Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 13 NO: 4 PAGE: 1023-1030 YEAR: 2024
DOI: 10.17798/bitlisfen.1495657

Numerical Solutions for Mixed Fractional Order Two-Dimensional Telegraph Equations

Fatih ÖZBAĞ[1*], Mahmut MODANLI[1], Sadeq Taha ABDULAZEEZ[2]

[1] *Harran University, Faculty of Arts and Sciences, Department of Mathematics, Şanlıurfa, Türkiye*
[2] *University of Duhok, College of Basic Education, Department of Mathematics, Duhok, Iraq*
(ORCID: 0000-0002-5456-4261) (ORCID: 0000-0002-7743-3512) (ORCID: 0000-0003-4515-1585)

Keywords: Two-Dimensional Telegraph Equation, Caputo and ABC Fractional Derivatives, Finite Difference Technique, Numerical Solution.

Abstract
This research presents an innovative numerical approach to solving two-dimensional telegraph equations of mixed fractional order by integrating the fractional derivatives of Caputo and Atangana-Baleanu Caputo (ABC) into a single model. Using MATLAB as its implementation, the research creates a customized first-order difference scheme and analyses stability. The ability to manage mixed fractional derivatives in 2D telegraph equations, a situation that has not been tackled in literature before, is the method's innovative aspect. This development shows that these complicated equations may be efficiently and reliably modelled, opening up new avenues for the study of complex physical phenomena. The work makes a substantial contribution to the numerical analysis of fractional differential equations with mixed derivative types and opens up possible applications in areas like wave propagation and anomalous diffusion processes.

1. Introduction

Initial value problems of fractional order can be used to model a variety of scientific disciplines, helping us better understand how to characterize natural phenomena in fields like engineering, physics, economics, biology, and seismology. According to its applications in numerous engineering and scientific domains, such as [1]-[4], fractional calculus theory has grown in popularity and importance during the last few decades. Additionally, it has demonstrated success in simulating actual issues that arise in the field of chemistry [5], physics [6], biology [7], and other fields.

Over the past few years, researchers have extensively studied different aspects of fractional models applied to the telegraph equation, for example, see [8]-[14]. The authors in [8] provided a hybrid approach built on the finite difference scheme and Sinc- Galerkin approach allows for a partial solution of the two-dimensional hyperbolic telegraph problem. The B-spline collocation method was introduced by the authors in [9] as a method for solving the 1D hyperbolic telegraph equation, additionally, they demonstrated how the suggested method converges. The authors in [10] used a Galerkin-like approach to numerically present the two-dimensional hyperbolic telegraph equations. To solve a two-dimensional hyperbolic telegraph equation with both Dirichlet and Neumann boundary conditions, the authors in [11] developed a numerical solution based on the polynomial differential quadrature method. In [12], Oruç used the Hermite wavelet-based method for obtaining 2D hyperbolic telegraph equation solutions in numerical form. The approximate numerical solution and stability estimates for the two-dimensional telegraph equation were obtained employing finite difference schemes in [13]. In [14], the authors proposed a semi-discrete method utilizing shifting shape functions of least squares to provide numerical results of complicated variable-order time fractional 2-dimensional telegraph problems. Through the use of the Legendre wavelet collocation technique, the authors in [15], a collection of time-fractional telegraph equations that take the Caputo fractional derivative. The authors of

[*]Corresponding author: *fozbag@harran.edu.tr* *Received: 04.06.2024, Accepted: 16.10.2024*

[16] employed a hybrid approach that incorporated the integration of the Legendre polynomials and the block-pulse functions.

