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Generation of Solitary Waves with Analytical Solution for The (3+1)-dimensional pKP-BKP Equation and Reductions

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Abstract
In this study, new solitary wave solutions are obtained for the combination of the B-type Kadomtsev-Petviashvili (BKP) equation and the potential Kadomtsev-Petviashvili (pKP) equation, called the integrable (3+1)-dimensional pKP-BKP equation, and its two reduced forms. For this purpose, the Bernoulli auxiliary equation method, which is an ansatz-based method, is used. As a result, kink, lump, bright soliton and breather wave solutions are observed. It is concluded that this method and the results obtained for the considered pKP -BKP equations are an important step for further studies in this field.

1. Introduction

Nonlinear partial differential equations (NPDEs) are used to model the complexity of many physical, engineering, and mathematical systems [1-10]. These equations, which allow nonlinear effects to be taken into account, are essential for more accurately describing many real-world phenomena. NPDEs arise in many fields, including fluid mechanics, electromagnetism, wave propagation, and chemical reactions [11-13]. It is important that these equations are integrable because this makes them more mathematically accessible and solvable, making them easier to use in various applications [14]. It also means that these equations can have analytical solutions. Analytical solutions represent the general solution of the equations and are often useful for understanding the behavior of the system. By combining one integrable system with another, many different unexpected results can be seen [15]. There are many examples in the literature [16-20,31-33]. Recently, the pKP-BKP equation also remains important for researchers to obtain more insights and solitary wave results [20-26].

The nonlinear pKP-BKP equation is obtained by considering the potential Kadomtsev-Petviashvili (pKP) equation and the B-type Kadomtsev-Petviashvili (BKP) equation [26]. The pKP (potential Kadomtsev–Petviashvili) equation [29], which models nonlinear and nonlinear waves in two-dimensional space with an additional degree of freedom, has been used in fields as diverse as plasma physics, fluid dynamics, and nonlinear optics [20,21]. The BKP (B-type Kadomtsev–Petviashvili) equation, which describes the interactions between exponentially localized structures, is an important model for the shallow water wave in fluids and the electrostatic wave potential in plasmas [27,28]. The pKP-BKP Eq.(1), which shows the double effect of the newly developed pKP equation and the BKP equation, has been a subject of curiosity and interest to researchers [20,21].

In this paper, the Bernoulli equation method is utilized to obtain solitary wave solutions for a general form of the (3+1)-dimensional pKP-BKP equation [26] as follows:

$$\begin{aligned}&\Delta_{xt}+\alpha\left(15(\Delta_x)^3+15\Delta_x\Delta_{xxx}+\Delta_{xxxxx}\right)_x\\&+\beta\left(6\Delta_x\Delta_{xx}+\Delta_{xxxx}\right)+\gamma\left(\Delta_{xxxy}+3(\Delta_x\Delta_y)_x\right)\\&+a\Delta_{xx}+b\Delta_{xy}+c\Delta_{xz}-\frac{\gamma^2}{5\alpha}\Delta_{yy}=0,\end{aligned} \tag{1}$$

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where some extra terms are inserted into the pKP-BKP equation, such as $b\Delta_{xy}$ and $c\Delta_{xz}$ and $\Delta = \Delta(x,y,z,t)$.When

a. $\gamma = 0$

$$\Delta_{xt} + \alpha\left(15\left(\Delta_x\right)^3 + 15\Delta_x\Delta_{xxx} + \Delta_{xxxxx}\right)_x + \beta\left(6\Delta_x\Delta_{xx} + \Delta_{xxxx}\right) + a\Delta_{xx} + b\Delta_{xy} + c\Delta_{xz} = 0, \tag{2}$$

eliminating three expressions from the pKP-BKP equation,

b. $\beta = 0$

$$\Delta_{xt} + \alpha\left(15\left(\Delta_x\right)^3 + 15\Delta_x\Delta_{xxx} + \Delta_{xxxxx}\right)_x + \gamma\left(\Delta_{xxxy} + 3\left(\Delta_x\Delta_y\right)_x\right) + a\Delta_{xx} + b\Delta_{xy} + c\Delta_{xz} + \mu\Delta_{yy} = 0, \tag{3}$$

eliminating two expressions from the pKP-BKP equation, reduction to two equations will be investigated.

Wazwaz [26] obtained breath-wave solutions for the (3+1)-dimensional pKP-BKP equation using the Hirota bilinear method for Eq.(1). In the study of Ma [21], multiple soliton solutions and lumped solutions were determined. And these newly designed equations were verified to be Painlevé integrable, yielding multiple soliton solutions and lumped solutions for each model developed [26]. For the (3+1)-dimensional pKP-BKP model, several studies have been conducted in the literature [20-26] and the aim of this study is to obtain new results to improve the previous findings.

In summary, the structure of this paper is as, In Section 2 the fundamental properties of the utilized method are desribed and the steps to be used. In Section 3, solutions for the (3+1)-dimensional pKP-BKP equation and its two reduced forms are presented and supported by figures for a better understanding of the dynamics of these solutions. In Section 4, we have given a comparison. In Section 5 the gained results are presented.

