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# Tzitzeica Smarandache Curves in Euclidean 3-Space

Research Article

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## Abstract

The aim of this study is to examine the relations between Tzitzeica curves and Smarandache curves in Euclidean space. In addition, the necessary and sufficient conditions for Smarandache curves to be Tzitzeica curves in 3-dimensional Euclidean space are investigated and examples are given.

**Keywords:** Tzitzeica Curves, Smarandache Curves, Planar Tzitzeica Curves

## 1. INTRODUCTION

Smarandache curves were first described by Turgut and Yılmaz in 2008 [1]. The authors named as the Smarandache curve in Minkowski space, whose position vector is formed by the Frenet frame vectors of the other regular curve. They defined special cases of these curves and expressed the $TB_2$ curve. In [2], the author studied some special Smarandache curves in Euclidean space. In [3,4,5], the authors studied Smarandache curves according to the Darboux frame in 3-dimensional Euclidean space and Minkowski space. In [6], the author studied Smarandache curves in 4-dimensional Euclidean space. The author obtained Frenet-Serret and Bishop invariants for Smarandache curves and calculated the first, second and third curvatures of the Smarandache curve. In [7], the authors calculated the curvature and torsion of the Smarandache curve when the Frenet vectors of the Anti-Salkowski curve were taken as position vectors. In [8], the author studied Smarandache curves obtained from a curve with by a parallel transport frame in 4-dimensional Euclidean space. In [9], the authors examined families of hypersurfaces with Smarandache curves in 4-dimensional Galilean space. In [10], the authors re-characterized the Smarandache curves with by an alternative frame which is different from the Frenet frame. In [11], the authors defined new conjugate curves by integrating the Smarandache curves and examined the relations between the main curve and the

Frenet vectors of the resulting curve. In [12], the authors studied Smarandache Ruled surfaces. In [13], the authors examined some special Smarandache curves according to the Flc-frame in 3-dimensional Euclidean space.

Romanian Mathematician Gheorghe Tzitzeica defined a class of curves that he called Tzitzeica curves in 1911 [14]. In [15], the authors examined the relations between Tzitzeica curves and surfaces in Minkowski space. In [16], the author showed that elliptic and hyperbolic cylindrical curves in Euclidean space satisfy the Tzitzeica condition. In [17,18], the authors examined hyperbolic and elliptic cylindrical curves in Minkowski space, respectively. In [19], the author gave the necessary and sufficient condition for a space curve to become a Tzitzeica curve. In [20], the authors studied Tzitzeica curves in 3-dimensional Euclidean space. In [21], the authors examined Tzitzeica surfaces in 3-dimensional Euclidean space. In additionally, the planar Tzitzeica curve definition was defined for the first time. In [22,23], the authors studied Tzitzeica curves in 4-dimensional Euclidean space. In [24], the authors studied osculating and rectifiying curve in 4-dimensional Galilean space.

## 2. BASIC NOTATIONS

For a regular curve $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$, if the $k_1$ curvature $k_1(s)$ and the $k_2$ curvature $k_2(s)$ of α are constant functions, then α is called a screw curve or helix curve [26]. Since these curves are traces of one-parameter groups of Euclidean transformations, they were named W-curves by F. Klein and S. Lie [27]. If the $\frac{k_1(s)}{k_2(s)}$ ratio of a curve in $\mathbb{E}^3$ is a non-zero constant, this curve is called a general Helix [28]. For a regular space curve $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$, the planes at each point of $\alpha(s)$ spanned by $\{T, N_1\}, \{T, N_2\}, \{N_1, N_2\}$ are know as the osculating plane, rectifying plane and normal plane, respectively. If the position vector α lies on its rectifying plane or osculating plane or normal plane then, $\alpha(s)$ is called rectifying curve, osculating curve and normal curve, respectively [29,30].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in three-dimensional Euclidean space. Let us denote $T(s) = \alpha'(s)$ and call $T(s)$ a unit tangent vector of $\alpha$ at $s$. We denote the curvature of $\alpha$ by $k_1(s) = \|\alpha''(s)\|$. If $k_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve $\alpha$ at $s$ is given by $T'(s) = k_1(s)N_1(s)$. The unit vector $N_2(s) = T(s) \times N_1(s)$ is called the unit binormal vector of $\alpha$ at $s$.
Then we have the Frenet-Serret formulae

$$\begin{aligned} T'(s) &= k_1(s)N_1(s) \\ N_1'(s) &= -k_1(s)T(s) + k_2(s)N_2(s) \\ N_2'(s) &= -k_2(s)N_1(s) \end{aligned} \tag{1}$$

where $k_2(s)$ is the torsion of the curve $\alpha$ at $s$ [25]