The finite difference method has been successfully used to solve a variety of issues, including fractional order telegraph equations [17], fractional chaotic systems in the sense of Caputo [18], and pseudo-hyperbolic telegraph equations based on the fractional operators [19]-[20], advection–dispersion equations [21], generalized fractional derivative terms in the fractional telegraph equation [22] and the higher order mixed fractional differential equations [23]. One dimensional linear and nonlinear hyperbolic telegraph equation is studied in [24-30]. Moreover, our work extends the application of finite difference methods to the particular case of mixed Caputo and ABC fractional derivatives in the context of 2D telegraph equations, whereas previous studies like [17]–[23] have used these methods to various fractional order equations. This development makes it possible to represent complex physical systems more accurately in fields like engineering, physics, and biology, where the interaction between different types of fractional derivatives can be important for modeling real-world phenomena. In the present study, our main goals are twofold: firstly, to introduce an innovative formulation of the fractional order two-dimensional telegraph partial differential equation, and secondly, to propose a finite difference technique for obtaining numerical approximations of these equations. Notably, since no previous research has directly addressed the fractional order two-dimensional telegraph equation using the fractional derivatives of Caputo and ABC, our approach is unique in this regard.

For this, we take into account the fractional order two-dimensional telegraph equation depending on the fractional derivatives, as follows

$$\begin{aligned} &{}_0^C D_t^{\alpha} v(t,x,y) + {}^{ABC}_{\ \ 0}D_t^{\beta} v(t,x,y) + v(t,x,y) = \\ &v_{xx}(t,x,y) + v_{yy}(t,x,y) + f(t,x,y) \qquad (1) \\ &0 < x, y < L,\ 0 < t, 1 < \alpha \leq 2, 0 < \beta \leq 1, \end{aligned}$$

with initial and boundary conditions

$$v(0,x,y) = v_t(0,x,y) = \varphi(x),\ 0 \leq x, y \leq L \qquad (2)$$

$$\begin{aligned} &v(t,0,y) = v_t(t,L,y) = \sigma(x), \\ &0 \leq t \leq C, 0 \leq y \leq L \qquad (3) \end{aligned}$$

$$\begin{aligned} &v(t,x,0) = v_t(t,x,L) = \mu(x), \\ &0 \leq t \leq C, 0 \leq x \leq L \qquad (4) \end{aligned}$$

where, $\varphi(x)$, $\mu(x)$, $\sigma(x)$, and $f(t,x,y)$ are known functions, but $v(t,x,y)$ is an unidentified function that should be examined. The initial and boundary conditions of the telegraph partial differential equation play a crucial role in determining the behavior of the solution, as they define how signals propagate in time and space along the transmission line. These are essential in ensuring the solution remains physically meaningful over the entire spatial domain.

${}_0^C D_t^{\alpha} v(t,x,y)$ is the Caputo fractional derivative that is defined as

$${}_0^C D_t^{\alpha} v(t,x,y) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{v^{(n)}(s,x,y)}{(t-s)^{\alpha-n+1}} ds,$$

$n - 1 < \alpha \leq n$ and $n = 1,2, \ldots \in N$. Also, ${}^{ABC}_{\ \ 0}D_t^{\beta} v(t,x,y)$ is the Atangana–Baleanu Caputo fractional derivative and defined as

$${}^{ABC}_{\ \ 0}D_t^{\beta} v(t,x,y) = \frac{B(\beta)}{1-\beta} \int_0^t v'(s,x,y) E_{\beta}\left[\frac{-\beta}{1-\beta}(t-s)^{\beta}\right] ds.$$

Here $B(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$ and $E_{\beta}\left[\frac{-\beta}{1-\beta}(t-s)^{\beta}\right] = \sum_{k=0}^{\infty} \frac{\left[\frac{-\beta}{1-\beta}(t-s)^{\beta}\right]^k}{\Gamma(\beta k+1)}$ is the Mittag-Leffler function.

The improved Caputo and ABC fractional derivatives facilitate the application of derivatives to non-integer orders, effectively representing memory effects in physical systems. The Caputo derivative is extensively utilized in domains such as viscoelasticity, anomalous diffusion, and control theory because it offers beginning conditions analogous to those of integer-order equations. The ABC fractional derivative improves this by incorporating a non-local and non-singular kernel, rendering it appropriate for intricate systems with long-range interactions. It is utilized in epidemiology, chaos theory, and financial modeling. By integrating these two derivatives in our examination of the two-dimensional telegraph equation, we develop a more adaptable model that encompasses many physical phenomena, tackling situations where conventional fractional derivatives are unsatisfactory.

