = h'(c_{d_1}\eta, c_{d_2}\eta, ..., c_{d_m}\eta) = (h(c_{d_1}, c_{d_2}, ..., c_{d_m}))\eta \\ & = (h(c_{d_{m+1}}, k_1^{m-1}))\eta = h'(c_{d_{m+1}}\eta, k_1\eta, ..., k_{m-1}\eta) = d_{m+1}. \end{aligned}$$

(2) We have

$k(c_{d_1}, c_{d_2}, ..., c_{d_n}) \in h(k(c_{d_1}, c_{d_2}, ..., c_{d_n}), ker(\eta)^{(m-1)}) \subseteq h(c_{d_{n+1}}, ker(\eta)^{(m-1)})$, so there exist $k_1, k_2, ..., k_{m-1} \in ker(\eta)$ such that $k(c_{d_1}, c_{d_2}, ..., c_{d_n}) = h(c_{d_{n+1}}, k_1^{m-1})$. Thus, we obtain

$$\begin{aligned} k'(d_1, d_2, ..., d_n) \quad & = k'(c_{d_1}\eta, c_{d_2}\eta, ..., c_{d_n}\eta) = (k(c_{d_1}, c_{d_2}, ..., c_{d_n}))\eta \\ & = (h(c_{d_{n+1}}, k_1^{m-1}))\eta = h'(c_{d_{n+1}}\eta, k_1\eta, ..., k_{m-1}\eta) = d_{n+1}. \end{aligned}$$

This completes the proof. □

Author Contribution Statements Conceptualization, Mohammadi and Davvaz; methodology, Mohammadi and Davvaz; formal analysis, Davvaz; investigation, Davvaz; resources, Mohammadi and Davvaz; writing-original draft preparation, Mohammadi; writing-review and editing, Mohammadi and Davvaz; supervision, Davvaz; All authors have read and agreed to the published version of the manuscript.

Declaration of Competing Interests The authors declare no conflict of interest.

References

[1] Alam, S., Rao, S., Davvaz, B., (m, n)-Semirings and a generalized fault- tolerance algebra of systems, *J. Appl. Math.*, (2013), Art. ID 482391, 10 pp. https://doi.org/10.1155/2013/482391

[2] Balakrishnan, R., Chelvam, T., α_1 , α_2-Near-rings, *International Journal of Algebra*, 4(2) (2010), 71–79.

[3] Chaudhari, J. N., Nemade, H., Davvaz, B., On partitioning ideals of (m, n)-semirings, *Asian European Journal of Mathematics*, 15(8) (2022), 2250144 (13 pages). https://doi.org/10.1142/S1793557122501443

[4] Clay, J., Near-rings: Geneses and Applications, Oxford, New York, 1992. https://doi.org/10.1093/oso/9780198533986.002.0001

[5] Crombez, G., On (m,n)-rings, *Abh. Math. Semin. Univ. Hambg.*, 37 (1972), 180–199. https://doi.org/10.1007/BF02999695
[6] Crombez, G., Timm, J., On (n, m)-quotient rings, *Abh. Math. Semin. Univ. Hambg.*, 37 (1972), 200–203. https://doi.org/10.1007/BF02999696
[7] Davvaz, B., Leoreanu-Fotea, V., Vougiouklis, T., A survey on the theory of n-hypergroups, *Mathematics*, 11 (2023), 551. https://doi.org/10.3390/math11030551
[8] Davvaz, B., Mohammadi, F., Different types of ideals and homomorphisms of (m,n)-semirings, *TWMS J. Pure Appl. Math.*, 12(2) (2021), 209-222.
[9] Dickson, L., Definitions of a group and a field by independent postulates, *Trans. Amer. Math. Soc.*, 6 (1905), 198-204.
[10] Dörnte, W., Untersuchungen uber einen verallgemeinerten Gruppenbegriff, *Math. Z.*, 29 (1929), 1-19.
[11] Dudek, W. A., Glazek, K., Around the Hosszu-Gluskin theorem for n-ary groups, *Discret. Math.*, 308 (2008), 4861-4876. https://doi.org/10.1016/j.disc.2007.09.005
[12] Post, E. L., Polyadic groups, *Trans. Amer. Math. Soc.*, 48 (1940), 208-350.
[13] Usan, J., Zizovic, M., Some remarks on (m,n)-rings, *Filomat*, 13 (1999), 53-57.
[14] Vasantha Kandasamy, W. B., Smarandache near-rings, American Research Press Rehoboth, NM, 2002.