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve, with a curvature $k_1(s) > 0$ and $k_2(s) \neq 0$ . If the torsion of $\alpha(s)$ satisfies the condition

$$\frac{k_2(s)}{d_{osc}^2} = \mathrm{a} \tag{2}$$

for some real non-zero constant a then $\alpha(s)$ is called the Tzitzeica curve (Tz-curve), where

$$d_{osc} = \langle \alpha(s), N_2(s) \rangle \tag{3}$$

is the orthogonal distance from the origin to the osculating plane of $\alpha(s)$ . Here, $N_2(s)$ is the binormal vector field of $\alpha$ [14].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^2$ be a unit speed planar curve $(k_1(s) > 0)$. In this case, if the curvature of $\alpha(s)$ satisfies the condition

$$k_1(s) = \mathrm{a}_1 . d_{osc}^2 \tag{4}$$

for some real non-zero constant $\mathrm{a}_1$ then $\alpha(s)$ is called the planar Tzitzeica curve (planar Tz-curve) where

$$d_{osc} = \langle \alpha(s), N_1(s) \rangle \tag{5}$$

and $N_1(s)$ is the unit normal vector field of $\alpha$ [21].
If the position vector of a regular curve in Minkowski space consists of by the Frenet frame vectors on another regular curve, this curve is called a Smarandache curve [1].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve and $\{T(s), N_1(s), N_2(s)\}$ be its Frenet frame. Then we have Smarandache curves of $\alpha(s)$ by with
1) $TN_1$- Smarandache curve are defined by

$$\beta_{TN_1}(s_\beta) = \frac{1}{\sqrt{2}}\big(T(s) + N_1(s)\big) \tag{6}$$

2) $TN_2$- Smarandache curve are defined by

$$\beta_{TN_2}(s_\beta) = \frac{1}{\sqrt{2}}\big(T(s) + N_2(s)\big) \tag{7}$$

3) $N_1N_2$- Smarandache curve are defined by

$$\beta_{N_1N_2}(s_\beta) = \frac{1}{\sqrt{2}}\big(N_1(s) + N_2(s)\big) \tag{8}$$

4) $TN_1N_2$- Smarandache curve are defined by

$$\beta_{TN_1N_2}(s_\beta) = \frac{1}{\sqrt{3}}\big(T(s) + N_1(s) + N_2(s)\big) \tag{9}$$

[2]. Here $s_\beta$ is the arc-length parameter of the curve.

## 3.TZITZEICA SMARANDACHE CURVES IN $\mathbb{E}^3$

### 3.1 $TN_1$- Smarandache Curve

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$. Let $\beta_{TN_1}$ Smarandache curve in $\mathbb{E}^3$given with the parametrization (6). If we denote the arc-length parameter of the $\beta_{TN_1}$ curve with $s_\beta$ and take the derivative of the $\beta_{TN_1}$ curve, we obtain

$$\beta'_{TN_1}(s_\beta) = \frac{d_{\beta_{TN_1}}}{d_{s_\beta}} \cdot \frac{d_{s_\beta}}{d_s} = \frac{1}{\sqrt{2}}(-k_1(s)T(s) + k_1(s)N_1(s) + k_2(s)N_2(s)). \tag{10}$$

For the norm of this expression is $\left\|\frac{d_{\beta_{TN_1}}}{d_{s_\beta}}\right\| = 1$, we get

$$\frac{d_{s_\beta}}{d_s} = \frac{\sqrt{2k_1^2 + k_2^2}}{\sqrt{2}} \neq 0. \tag{11}$$

In this case, from the expression (10), the tangent vector field of the $\beta_{TN_1}$ curve becomes

$$T_{\beta_{TN_1}} = \frac{d_{\beta_{TN_1}}}{d_{s_\beta}} = \frac{1}{\sqrt{2k_1^2 + k_2^2}}(-k_1 T + k_1 N_1 + k_2 N_2). \tag{12}$$