To solve problem (1), the finite difference technique is implemented. First-order difference schemes were built for the suggested model. The explicit finite difference method is used to analyze the error estimates for the two-dimensional telegraph equation dependent on the Caputo and ABC fractional derivatives. The Von-Neumann analysis approach is then provided to generate stability estimations for the stated problem.

This methodology provides a reliable and efficient mechanism for modeling these complex equations, as highlighted in the papers focused on stability analysis. Given that the existing literature has not thoroughly explored this specific issue, the finite difference method's capacity to accommodate fractional derivatives of Caputo and Atangana-Baleanu Caputo (ABC) makes it an appropriate selection for this research.

The following is how the current paper is structured: Part 2 presents a finite difference technique for the proposed model with the addition of a stability analysis. Part 3 discusses the numerical calculations of the introduced model. The article's conclusions are provided in Part 4.

2. Material and Method

2.1. Finite Difference Technique and Stability Analysis

We analyze the stability of the proposed model and devise a numerical method known as a finite difference scheme. The scheme is designed to solve a mixed fractional order two-dimensional telegraph partial differential equation. It incorporates the use of Caputo and ABC fractional derivative.

To compute the numeric solution for the problem (1), the initial stage involves creating a difference scheme with first-order accuracy. This is achieved by adopting a specific grid spacing as $G_{\tau,h} = [0,C]_\tau \times [0,L]_h$ with $t_k = k\tau, x_n = nh, y_m = mh$ and $k = 1,2,\ldots,N, n,m = 1,2,\ldots,M$. Also for x and y axes, we take $h = \frac{L}{M}$ and for t axes, $\tau = \frac{C}{N}$. The finite difference techniques for Caputo and ABC fractional derivatives are then demonstrated. The following is how the Caputo fractional derivative of order $1 < \alpha \leq 2$ is constructed using a first-order difference technique.

$$
{}_0^C D_t^\alpha v(t_k, x_n, y_m) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} ((j+1)^{2-\alpha} - j^{2-\alpha})\left(v_{n,m}^{k-j+1} - 2v_{n,m}^{k-j} + v_{n,m}^{k-j-1}\right).
$$

For simplicity, we call $b_j^\alpha = ((j+1)^{2-\alpha} - j^{2-\alpha})$.

Next, the ABC fractional derivative of order $0 < \beta \leq 1$ is thus supplied with a first-order difference technique as

$$
{}_0^{ABC}D_t^\beta v(t_k, x_n, y_m) = \frac{1}{\Gamma(\beta)} \left(\sum_{j=0}^{k} \frac{v_{n,m}^{k+1} - v_{n,m}^k}{\tau} ((t_j - t_{k+1})^{1-\beta} - \left(t_j - t_k\right)^{1-\beta}) \right).
$$

For simplicity, we call

$$
d_j^k = ((t_j - t_{k+1})^{1-\beta} - \left(t_j - t_k\right)^{1-\beta}).
$$

Then, we need to give the following difference formula of the model (1), as

$$
\begin{cases}
\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} b_j^\alpha \left(v_{n,m}^{k-j+1} - 2v_{n,m}^{k-j} + v_{n,m}^{k-j-1}\right) \\
+ \frac{1}{\Gamma(\beta)} \left(\sum_{j=0}^{k} \frac{v_{n,m}^{k+1} - v_{n,m}^k}{\tau} d_j^k \right) + v_{n,m}^k \\
\quad = \frac{v_{n+1,m}^k - 2v_{n,m}^k + v_{n-1,m}^k}{h^2} + \\
\frac{v_{n,m+1}^k - 2v_{n,m}^k + v_{n,m-1}^k}{h^2} + f_n^k.
\end{cases} \tag{5}
$$