2. Bernoulli Auxiliary Equation Method

In this part of the study, we will use the Bernoulli auxiliary equation method to obtain analytical solutions to nonlinear equations. To obtain a solution using this method, the following steps must be followed. Suppose to solve the nonlinear partial differential equation given in the following form:

$$P\left(\Delta, \Delta_x, \Delta_y, \Delta_z, \Delta_t, \Delta_{xx}, \ldots\right) = 0, \tag{4}$$

where P is a function of $\Delta(x, y, z, t)$. Assume the following solitary wave transform structure: $\Delta(x, y, z, t) = u(\xi), \;\; \xi = \tau x + \rho y + \varphi z - \mathrm{d}t$, $d \neq 0$,

Eq. (4) reduces to the following ordinary differential equation (ODE) using the above transformation,

$$O\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \xi^2}, \frac{\partial u}{\partial \xi^3}, \ldots\right) = 0. \tag{5}$$

Assume that the Eq. (5) has the exactly the following solution:

$$u(\xi) = \sum_{i=0}^{N} g_i z^i(\xi), \tag{6}$$

where g_i are the parameters to be determined later. According to the homogeneous equilibrium principle, the value of N is found by adjusting the nonlinear term and the highest order subterms in Eq. (5). The Bernoulli auxiliary equation method [30] is used to determine the function $z(\xi)$ in Eq. (6):

$$\frac{dz}{d\xi} = Pz(\xi) + Qz^2(\xi), \tag{7}$$

where P and Q are constants other than zero and its solution is $z(\xi) = -\dfrac{P}{e^{-P\xi}PC_1 - Q}$ where C_1 is a constant of integration. Substituting Eq. (6) and Eq. (7) into Eq. (5) and setting the coefficients of $z(\xi)$ equal to zero gives a set of algebraic equations that, when solved, give the values of the elements. By calculating the results of these algebraic equations and substituting the aforementioned solution sets into Eq. (6), we can easily obtain the explicit solutions of Eq. (4).

3. Implementation Of Generalized (3+1)-Dimensional pKP-BKP Equation And Reduced Forms

3.1. The application of Eq. (1)

First of all, to reduce Eq.(1) to an ODE, using the classical wave transformation

$\Delta(x, y, z, t) = u(\xi), \;\; \xi = \tau x + \rho y + \varphi z - \mathrm{d}t$ where $\tau \neq 0$, $\rho \neq 0$, $\varphi \neq 0$ and $d \neq 0$, the Eq.(1) becomes:

$$\alpha(45\tau^4\Delta'^2\Delta''+15\tau^5\Delta''\Delta'''+15\tau^5\Delta'\Delta^{iv}+\tau^6\Delta^{vi})$$
$$+\beta(6\tau^3\Delta'\Delta''+\tau^4\Delta^{iv})+\gamma(\tau^3\rho\Delta^{iv}+6\tau^2\rho\Delta'\Delta'')$$
$$+\left(a\tau^2+b\tau\rho+c\tau\varphi-\frac{\gamma^2\rho^2}{5\alpha}-\tau d\right)\Delta''=0. \quad (8)$$

When the $\Delta^{(v)}=N+5$ and $(\Delta')^3=3(N+1)$ terms in Eq.(1) are balancing, $N=1$ is obtained. And from here we found ,

$$u(\xi)=\sum_{i=0}^{1}g_iz^i(\xi)=g_0+g_1z(\xi), \quad (9)$$

where g_0 and g_1 are constants. Substituting Eq.(7) and Eq.(9) into Eq.(8), equating all coefficients of $z^i(\xi)$, $i=1,\ldots,7$ to zero, gives the following set of algebraic equations:

$$z^6(\xi):8Q^2\tau^2+6Q\tau g_1+g_1^2=0\,,$$

$$z^5(\xi):(2Q\tau+g_1)\left(560P^2Q\tau^3\alpha+135P^2\tau^2\alpha g_1+4Q\tau\beta+4Q\rho\gamma\right)=0,$$

$$z^4(\xi):(2Q\tau+g_1)\left(70P^2Q\tau^3\alpha+15P^2\tau^2\alpha g_1+2Q\tau\beta+2Q\rho\gamma\right)=0,$$

$$z^3(\xi):24\left(\frac{25P^2Q\tau^3\beta}{12}+\left(\frac{25}{12}P^2\gamma Q\rho+P^2g_1\beta\right)\tau^2+\left(P^2g_1\rho\gamma+\frac{Qa}{12}\right)\tau+\frac{Q(b\rho+c\varphi-d)}{12}\right)Q\tau$$
$$+45\left(\frac{602}{45}Q^2\tau^2+\frac{26}{3}Q\tau g_1+g_1^2\right)\tau^4P^4\alpha-\frac{2Q^2\gamma^2\rho^2}{5\alpha}=0,$$