Again, by taking derivative of (12) and using (11), we obtain

$$(T_{\beta_{TN_1}})' = \left(\frac{d_{\beta_{TN_1}}}{d_{s_\beta}}\right)' \cdot \frac{d_{s_\beta}}{d_s} = \frac{\sqrt{2}}{(2k_1^2 + k_2^2)^2}(AT + BN_1 + CN_2) \tag{13}$$

where

$$A(s) = -k_1^2(2k_1^2 + k_2^2) - k_2(k_1' k_2 - k_1 k_2')$$

$$B(s) = -k_1^2(2k_1^2 + 3k_2^2) + k_2(k_1' k_2 - k_1 k_2' - k_2^3)$$

$$C(s)=k_1k_2(2k_1^2+k_2^2)-2k_1(k_1'k_2-k_1k_2').$$

The curvature of the $\beta_{TN_1}$ curve is

$$k_{1\beta_{TN_1}}=\left\|T'_{\beta_{TN_1}}\right\|=\frac{\sqrt{2}}{(2k_1^2+k_2^2)^2}\sqrt{A^2+B^2+C^2}. \tag{14}$$

The principal normal vector field of the $\beta_{TN_1}$ curve is

$$N_{1\beta_{TN_1}}=\frac{T'_{\beta_{TN_1}}}{k_{1\beta_{TN_1}}}=\frac{1}{\sqrt{A^2+B^2+C^2}}(AT+BN_1+CN_2). \tag{15}$$

By using (13) and (14). The binormal vector field of the $\beta_{TN_1}$ curve is

$$N_{2\beta_{TN_1}}=T_{\beta_{TN_1}}\,\mathrm{x}\,N_{1\beta_{TN_1}}=\frac{1}{DE}[(Ck_1-Bk_2)T+(Ck_1+Ak_2)N_1+(-Bk_1-Ak_1)N_2]. \tag{16}$$

By using (12) and (15), where

$$D(s)=\sqrt{2k_1^2+k_2^2}$$
$$E(s)=\sqrt{A^2+B^2+C^2}\,.$$

By taking derivative of (16) and using (11), we obtain

$$(N_{2\beta_{TN_1}})'=\frac{\sqrt{2}}{D^3E^2}\left\{\begin{matrix}[-(D'E+DE')(Ck_1-Bk_2)+DE(C'k_1+Ck_1'-B'k_2-Bk_2'-Ck_1^2-Ak_1k_2)]T\\+[-(D'E+DE')(Ck_1+Ak_2)+DE(Ck_1^2+C'k_1+Ck_1'+A'k_2+Ak_2'+Ak_1k_2)]N_1\\+[(D'E+DE')(k_1A+k_1B)+DE(Ck_1k_2+Ak_2^2-Ak_1'-Bk_1'-A'k_1-B'k_1)]N_2\end{matrix}\right\}. \tag{17}$$

By using (15) and (17), we obtain

$$k_{2\beta_{TN_1}}=-\langle N'_{2\beta_{TN_1}},N_{1\beta_{TN_1}}\rangle=\frac{-\sqrt{2}}{D^2E^2}\begin{bmatrix}k_1\big(C'(A+B)-C(A'+B')\big)+k_2(-AB'+A'B)\\+k_1^2C(-A+B)+k_2^2AC+k_1k_2(-A^2+AB+C^2)\end{bmatrix}. \tag{18}$$

**Theorem 1:** Let $\beta_{TN_1}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1,k_2\neq 0$ are non-constant curvatures. If $\beta_{TN_1}$ Smarandache curve is a Tz-curve then, the equation must be

$$\frac{k_{2\beta_{TN_1}}}{d_{osc}^2}=\frac{-2\sqrt{2}}{4C^2k_1^2+4Ck_1k_2(A-B)+k_2^2(A-B)^2}\left\{\begin{matrix}k_1[C'(A+B)-C(A'+B')]\\+k_2(-AB'+A'B)\\+k_1^2C(B-A)+k_2^2AC\\+k_1k_2(-A^2+AB+C^2)\end{matrix}\right\}=\mathrm{a} \tag{19}$$

where a is nonzero constant [31].

**Proof.** By using (6) and (16), we get

$$d_{osc}=\langle\beta_{TN_1},N_{2\beta_{TN_1}}\rangle=\frac{1}{\sqrt{2}DE}\big(2Ck_1+k_2(A-B)\big). \tag{20}$$

Then, using equations (18) and (20), the expression (19) is obtained.

**Corollary 2:** Let $\beta_{TN_1}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1,k_2\neq 0$ are non-constant curvatures. If $k_1=ck_2$ $(c\neq 0\ constant)$, $\beta_{TN_1}$ Smarandache curve becomes a planar curve.

**Theorem 3:** Let $\beta_{TN_1}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$, where $k_1,k_2\neq 0$ are non-constant curvatures. If $k_1=ck_2$ $(c\neq 0\ constant)$, $\beta_{TN_1}$ Smarandache curve becomes a planar Tz-curve [31].