After rewriting the formula (5), we obtain

$$
\begin{cases}
\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{k-1} b_j^\alpha \left(v_{n,m}^{k-j+1} - 2v_{n,m}^{k-j} + v_{n,m}^{k-j-1}\right) \\
-\frac{1}{h^2} v_{n+1,m}^k - \frac{1}{h^2} v_{n-1,m}^k + \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{d_j^k}{\tau\Gamma(\beta)}\right) v_{n,m}^{k+1} \\
+ \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} v_{n,m}^{k-1} + \\
\left(\frac{-2\tau^{-\alpha}}{\Gamma(3-\alpha)} - \frac{d_j^k}{\tau\Gamma(\beta)} + 1 + \frac{4}{h^2}\right) v_{n,m}^k \\
-\frac{1}{h^2} v_{n,m+1}^k - \frac{1}{h^2} v_{n,m-1}^k = f_n^k.
\end{cases} \tag{6}
$$

For the stability of scheme (6) we give the following theorem.

Theorem 2.1. If the following condition is satisfied, the formula (6) have the stability estimates:

$$
\tau^{-\alpha} < \Gamma(3-\alpha)(1 + \tfrac{8}{h^2}).
$$

Proof. Using the Von-Neumann analysis method for the formula (6) and with the given condition, the proof of Theorem 2.1. can be shown easily.

2.2. Numerical Results

The present section offers the substantially numerical solutions for the mixed fractional order two-dimensional telegraph problem that depend on the Caputo and ABC fractional derivatives. In the section that follows, we solve a test problem using the numerical method to demonstrate the usefulness of the strategy. We assess the method's performance by computing the maximum norm errors. Our focus is a computational evaluation of the mixed fractional order two-dimensional telegraph equation's initial boundary value problem.

Example 1. We examine a set of mixed fractional order partial differential equations in a two-dimensional telegraph system.

$$\begin{cases} {}_{0}^{C}D_t^{\alpha} v(t,x,y) + {}^{ABC}_{\ \ \ 0}D_t^{\beta} v(t,x,y) + v(t,x,y) \\ = v_{xx}(t,x,y) + v_{yy}(t,x,y) + f(t,x,y) \\ f(t,x,y) = \frac{1}{B(\beta)}\left(\frac{6(1-\beta)}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{6\beta t^{\beta+3-\alpha}}{\Gamma(\beta+4-\alpha)}\right) sin(x)\sin(y) \\ +t^3 sin(x) sin(y) \\ +\frac{3}{B(\beta)}\left((1-\beta)t^3 + 6\beta \frac{t^{\beta+3}}{\Gamma(\beta+4)}\right) sin(x)\sin(y) \qquad (7) \\ 1 < \alpha \le 2, 0 < \beta \le 1, \\ v(0,x,y) = v_t(0,x,y) = 0, \ \ 0 \le x,y \le \pi, \\ v(t,0,y) = v_t(t,L,y) = 0, \ \ 0 \le t \le 1, 0 \le y \le \pi, \\ v(t,x,0) = v_t(t,x,L) = 0, \ \ 0 \le t \le 1, 0 \le x \le \pi. \end{cases}$$

One can easily check that the exact solution of (7) is

$$v(t,x,y) = \frac{1}{B(\beta)}\left((1-\beta)t^3 + 6\beta \frac{t^{\beta+3}}{\Gamma(\beta+4)}\right) sin(x) sin(y).$$

We use the formula (6), apply the similar numerical procedure in [13] and utilize a modified Gauss elimination method to solve the problem (7) for n and m. The MATLAB software is employed to acquire solutions for equation (7). We compute numerical solutions for various grid points of N and M, and the error is calculated using the subsequent formula;

$$\varepsilon = \max_{1\le k\le N-1, 1\le n\le M-1} |v(t_k,x_n,y_m) - v(t,x,y)|$$

where $v(t_k,x_n,y_m)$ is approximate and $v(t,x,y)$ is exact solution. In problem (7), we take $\alpha = 1.9$ and $\beta = 0.9$ then calculate the error values in Table 1.