$$z^2(\xi):10\left(\frac{21Q\tau}{10}+g_1\right)\tau^5P^4\alpha+-\frac{Q\gamma^2\rho^2}{5\alpha}$$
$$+2\tau\left(\frac{5P^2Q\tau^3\beta}{2}+\left(\frac{5}{2}P^2Q\gamma\rho+P^2g_1\beta\right)\tau^2+\left(P^2\gamma\rho g_1+\frac{Qa}{2}\right)\tau+\frac{Q(b\rho+c\varphi-d)}{2}\right)=0,$$

$$z^1(\xi):P^4\tau^6\alpha+\tau\left(P^2\beta\tau^3+P^2\gamma\rho\tau^2+a\tau+b\rho+c\varphi-d\right)-\frac{\gamma^2\rho^2}{5\alpha}=0.$$

The following results are obtained by solving the above equations:

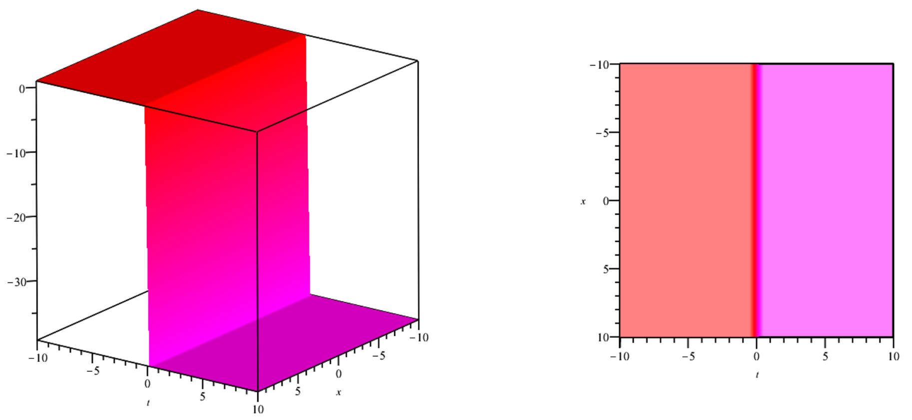
Case 1:

$$d=-\frac{20\alpha^2P^4\tau^6-5a\tau^2\alpha-5b\tau\rho\alpha-5c\tau\varphi\alpha+\gamma^2\rho^2}{5\alpha\tau},\ P=\frac{\sqrt{-5\tau\alpha(\beta\tau+\gamma\rho)}}{5\tau^2\alpha}, \quad (10)$$

$$\Delta(x,y,z,t)=g_0-\frac{2Q\sqrt{-5\tau\alpha(\beta\tau+\gamma\rho)}}{5\tau\alpha\left(\exp\left(-P(-dt+\rho y+\tau x+\varphi z)\right)\frac{C_1\sqrt{-5\tau\alpha(\beta\tau+\gamma\rho)}}{5\tau^2\alpha}-Q\right)}. \quad (11)$$

Figure 1 plots the 3D and contour profiles of solution Eq. (11) with various values of parameters

(a) $y=1$, $z=2$, $\tau=0.05$, $\alpha=0.5$, $g_1=1$, $a=1$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.1$, $Q=1.7$, $\gamma=-1$, $\beta=0.78$, $\rho=1.5$, $\varphi=0.07$ and (b) is a contour map that can be useful for analysis of the difference between two shapes. The resulting shape is a kink soliton, which shows a localized perturbation of the wave along a phase shift.



(a) (b)

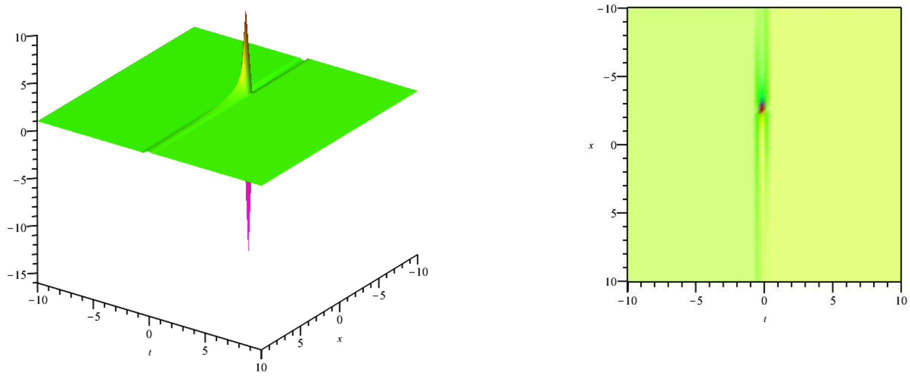
Figure 1. (a) 3D and (b) contour plots of the kink solution.