**Proof:** By using (6) and (15), we obtain

$$d_{osc}=\langle\beta_{TN_1},N_{1\beta_{TN_1}}\rangle=\frac{A+B}{\sqrt{2}\sqrt{A^2+B^2+C^2}}\,. \tag{21}$$

Using the expressions (4), (14) and (21), we get

$$\frac{k_{1\beta_{TN_1}}}{d_{osc}^2}=\frac{2\sqrt{2}}{\left(\sqrt{2k_1^2+k_2^2}\right)^9}\begin{bmatrix}(2k_1^2+k_2^2)^2(k_1^2+k_2^2)\\+2(k_1'k_2-k_1k_2')[k_1'k_2-k_1k_2'-k_2(2k_1^2+k_2^2)]\end{bmatrix}^{\frac{3}{2}}. \tag{22}$$

If $k_1=ck_2$ is used in (22), we obtain

$$\frac{k_{1\beta_{TN_1}}}{d_{osc}^2}=2\sqrt{2}\left(\frac{c^2+1}{2c^2+1}\right)^{\frac{3}{2}}=\text{constant}.$$

Therefore, $\beta_{TN_1}$ Smarandache curve is a planar Tz-curve.

**Theorem 4:** Let $\beta_{TN_1}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$. If $\alpha(s)$ is W-curve ($i.e.\ k_1, k_2 \neq 0\ constant$) then, $\beta_{TN_1}$ Smarandache curve is a planar Tz-curve [31].
**Proof:** $k_1, k_2 \neq 0\ (constant)$, from equation (17), we obtain $(N_{2\beta_{TN_1}})' = 0$. Then from equation (18), we get $k_{2\beta_{TN_1}} = 0$. This means that the $\beta_{TN_1}$ Smarandache curve is a planar curve. Then substituting $k_1, k_2 \neq 0\ (constant)$ into (22), we obtain

$$\frac{k_{1\beta_{TN_1}}}{d_{osc}^2} = \frac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{(2k_1^2+k_2^2)^{\frac{3}{2}}} \neq 0\ (constant).$$

Therefore, $\beta_{TN_1}$ Smarandache curve is a planar Tz-curve.

### 3.2 $TN_2$- Smarandache Curve

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$. Let $\beta_{TN_2}$ be a Smarandache curve in $\mathbb{E}^3$ given with the parametrization (7). If we denote the arc-length parameter of the $\beta_{TN_2}$ curve with $s_\beta$ and take the derivative of the $\beta_{TN_2}$ curve, we obtain

$$\beta'_{TN_2}(\mathrm{s}_\beta) = \frac{d_{\beta_{TN_2}}}{d_{s_\beta}} \cdot \frac{d_{s_\beta}}{d_s} = \frac{1}{\sqrt{2}}\big(k_1(s) - k_2(s)\big)N_1(s) \tag{23}$$

For the norm of this expression is $\left\|\frac{d_{\beta_{TN_2}}}{d_{s_\beta}}\right\| = 1$, we get

$$\frac{d_{s_\beta}}{d_s} = \frac{\sqrt{(k_1-k_2)^2}}{\sqrt{2}} = \frac{|k_1-k_2|}{\sqrt{2}} \neq 0. \tag{24}$$

In this case, from the expression (23), the tangent vector field of the $\beta_{TN_2}$ curve becomes

$$T_{\beta_{TN_2}} = \frac{d_{\beta_{TN_2}}}{d_{s_\beta}} = \frac{(k_1-k_2)N_1}{|k_1-k_2|} = \begin{cases} N_1,\ k_1 > k_2 \\ -N_1,\ k_1 < k_2 \end{cases} \tag{25}$$

Again, by taking derivative of (25) and using (24), we obtain

$$(T_{\beta_{TN_2}})' = \left(\frac{d_{\beta_{TN_2}}}{d_{s_\beta}}\right)' \cdot \frac{d_{s_\beta}}{d_s} = \frac{-\sqrt{2}k_1}{(k_1-k_2)}T + \frac{\sqrt{2}k_2}{(k_1-k_2)}N_2. \tag{26}$$

The curvature of the $\beta_{TN_2}$ curve is

$$k_{1\beta_{TN_2}} = \left\|T'_{\beta_{TN_2}}\right\| = \frac{\sqrt{2}\sqrt{k_1^2+k_2^2}}{|k_1-k_2|}. \tag{27}$$