Table 1. Error table for problem (7)

N, M	Error values
N=M=5	0.1873
N=M=10	0.0869
N=M=15	0.0507
N=M=20	0.0358
N=M=25	0.0282
N=M=30	0.0246
N=M=35	0.0228
N=M=40	0.0221
N=M=50	0.0221

The error margin can be seen in the error analysis table in the numerical calculations for the created difference scheme that existed less than one. The decreasing trend in maximum norm errors as the number of grid points increases highlights the sensitivity of the scheme. Table 1 confirms the accuracy of the generated difference scheme. Furthermore, numerical simulations are provided for N and M values to demonstrate the similarity between the exact and approximate solutions. Figure 1 and Figure 2 represent the exact solution and numerical solution of (7) respectively. From these figures, one can conclude that the solutions are almost identical.

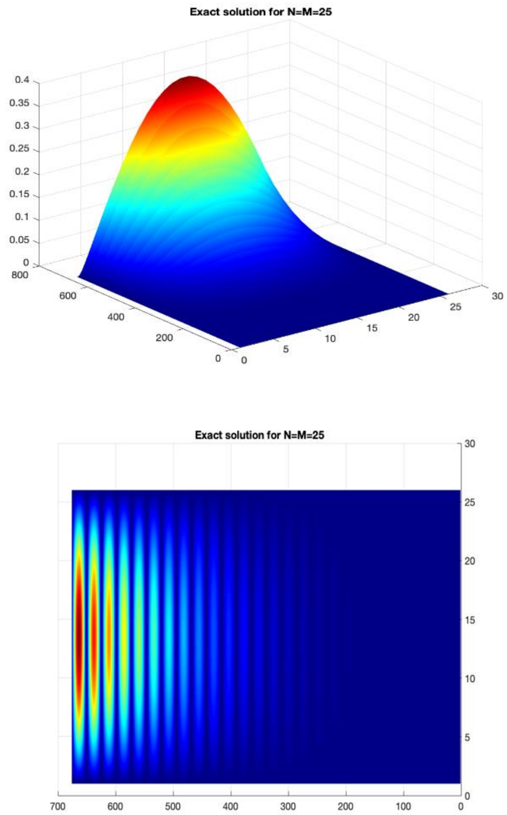

Figure 1. Gives the exact solution of (7) from different angles when $N = M = 25$, $\beta = 0.9$, and $\alpha = 1.9$.

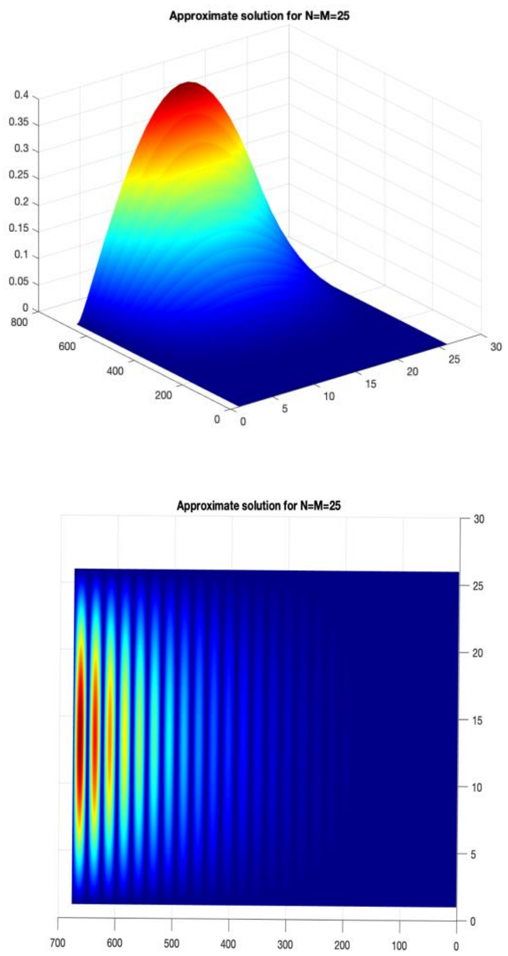

Figure 2. Gives the approximate solutions of (7) from different angles when $N = M = 25$, $\beta = 0.9$, and $\alpha = 1.9$.