Case 2:

$$g_1=-2Q\tau\,,\ d=-\frac{-5\alpha^2P^4\tau^6-5\alpha\beta P^2\tau^4-5\alpha P^2\gamma\rho\tau^3-5a\tau^2\alpha-5b\tau\rho\alpha-5c\tau\varphi\alpha+\gamma^2\rho^2}{5\alpha\tau}, \tag{12}$$

$$\Delta(x,y,z,t)=g_0-\frac{2QP\tau}{\exp\left(-P\left(-dt+\rho y+\tau x+\varphi z\right)C_1P-Q\right)}. \tag{13}$$

Figure 2 shown the 3D and contour profiles of solution Eq.(13) with various values of parameters (a) $y=1$, $z=2$, $\tau=0.05$, $\alpha=0.5$, $P=2$, $a=1$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.1$, $Q=1.7$, $\gamma=-1$, $\beta=0.78$, $\rho=1.5$, $\varphi=0.07$ and (b) is a contour plot that can be useful for analysis of the difference between two shapes. The resulting shape is a lump soliton.



(a) (b)

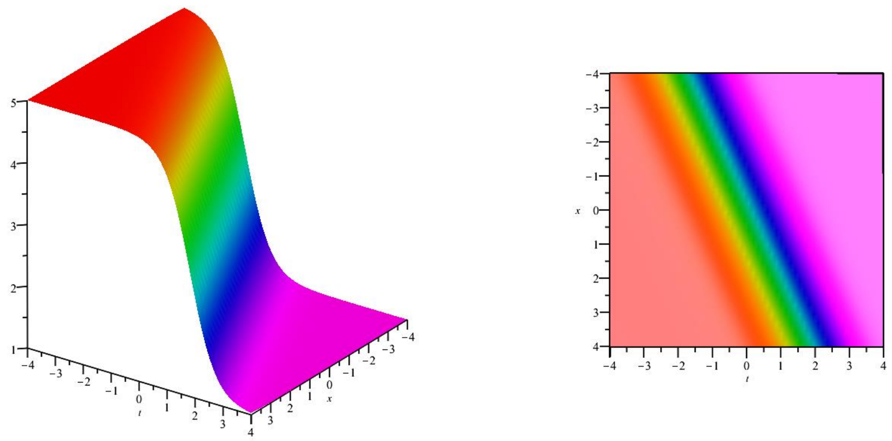
Figure 2. (a) 3D and (b) contour plots of the lump solution.

Case 3:

$$\alpha=\frac{4\beta^2\tau^2+8\beta\gamma\rho\tau+9\gamma^2\rho^2}{25\tau\left(a\tau+b\rho+c\varphi-d\right)},\ Q=-\frac{g_1}{4\tau},\ P=\frac{\sqrt{-5\tau\alpha\left(\beta\tau+\gamma\rho\right)}}{5\tau^2\alpha}, \tag{14}$$

$$\Delta(x,y,z,t)=g_0+\frac{4g_1\sqrt{-5\tau\alpha\left(\beta\tau+\gamma\rho\right)}}{4\exp\left(-P\left(-dt+\rho y+\tau x+\varphi z\right)\right)C_1\sqrt{-5\tau\alpha\left(\beta\tau+\gamma\rho\right)}+5g_1\tau\alpha}. \tag{15}$$

Figure 3 shown the 3D and contour profiles of solution Eq.(15) with various values of parameters (a) $y=0$, $z=0$, $\tau=0.5$, $\alpha=0.02$, $g_1=2$, $a=3$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.5$, $\gamma=-1.1$, $\beta=0.78$, $\rho=1.5$, $\varphi=0.07$, $d=1.1$ and (b) is a contour plot that can be useful for analysis of the between two shapes. Such solitons is determined by the local structure of the soliton.



(a) (b)

Figure 3. (a) 3D and (b) contour plots of the solution Eq.(15).

3.2. The application of Eq. (2)

To reduce Eq.(2) to an ODE, using the classical wave transformation

$\Delta(x,y,z,\ t)=u(\xi),\ \ \xi=\tau x+\rho y+\varphi z-\ dt$ where $\tau\neq 0$, $\rho\neq 0$, $\varphi\neq 0$ and $d\neq 0$, the Eq.(2) becomes:

$$\begin{aligned}&\alpha(45\tau^4\Delta'^2\Delta''+15\tau^5\Delta''\Delta'''+15\tau^5\Delta'\Delta^{iv}+\tau^6\Delta^{vi})\\&+\beta(6\tau^3\Delta'\Delta''+\tau^4\Delta^{iv})\\&+\left(a\tau^2+b\tau\rho+c\tau\varphi-\tau d\right)\Delta''=0.\end{aligned} \tag{16}$$