The principal normal vector field of the $\beta_{TN_2}$ curve is

$$N_{1\beta_{TN_2}} = \frac{T'_{\beta_{TN_2}}}{k_{1\beta_{TN_2}}} = \begin{cases} \frac{-k_1T+k_2N_2}{\sqrt{k_1^2+k_2^2}},\ k_1 > k_2 \\ \frac{k_1T-k_2N_2}{\sqrt{k_1^2+k_2^2}},\ k_1 < k_2 \end{cases}. \tag{28}$$

By using (26) and (27). The binormal vector field of the $\beta_{TN_2}$ curve is

$$N_{2\beta_{TN_2}} = T_{\beta_{TN_2}} \mathrm{x}\, N_{1\beta_{TN_2}} = \frac{1}{\sqrt{k_1^2+k_2^2}}(k_2T+k_1N_2). \tag{29}$$

By using (25) and (28). By taking derivative of (29) and using (24), we obtain

$$(N_{2\beta_{TN_2}})' = \begin{cases} \frac{\sqrt{2}(k_1'k_2-k_1k_2')}{(k_1-k_2)(k_1^2+k_2^2)^{\frac{3}{2}}}(-k_1T+k_2N_2),\ k_1 > k_2 \\ \frac{-\sqrt{2}(k_1'k_2-k_1k_2')}{(k_1-k_2)(k_1^2+k_2^2)^{\frac{3}{2}}}(-k_1T+k_2N_2),\ k_1 < k_2 \end{cases}. \tag{30}$$

By using (28) and (30), we obtain

$$k_{2\beta_{TN_2}} = -\langle N'_{2\beta_{TN_2}}, N_{1\beta_{TN_2}} \rangle = \frac{-\sqrt{2}(k_1'k_2 - k_1k_2')}{(k_1-k_2)(k_1^2+k_2^2)}. \tag{31}$$

**Theorem 5:** Let $\beta_{TN_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. If $\beta_{TN_2}$ Smarandache curve is a Tz-curve then, the equation must be

$$\frac{k_{2\beta_{TN_2}}}{d_{osc}^2} = \frac{-2\sqrt{2}(k_1'k_2 - k_1k_2')}{(k_1-k_2)(k_1+k_2)^2} \neq 0 \ (constant) \ [31]. \tag{32}$$

**Proof:** By using (7) and (29), we get

$$d_{osc} = \langle \beta_{TN_2}, N_{2\beta_{TN_2}} \rangle = \frac{k_1+k_2}{\sqrt{2}\sqrt{k_1^2+k_2^2}} \,. \tag{33}$$

Then, using equations (31) and (33), the expression (32) is obtained.

**Corollary 6:** Let $\beta_{TN_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. $k_1 = ck_2$ $(c \neq 0 \ constant)$ if and only if $\beta_{TN_2}$ Smarandache curve becomes a planar curve.

**Theorem 7:** Let $\beta_{TN_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$, where $k_1, k_2 \neq 0$ are non-constant curvatures. If $k_1 = ck_2$ $(c \neq 0 \ constant)$, $\beta_{TN_2}$ Smarandache curve becomes a planar Tz-curve [31].

**Proof:** By using (7) and (28), we obtain

$$d_{osc} = \langle \beta_{TN_2}, N_{1\beta_{TN_2}} \rangle = \begin{cases} \dfrac{-(k_1-k_2)}{\sqrt{2}\sqrt{(k_1^2+k_2^2)}}, & k_1 > k_2 \\ \dfrac{(k_1-k_2)}{\sqrt{2}\sqrt{(k_1^2+k_2^2)}}, & k_1 < k_2 \end{cases}. \tag{34}$$

Using the expressions (4), (27) and (34), we get

$$\frac{k_{1\beta_{TN_2}}}{d_{osc}^2} = \begin{cases} \dfrac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{(k_1-k_2)^3}, & k_1 > k_2 \\ \dfrac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{-(k_1-k_2)^3}, & k_1 < k_2 \end{cases}. \tag{35}$$

If $k_1 = ck_2$ is used in (35), we obtain

$$\frac{k_{1\beta_{TN_2}}}{d_{osc}^2} = \mp \frac{2\sqrt{2}(c^2+1)^{\frac{3}{2}}}{(c-1)^3} = \text{constant.}$$

Therefore, $\beta_{TN_2}$ Smarandache curve is a planar Tz-curve.