Example 2. Consider the following mixed fractional order partial differential equations in a two-dimensional telegraph system.

$$\begin{cases} {}_0^CD_t^{\alpha} v(t,x,y) + {}^{ABC}_{\ \ 0}D_t^{\beta} v(t,x,y) + v(t,x,y) \\ = v_{xx}(t,x,y) + v_{yy}(t,x,y) + f(t,x,y) \\ f(t,x,y) = \frac{1}{B(\beta)}\left(\frac{6(1-\beta)}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{6\beta t^{\beta+3-\alpha}}{\Gamma(\beta+4-\alpha)}\right)(x-x^2)(y-y^2) \\ +t^3(x-x^2)(y-y^2) + \frac{1}{B(\beta)}\left((1-\beta)t^3 + 6\beta\frac{t^{\beta+3}}{\Gamma(\beta+4)}\right) \\ \left((x-x^2)(y-y^2) + 2(x-x^2) + 2(y-y^2)\right) \qquad (8) \\ 1 < \alpha \le 2, 0 < \beta \le 1, \\ v(0,x,y) = v_t(0,x,y) = 0, \ \ 0 \le x,y \le 1, \\ v(t,0,y) = v_t(t,L,y) = 0, \ \ 0 \le t \le 1, 0 \le y \le 1, \\ v(t,x,0) = v_t(t,x,L) = 0, \ \ 0 \le t \le 1, 0 \le x \le 1. \end{cases}$$

The exact solution of (8) is

$$v(t,x,y) = \frac{1}{B(\beta)}\left((1-\beta)t^3 + 6\beta\frac{t^{\beta+3}}{\Gamma(\beta+4)}\right)(x-x^2)(y-y^2).$$

By applying the same procedure as in the first example, we compute numerical solutions and the error is calculated using the subsequent formula;

$$\varepsilon = \max_{1\le k\le N-1, 1\le n\le M-1} |v(t_k, x_n, y_m) - v(t,x,y)|$$

where $v(t_k, x_n, y_m)$ is approximate and $v(t,x,y)$ is exact solution. We take $\alpha = 1.9$ and $\beta = 0.9$ then calculate the error values in Table 2.

Table 2. Error table for problem (8)

N, M	Error values
N=10, M=5	0.0073
N=20, M=10	0.0033
N=30, M=15	0.0020
N=40, M=20	0.0015
N=50, M=25	0.0011
N=60, M=30	0.0010
N=70, M=35	0.0009

In Table 2, again the error margin can be seen that they are less than 1. Moreover, error values keep decreasing as grid values keep increasing. Table 2 also confirms the accuracy of the generated difference scheme.

We conclude that the empirical data gathered in this study provides strong confirmation for the suggested methodology based on what is evident from example 1 and 2. The accuracy and dependability of the recommended technique are highlighted by the strong agreement between the numerical results and the predicted theoretical outcomes. This innovation not only solves the issue at hand but also paves the way for prospective applications in related fields. Its accuracy and applicability represent a substantial improvement in the discipline and provide both researchers and practitioners with an invaluable tool. This novel strategy has the potential to completely alter how we tackle comparable problems in the future.

3. Conclusion and Suggestions

This research paper focuses on evaluating mixed fractional order two-dimensional telegraph equations using a combination of Caputo and ABC fractional derivatives. The authors employed a finite difference approach to analyze the problem presented in the study. Stability estimates were obtained via Von-Neumann analysis method. Error analysis data was computed by using the finite difference technique. MATLAB applications were used for calculating the error analysis results.