Balancing Eq.(8), we found $N=1$. Apply the transform,

$$u(\xi)=g_0+g_1z(\xi), \tag{17}$$

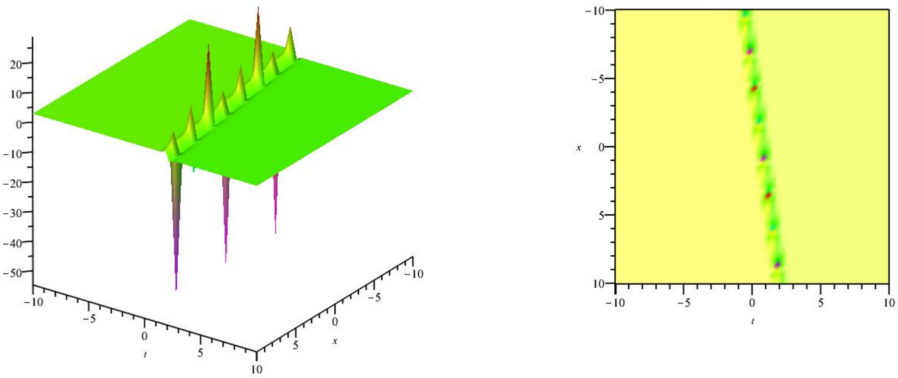
where g_0 and g_1 are constants. Inserting Eq. (7) and Eq. (17) into Eq. (16) and setting all coefficients of $z^i(\xi)$, $i=1,\ldots,7$ to zero results in the set of algebraic equations. Three cases of the parameters obtained by solving this algebraic equation are discussed as follows:

Case 1:

$$g_1=-2Q\tau\,,\ d=P^4\alpha\tau^5+P^2\beta\tau^3+a\tau+b\rho+c\varphi, \tag{18}$$

$$\Delta(x,y,z,t)=g_0+\frac{4Q\beta}{5\alpha P\tau\left(\exp\left(-\frac{\sqrt{-\frac{5\beta}{\alpha}}\left(-dt+\rho y+\tau x+\varphi z\right)}{5\tau}\right)\frac{C_1\sqrt{-\frac{5\beta}{\alpha}}}{5\tau}-Q\right)}. \tag{19}$$

Figure 4 shown the 3D and contour profiles of solution Eq.(19) with various values of parameters (a) $y=1$, $z=2$, $\tau=0.5$, $\alpha=0.5$, $a=1$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.1$, $Q=1.7$, $\gamma=-1$, $\beta=0.78$, $\rho=1.5$, $\varphi=0.07$ and (b) is a contour plot that can be useful for analysis of the between two shapes. Breather wave solution is obtained for the (3 + 1) dimensional pKP - BKP Eq.(2).



(a) (b)

Figure 4. (a) 3D and (b) contour plots of the breather solution.

Case 2:

$$d=\frac{25P\alpha b\rho+25P\alpha c\varphi+25a\alpha\sqrt{-\frac{\beta}{5\alpha}}-4\beta^{2}\sqrt{-\frac{\beta}{5\alpha}}}{25\alpha P},\ P=\frac{\sqrt{-\frac{5\beta}{\alpha}}}{5\tau},\ g_{1}=-\frac{4Q\sqrt{-\frac{\beta}{5\alpha}}}{P},\tag{20}$$

$$\Delta\left(x,y,z,t\right)=g_{0}+\frac{4Q\beta}{\alpha\sqrt{-\frac{5\beta}{\alpha}}\left(\exp\left(-\frac{\sqrt{-\frac{5\beta}{\alpha}}\left(-dt+\rho y+\tau x+\varphi z\right)}{5\tau}\right)\frac{C_{1}\sqrt{-\frac{5\beta}{\alpha}}}{5\tau}-Q\right)}.\tag{21}$$

Figure 5 plots the 3D and contour maps of solution Eq.(17) with different values of parameters (a) $y=0$, $z=0$, $\tau=0.82$, $\alpha=0.5$, $a=1$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.3$, $Q=1.7$, $\gamma=1$, $\beta=-0.78$, $\rho=1.5$, $\varphi=0.07$ and (b) is a contour plot that can be useful for analysis of the between two shapes. One bright soliton solution is found for the (3+ 1) dimensional pKP - BKP Eq.(2).