**Theorem 8:** Let $\beta_{TN_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$. If $\alpha(s)$ is W-curve ($i.e.$ $k_1, k_2 \neq 0$ $constant$) then, $\beta_{TN_2}$ Smarandache curve is a planar Tz-curve [31].

**Proof:** $k_1, k_2 \neq 0$ $(constant)$, from equation (30), we obtain $(N_{2\beta_{TN_2}})' = 0$. Then from equation (31), we get $k_{2\beta_{TN_2}} = 0$. This means that the $\beta_{TN_2}$ Smarandache curve is a planar curve. Then substituting $k_1, k_2 \neq 0$ $(constant)$ into (35), we obtain

$$\frac{k_{1\beta_{TN_2}}}{d_{osc}^2} = \begin{cases} \dfrac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{(k_1-k_2)^3}, & k_1 > k_2 \\ \dfrac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{-(k_1-k_2)^3}, & k_1 < k_2 \end{cases} \neq 0 \ (constant).$$

Therefore, $\beta_{TN_2}$ Smarandache curve is a planar Tz-curve.

### 3.3 $N_1N_2$- Smarandache Curve

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$. Let $\beta_{N_1N_2}$ Smarandache curve in $\mathbb{E}^3$ given with the parametrization (8). If we denote the arc-length parameter of the $\beta_{N_1N_2}$ curve with $s_\beta$ and take the derivative of the $\beta_{N_1N_2}$ curve, we obtain

$$\beta'_{N_1N_2}(s_\beta) = \frac{d_{\beta_{N_1N_2}}}{d_{s_\beta}} \cdot \frac{d_{s_\beta}}{d_s} = \frac{1}{\sqrt{2}}\left(-k_1(s)T(s) - k_2(s)N_1(s) + k_2(s)N_2(s)\right). \quad (36)$$

For the norm of this expression is $\left\|\frac{d_{\beta_{N_1N_2}}}{d_{s_\beta}}\right\| = 1$, we get

$$\frac{d_{s_\beta}}{d_s} = \frac{\sqrt{k_1^2+2k_2^2}}{\sqrt{2}} \neq 0. \quad (37)$$

In this case, from the expression (36), the tangent vector field of the $\beta_{N_1N_2}$ curve becomes

$$T_{\beta_{N_1N_2}} = \frac{d_{\beta_{N_1N_2}}}{d_{s_\beta}} = \frac{1}{\sqrt{k_1^2+2k_2^2}}(-k_1T - k_2N_1 + k_2N_2). \quad (38)$$

Again, by taking derivative of (38) and using (37), we obtain

$$(T_{\beta_{N_1N_2}})' = \left(\frac{d_{\beta_{N_1N_2}}}{d_{s_\beta}}\right)' \cdot \frac{d_{s_\beta}}{d_s} = \frac{\sqrt{2}}{B^2}\begin{bmatrix}(-2k_2A + k_1k_2B)T \\ +(k_1A - (k_1^2 + k_2^2)B)N_1 \\ +(-k_1A - k_2^2B)N_2\end{bmatrix} \quad (39)$$

where

$A(s) = k_1'k_2 - k_1k_2'$

$B(s) = k_1^2 + 2k_2^2.$

The curvature of the $\beta_{N_1N_2}$ curve is

$$k_{1\beta_{N_1N_2}} = \left\|T'_{\beta_{N_1N_2}}\right\| = \frac{\sqrt{2}}{B^{\frac{3}{2}}}C \quad (40)$$

where

$C(s) = \sqrt{2A^2 + (k_1^2 + k_2^2)B^2 - 2k_1AB}.$

The principal normal vector field of the $\beta_{N_1N_2}$ curve is

$$N_{1\beta_{N_1N_2}} = \frac{T'_{\beta_{N_1N_2}}}{k_{1\beta_{N_1N_2}}} = \frac{1}{\sqrt{B}C}\begin{bmatrix}(-2k_2A + k_1k_2B)T \\ +(k_1A - (k_1^2 + k_2^2)B)N_1 \\ +(-k_1A - k_2^2B)N_2\end{bmatrix}. \quad (41)$$

By using (39) and (40). The binormal vector field of the $\beta_{N_1N_2}$ curve is

$$N_{2\beta_{N_1N_2}} = T_{\beta_{N_1N_2}} \text{x}\, N_{1\beta_{N_1N_2}} = \frac{1}{C}[(k_2B)T - AN_1 + (-A + k_1B)N_2]. \quad (42)$$