The results of our analysis illustrate the suggested method's exceptional accuracy and underscore its adaptability in addressing a range of intricate issues. This efficacy highlights the method's prospective uses in domains such as wave propagation, anomalous diffusion processes, and other intricate physical phenomena that incorporate mixed fractional derivatives. This strategy is especially effective in situations when conventional procedures are inadequate for addressing the complexities of fractional calculus.

This approach might be expanded to higher-dimensional systems, nonlinear situations, and various forms of fractional derivatives in future study to enhance its application. Furthermore, investigating actual applications in telecommunications, materials science, and signal processing may enhance understanding of its importance and promote additional improvements to the approach.

Contributions of the authors

F. Ozbag: software development, methodology, visualization, revision. M. Modanlı: designing the study, methodology, stability analysis. S.T. Adulazeez: literature review, writing, editing and resources.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics

References

[1] A. Atangana and J. F. Gómez-Aguilar, “Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena,” *Eur. Phys. J. Plus*, vol. 133, no. 4, 2018.

[2] A. Atangana, “Blind in a commutative world: simple illustrations with functions and chaotic attractors, Chaos,” *Chaos, Solitons & Fractals*, vol. 114, pp. 347–363, 2018.

[3] A. Atangana, “RETRACTED ARTICLE: Derivative with two fractional orders: A new avenue of investigation toward revolution in fractional calculus,” *Eur. Phys. J. Plus*, vol. 131, no. 10, 2016.

[4] M. Modanli, K. Karadag, and S. T. Abdulazeez, “Solutions of the mobile–immobile advection–dispersion model based on the fractional operators using the Crank–Nicholson difference scheme,” *Chaos Solitons Fractals*, vol. 167, no. 113114, p. 113114, 2023.

[5] S. B. Yuste, L. Acedo, and K. Lindenberg, “Reaction front in an A+ B→ C reaction-subdiffusion process,” *Physical Review* E, vol. 69, no. 3, 2004.

[6] A. Atangana, “Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties,” Physica A, vol. 505, pp. 688–706, 2018.

[7] S. B. Yuste and K. Lindenberg, “Subdiffusion-limited A+A reactions,” *Phys. Rev. Lett*., vol. 87, no. 11, p. 118301, 2001.

[8] E. Hesameddini and E. Asadolahifard, “A new spectral Galerkin method for solving the two dimensional hyperbolic telegraph equation,” *Comput. Math. Appl*., vol. 72, no. 7, pp. 1926–1942, 2016.

[9] M. Zarebnia and R. Parvaz, “A new approach for solution of telegraph equation,” *International Journal of Nonlinear Analysis and Applications,* vol. 12, no. 1, pp. 385–396, 2021.

[10] Ş. Yüzbaşı and M. Karaçayır, “A Galerkin-like scheme to solve two-dimensional telegraph equation using collocation points in initial and boundary conditions,” *Comput. Math. Appl*., vol. 74, no. 12, pp. 3242–3249, 2017.

[11] R. Jiwari, S. Pandit, and R. C. Mittal, "A differential quadrature algorithm to solve the two dimensional linear hyperbolic telegraph equation with Dirichlet and Neumann boundary conditions," *Appl. Math. Comput.*, vol. 218, no. 13, pp. 7279–7294, 2012.

[12] Ö. Oruç, "A numerical procedure based on Hermite wavelets for two-dimensional hyperbolic telegraph equation," *Eng. Comput.*, vol. 34, no. 4, pp. 741–755, 2018.

[13] M. Modanli and F. Ozbag, "Stability of finite difference schemes for two-space dimensional telegraph equation," *Pramana*, vol. 96, no. 4, 2022.