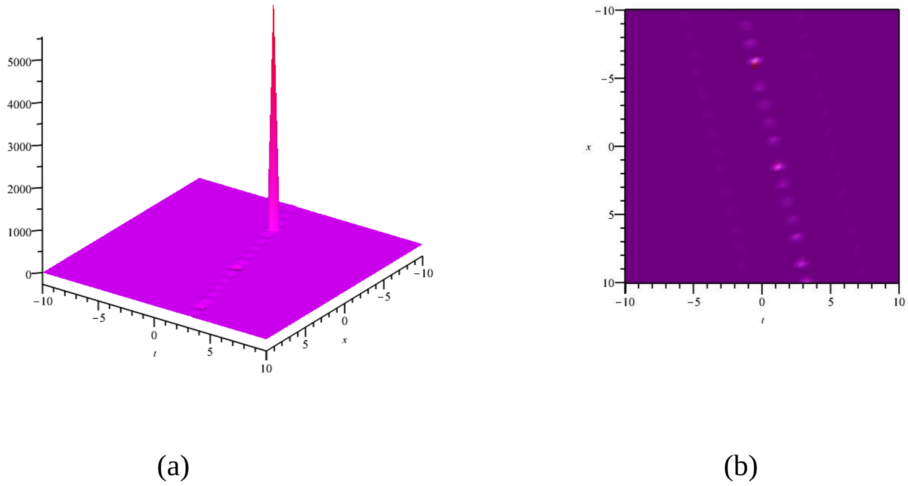


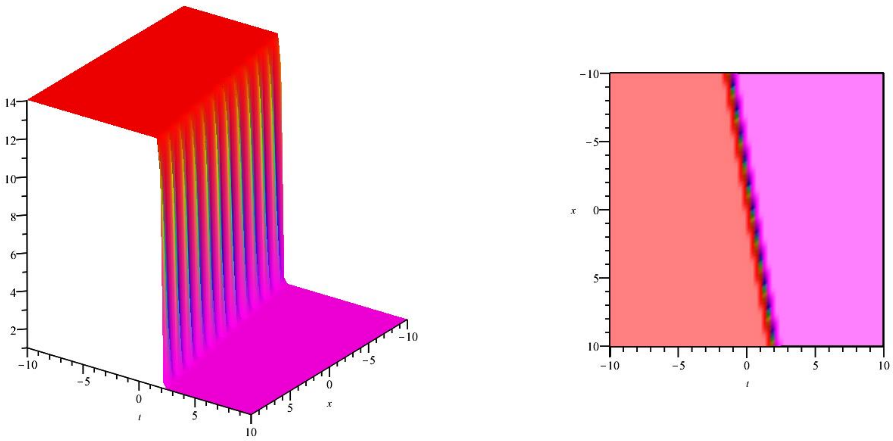
Figure 5. (a) 3D graph of the solution Eq.(17) and (b) contour graph of the solution Eq.(21).

Case 3:

$$\alpha = \frac{4\beta^2}{25(a\tau + b\rho + c\varphi - d)},\ Q = -\frac{g_1}{4\tau},\ P = \frac{\sqrt{-5\alpha\beta}}{5\tau\alpha}, \tag{22}$$

$$\Delta(x,y,z,t) = g_0 - \frac{4g_1\beta}{-4\exp\left(-\frac{\sqrt{-5\alpha\beta}(-dt+\rho y+\tau x+\varphi z)}{5\alpha\tau}\right)C_1\beta + \sqrt{-5\alpha\beta}g_1}. \tag{23}$$

Figure 6 shows the 3D and contour maps of solution Eq.(23) with different values of parameters (a) $y=0$, $z=5$, $\tau=0.08$, $d=0.5$, $a=1$, $b=1.8$, $c=1.3$, $g_0=1$, $C_1=0.1$, $Q=1.7$, $\gamma=-1$, $\beta=-0.78$, $\rho=0.5$, $\varphi=0.04$, $g_1=1$ and (b) is a contour plot that can be useful for analysis of the between two shapes. Kink solution is obtained for the (3+ 1) dimensional pKP - BKP Eq.(2).



(a) (b)

Figure 6. (a) 3D graph of the kink solution Eq.(23) and (b) contour graph of the kink solution Eq.(23).

3.3. The application of Eq. (3)

Reducing Eq.(3) to an ODE, using the classical wave transformation

$\Delta(x, y, z, t) = u(\xi), \quad \xi = \tau x + \rho y + \varphi z - dt$ where $\tau \neq 0$, $\rho \neq 0$, $\varphi \neq 0$ and $d \neq 0$, the Eq.(3) is transformed into:

$$\begin{aligned}&\alpha(45\tau^4\Delta'^2\Delta'' + 15\tau^5\Delta''\Delta''' + 15\tau^5\Delta'\Delta^{iv} + \tau^6\Delta^{vi}) \\ &+\gamma(\tau^3\rho\Delta^{iv} + 6\tau^2\rho\Delta'\Delta'') \\ &+\left(a\tau^2 + b\tau\rho + c\tau\varphi + \mu\rho^2 - \tau d\right)\Delta'' = 0.\end{aligned} \tag{24}$$

If we balance Eq.(8), we find $N = 1$. The use of the conversion,

$$u(\xi) = g_0 + g_1 z(\xi), \tag{25}$$

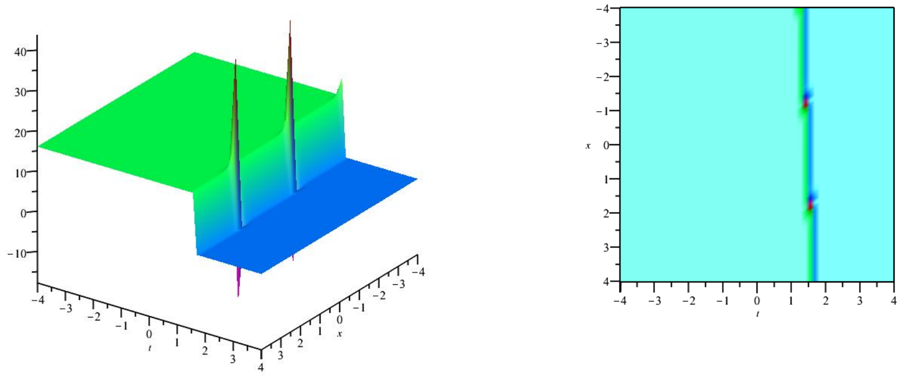
where g_0 and g_1 are constants. Exchange Eq.(7) and Eq.(25) into Eq.(24), equating all coefficients of $z^i(\xi)$, $i = 1, \ldots, 7$ to zero, provides the set of algebraic equation systems. We found parameters corresponding to the following cases when solving these systems:

Case 1:

$$g_1 = -2Q\tau\,, d = \frac{P^4\alpha\tau^6 + P^2\gamma\rho\tau^3 + a\tau^2 + b\tau\rho + c\tau\varphi + \mu\rho^2}{\tau}, \tag{26}$$

$$\Delta(x, y, z, t) = g_0 - \frac{2QP\tau}{\exp\left(-P\left(-dt + \rho y + \tau x + \varphi z\right)C_1 P - Q\right)}. \tag{27}$$

Figure 7 plots the 3D and contour profiles of solution Eq.(27) with various values of parameters (a) $y = 1$, $z = 2$, $\tau = 0.05$, $\alpha = 0.5$, $g_1 = 2$, $a = 1$, $b = 1.8$, $c = 1.3$, $g_0 = 1$, $C_1 = 0.1$, $Q = 2$, $\gamma = -1$, $\mu = 0.06$, $\rho = 1.5$, $\varphi = 0.07$, $d = 1.1$ and (b) is a contour map that can be useful for analysis of the difference between two shapes. Two lump soliton solution is found for the (3 + 1) dimensional pKP - BKP Eq.(3).



(a) (b)

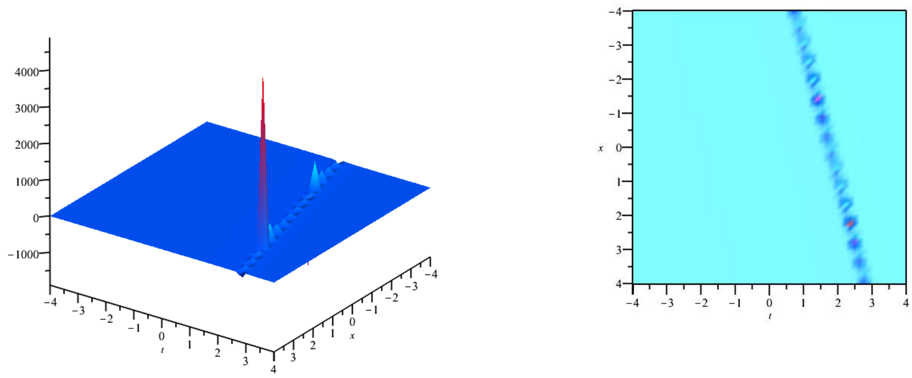
Figure 7. (a) 3D and (b) contour plots of the two lump solution Eq.(27).

Case 2:

$$g_1=-4Q\tau\,,\ d=-\frac{-25P^4\alpha^2\mu\tau^5+4P^4\alpha\gamma^2\tau^5+5P^2\alpha b\gamma\tau^3-a\tau\gamma^2-c\gamma^2\varphi}{\gamma^2}\,,\ \rho=-\frac{5P^2\alpha\tau^3}{\gamma}, \tag{28}$$

$$\Delta\left(x,y,z,t\right)=g_0-\frac{2QP\tau}{\exp\left(-P\left(-dt-\frac{5P^2\alpha\tau^3}{\gamma}y+\tau x+\varphi z\right)C_1P-Q\right)}. \tag{29}$$

Figure 8 plots the (a) 3D and (b) contour maps of solution Eq.(29) with different values of parameters for $y=0$, $z=0$, $P=2$, $\tau=0.05$, $\alpha=0.5$, $g_1=1$, $a=2$, $b=1.8$, $c=1.3$, $g_0=-1$, $C_1=0.5$, $Q=2$, $\gamma=-1$, $\mu=0.06$, $\varphi=0.07$, $d=1.1$. The one bright soliton shape and amplitude modulation of the soliton are shown in the graph.



(a) (b)

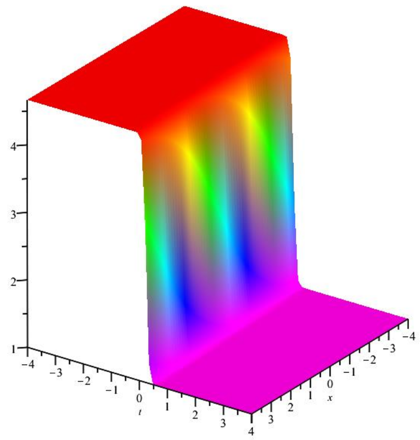
Figure 8. (a) 3D and (b) contour plots of Eq.(29) when $y=0$, $z=0$, $P=2$, $\tau=0.05$, $\alpha=0.5$, $g_1=1$, $a=2$, $b=1.8$, $c=1.3$, $g_0=-1$, $C_1=0.5$, $Q=2$, $\gamma=-1$, $\mu=0.06$, $\varphi=0.07$, $d=1.1$.