By using (38) and (41). By taking derivative of (42) and using (37), we obtain

$$(N_{2\beta_{N_1N_2}})' = \frac{\sqrt{2}}{\sqrt{B}C^2}\begin{Bmatrix}[-k_2BC' + C(k_2'B + k_2B' + k_1A)]T \\ +[AC' + C(k_2A - A')]N_1 \\ +[C'(A - k_1B) + C(-k_2A - A' + k_1'B + k_1B')]N_2\end{Bmatrix}. \quad (43)$$

By using (41) and (43), we obtain

$$k_{2\beta_{N_1N_2}} = -\langle N'_{2\beta_{N_1N_2}}, N_{1\beta_{N_1N_2}}\rangle = \frac{-1}{\sqrt{2}C^2}[-3AB' + 2B(A' - k_2A)] \quad (44)$$

**Theorem 9:** Let $\beta_{N_1N_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. If $\beta_{N_1N_2}$ Smarandache curve is a Tz-curve then, the equation must be

$$\frac{k_{2\beta_{N_1N_2}}}{d_{osc}^2} = \frac{-\sqrt{2}[-3AB' + 2B(A' - k_2A)]}{(-2A + k_1B)^2} \neq 0\ (constant)\ [31]. \quad (45)$$

**Proof:** By using (8) and (42), we get

$$d_{osc}=\langle \beta_{N_1N_2}, N_{2\beta_{N_1N_2}}\rangle=\frac{1}{\sqrt{2C}}(-2A+k_1B). \quad (46)$$

Then, using equations (44) and (46), the expression (45) is obtained.

**Corollary 10:** Let $\beta_{N_1N_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. If $k_1 = ck_2$ ($c \neq 0$ $constant$), $\beta_{N_1N_2}$ Smarandache curve becomes a planar curve.

**Theorem 11:** Let $\beta_{N_1N_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$, where $k_1, k_2 \neq 0$ are non-constant curvatures. If $k_1 = ck_2$ ($c \neq 0$ $constant$), $\beta_{N_1N_2}$ Smarandache curve becomes a planar Tz-curve [31].

**Proof:** By using (8) and (41), we obtain

$$d_{osc}=\langle \beta_{N_1N_2}, N_{1\beta_{N_1N_2}}\rangle=\frac{-B\sqrt{B}}{\sqrt{2C}}. \quad (47)$$

Using the expressions (4), (40) and (47), we get

$$\frac{k_{1\beta_{N_1N_2}}}{d_{osc}^2}=\frac{2\sqrt{2}C^3}{B^{\frac{9}{2}}}. \quad (48)$$

If $k_1 = ck_2$ is used in (48), we obtain

$$\frac{k_{1\beta_{N_1N_2}}}{d_{osc}^2}=2\sqrt{2}\left(\frac{c^2+1}{c^2+2}\right)^{\frac{3}{2}}=\text{constant}.$$

Therefore, $\beta_{N_1N_2}$ Smarandache curve is a planar Tz-curve.

**Theorem 12:** Let $\beta_{N_1N_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$. If $\alpha(s)$ is W-curve ($i.e.$ $k_1, k_2 \neq 0$ $constant$) then, $\beta_{N_1N_2}$ Smarandache curve is a planar Tz-curve [31].

**Proof:** $k_1, k_2 \neq 0$ ($constant$), from equation (43), we obtain $(N_{2\beta_{N_1N_2}})' = 0$. Then from equation (44), we get $k_{2\beta_{N_1N_2}} = 0$. This means that the $\beta_{N_1N_2}$ Smarandache curve is a planar curve. Then substituting $k_1, k_2 \neq 0$ ($constant$) into (48), we obtain

$$\frac{k_{1\beta_{N_1N_2}}}{d_{osc}^2}=\frac{2\sqrt{2}(k_1^2+k_2^2)^{\frac{3}{2}}}{(k_1^2+2k_2^2)^{\frac{3}{2}}}\neq 0\ (constant).$$

Therefore, $\beta_{N_1N_2}$ Smarandache curve is a planar Tz-curve.