[14] M. Hosseininia and M. H. Heydari, "Meshfree moving least squares method for nonlinear variable-order time fractional 2D telegraph equation involving Mittag–Leffler non-singular kernel," *Chaos Solitons Fractals*, vol. 127, pp. 389–399, 2019.

[15] X. Xu and D. Xu, "Legendre wavelets direct method for the numerical solution of time-fractional order telegraph equations," *Mediterr. J. Math.*, vol. 15, no. 1, 2018.

[16] N. Mollahasani, M. M. (mohseni) Moghadam, and K. Afrooz, "A new treatment based on hybrid functions to the solution of telegraph equations of fractional order," *Appl. Math. Model.*, vol. 40, no. 4, pp. 2804–2814, 2016.

[17] F. Ozbag and M. Modanli, "On the stability estimates and numerical solution of fractional order telegraph integro-differential equation," *Phys. Scr.*, vol. 96, no. 9, p. 094008, 2021.

[18] D. Baleanu, S. Zibaei, M. Namjoo, and A. Jajarmi, "A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system," *Adv. Differ. Equ.*, vol. 2021, no. 1, 2021.

[19] F. Ozbag and M. Modanli, "Numerical solutions of fractional order pseudo hyperbolic differential equations by finite difference method," *Afyon Kocatepe Univ. J. Sci. Eng.*, vol. 22, no. 5, pp. 998–1004, 2022.

[20] M. Modanli, F. Ozbag, and A. Akgülma, "Finite difference method for the fractional order pseudo telegraph integro-differential equation," *J. Appl. Math. Comput. Mech.*, vol. 21, no. 1, pp. 41–54, 2022.

[21] T. Liu and M. Hou, "A fast implicit finite difference method for fractional advection-dispersion equations with fractional derivative boundary conditions," *Adv. Math. Phys.*, vol. 2017, pp. 1–8, 2017.

[22] K. Kumar, R. K. Pandey, and S. Yadav, "Finite difference scheme for a fractional telegraph equation with generalized fractional derivative terms," *Physica A*, vol. 535, no. 122271, p. 122271, 2019.

[23] S. O. Abdulla, S. T. Abdulazeez, and M. Modanli, "Comparison of third-order fractional partial differential equation based on the fractional operators using the explicit finite difference method," *Alex. Eng. J.*, vol. 70, pp. 37–44, 2023.

[24] M. Modanli and F. Ozbag, "Stability of finite difference schemes to pseudo-hyperbolic telegraph equation," *Journal of Mathematical Sciences and Modelling*, vol. 5, no. 3, pp. 92–98, 2022.

[25] W. M. Abd-Elhameed, E. H. Doha, Y. H. Youssri, and M. A. Bassuony, "New Tchebyshev-Galerkin operational matrix method for solving linear and nonlinear hyperbolic telegraph type equations," *Numer. Methods Partial Differ. Equ.*, vol. 32, no. 6, pp. 1553–1571, 2016.

[26] E. H. Doha, W. M. Abd-Elhameed, and Y. H. Youssri, "Fully Legendre spectral Galerkin algorithm for solving linear one-dimensional telegraph type equation," *Int. J. Comput. Methods*, vol. 16, no. 08, p. 1850118, 2019.

[27] Y. H. Youssri, W. M. Abd-Elhameed, and A. G. Atta, "Spectral Galerkin treatment of linear one-dimensional telegraph type problem via the generalized Lucas polynomials," *Arab. J. Math*., vol. 11, no. 3, pp. 601–615, 2022.

[28] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, "Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem," *Math. Sci*., vol. 17, no. 4, pp. 415–429, 2023.

[29] H. T. Taghian, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, "Shifted Gegenbauer–Galerkin algorithm for hyperbolic telegraph type equation," *Int. J. Mod. Phys. C*., vol. 32, no. 09, p. 2150118, 2021.

[30] R. M., Hafez, and Y. H. Youssri. "Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation." *Iranian Journal of Numerical Analysis and Optimization,* 10(1), 195-223, 2020.