Case 3:

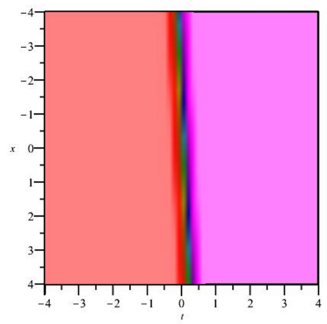
$$a=\frac{-25\alpha b\rho\tau-25\alpha c\tau\varphi-25\alpha\mu\rho^2+4\gamma^2\rho^2+25\alpha d\tau}{25\alpha\tau^2},\ Q=-\frac{g_1}{4\tau},\ P=\frac{\sqrt{-5\alpha\tau\gamma\rho}}{5\tau^2\alpha},\tag{30}$$

$$\Delta(x,y,z,t)=g_0-\frac{4g_1\sqrt{-5\alpha\tau\gamma\rho}}{4\exp\left(-\frac{\sqrt{-5\alpha\tau\gamma\rho}\left(-dt+\rho y+\tau x+\varphi z\right)}{5\alpha\tau^2}\right)C_1\sqrt{-5\alpha\tau\gamma\rho}+5\alpha\tau g_1}.\tag{31}$$

Figure 9 shows the (a) 3D and (b) contour maps of solution Eq.(31) with different values of parameters for $y=0$, $z=2$, $\tau=0.07$, $\alpha=1.7$, $g_1=2$, $a=3$, $b=0.02$, $c=2$, $g_0=1$, $C_1=0.8$, $Q=1$, $\gamma=-1$, $\mu=0.2$, $\varphi=0.07$, $d=1.7$, $\rho=0.5$. The obtained kink solution is important for the understanding of the dynamics and the explanation of complex phenomena such as phases and pattern formation.



(a)



(b)

Figure 9. Select $y=0$, $z=2$, $\tau=0.07$, $\alpha=1.7$, $g_1=2$, $a=3$, $b=0.02$, $c=2$, $g_0=1$, $C_1=0.8$, $Q=1$, $\gamma=-1$, $\mu=0.2$, $\varphi=0.07$, $d=1.7$, $\rho=0.5$ for (a) 3D and (b) contour plots of Eq.(31).

4. Comparison

In this section, we compare the obtained solutions with the solutions obtained by Wazwaz [26] using the Hirota bilinear method, the tan method, and the tanh method. Wazwaz used the Hirota bilinear method, the tan method, and the tanh method to search for soliton solutions of the (3+1) dimensional pKP-BKP equation for which the kink, singular, periodic, and exponential solutions exist. In this context, new solutions with many physical properties such as kink, lump, breather and bright soliton have been shown. On the other hand, using the Bernoulli auxiliary equation method, we found nine exact solutions of the (3+1) dimensional pKP-BKP equation. [26] shows that there are significant differences between them. This proves the superiority of the Bernoulli auxiliary equation method over other computational methods.

Also a new extended direct algebraic method proposed in [34] focuses on providing closed-form solutions for a wide class of nonlinear pseudo-parabolic models by direct algebraic manipulations, while the Bernoulli auxiliary equation method can provide fast and efficient solutions for specific models and conditions, since it is usually applied under specific conditions. Thus, we can state that this method provides a powerful, efficient and easy-to-use option for all NPDEs.

5. Conclusion

In this study we derive solutions for the (3+1)-dimensional pKP-BKP equation and its two reduced forms. We derive different types of wave solutions for each of the equations studied. Solitons are types of waves in which energy is transported in a localized form that remains unchanged in time and space. They include bright solitons, dark solitons, and kink solitons. By giving different values to the parameters in the master equation, different types of soliton solutions are obtained. Bright soliton waves, which occur in many nonlinear systems and are waves in which the energy or intensity increase is localized in a specific region, are constructed in Figs. 5 and 8. Kink soliton waves that result from changes in the environment or from local changes in the structure of the object are observed in Figs. 1, 6, and 9. Also breather waves, which are waves that occur in a given medium and spontaneously change their wavelength or form certain periodic or quasiperiodic shapes, are observed in Figure 4. Figure 7 shows a lump wave, which is defined as a wave with a solitonic structure, usually carrying a localized energy density. It is believed that these solutions will make an important contribution to the understanding of nonlinear phenomena. 3D and contour plots are presented together to understand the dynamics of these wave solutions. 3-D graphs show how the wave varies in space and time, while contour graphs show the wave as an array of lines connecting points of the same magnitude. It has been observed that different solutions emerge from those obtained in previous studies [20-26]. This will provide a basis for future studies on the pKP-BKP equation.

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Statement of Research and Publication Ethics

The study complies with the ethics of research and publication.

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