### 3.4 $TN_1N_2$- Smarandache Curve

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$. Let $\beta_{TN_1N_2}$ Smarandache curve in $\mathbb{E}^3$ given with the parametrization (9). If we denote the arc-length parameter of the $\beta_{TN_1N_2}$ curve with $s_\beta$ and take the derivative of the $\beta_{TN_1N_2}$ curve, we obtain

$$\beta'_{TN_1N_2}(s_\beta)=\frac{d_{\beta_{TN_1N_2}}}{d_{s_\beta}}\cdot\frac{d_{s_\beta}}{d_s}=\frac{1}{\sqrt{3}}(-k_1(s)T(s)+(k_1(s)-k_2(s))N_1(s)+k_2(s)N_2(s). \quad (49)$$

For the norm of this expression is $\left\|\frac{d_{\beta_{TN_1N_2}}}{d_{s_\beta}}\right\|=1$, we get

$$\frac{d_{s_\beta}}{d_s}=\frac{\sqrt{2}\sqrt{k_1^2+k_2^2-k_1k_2}}{\sqrt{3}}\neq 0. \quad (50)$$

In this case, from the expression (49), the tangent vector field of the $\beta_{TN_1N_2}$ curve becomes

$$T_{\beta_{TN_1N_2}}=\frac{d_{\beta_{TN_1N_2}}}{d_{s_\beta}}=\frac{1}{\sqrt{2A}}(-k_1T+(k_1-k_2)N_1+k_2N_2) \quad (51)$$

Again, by taking derivative of (51) and using (50), we obtain

$$(T_{\beta_{TN_1N_2}})'=\left(\frac{d_{\beta_{TN_1N_2}}}{d_{s_\beta}}\right)'\cdot\frac{d_{s_\beta}}{d_s}=\frac{\sqrt{3}}{4(A)^4}(BT+CN_1+DN_2) \quad (52)$$

where
$A(s) = \sqrt{k_1^2 + k_2^2 - k_1 k_2}$
$B(s) = (k_1' k_2 - k_1 k_2')(k_1 - 2k_2) - 2k_1(k_1^2 + k_2^2 - k_1 k_2)(k_1 - k_2)$
$C(s) = (k_1 + k_2)(k_1' k_2 - k_1 k_2') - 2(k_1^2 + k_2^2 - k_1 k_2)(k_1^2 + k_2^2)$
$D(s) = -(k_1' k_2 - k_1 k_2')(2k_1 - k_2) + 2k_2(k_1^2 + k_2^2 - k_1 k_2)(k_1 - k_2).$
The curvature of the $\beta_{TN_1N_2}$ curve is

$$k_{1\beta_{TN_1N_2}} = \left\| T'_{\beta_{TN_1N_2}} \right\| = \frac{\sqrt{3}}{4(A)^4}\sqrt{B^2 + C^2 + D^2} \tag{53}$$

The principal normal vector field of the $\beta_{TN_1N_2}$ curve is

$$N_{1\beta_{TN_1N_2}} = \frac{T'_{\beta_{TN_1N_2}}}{k_{1\beta_{TN_1N_2}}} = \frac{1}{\sqrt{B^2+C^2+D^2}}(BT + CN_1 + DN_2). \tag{54}$$

By using (52) and (53). The binormal vector field of the $\beta_{TN_1N_2}$ curve is

$$N_{2\beta_{TN_1N_2}} = T_{\beta_{TN_1N_2}} \text{x}\, N_{1\beta_{TN_1N_2}} = \frac{1}{\sqrt{2}A\sqrt{B^2+C^2+D^2}}\begin{Bmatrix} (Dk_1 - (D + C)k_2)T \\ +(Dk_1 + Bk_2)N_1 \\ +(-(B + C)k_1 + Bk_2)N_2 \end{Bmatrix}. \tag{55}$$

By using (51) and (54). By taking derivative of (55) and using (50), we obtain

$$(N_{2\beta_{TN_1N_2}})' = \frac{\sqrt{3}}{2A^3(B^2+C^2+D^2)^{\frac{3}{2}}}\begin{Bmatrix} \begin{bmatrix} EDk_1 - (D + C)Ek_2 \\ +A(B^2 + C^2 + D^2)[D(k_1' - k_2' - k_1^2) + D'(k_1 - k_2) - Ck_2' - C'k_2 - k_1 k_2 B] \end{bmatrix} T \\ \\ + \begin{bmatrix} EDk_1 + EBk_2 \\ +A(B^2 + C^2 + D^2)[D(k_1^2 - k_1 k_2 + k_1') + B(-k_2^2 + k_1 k_2 + k_2') + D'k_1 + B'k_2] \end{bmatrix} N_1 \\ \\ + \begin{bmatrix} -E(B + C)k_1 + EBk_2 \\ +A(B^2 + C^2 + D^2)[D(k_1 k_2) + B(k_2^2 - k_1' + k_2') + B'(-k_1 + k_2) - Ck_1' - C'k_1] \end{bmatrix} N_2 \end{Bmatrix} \tag{56}$$

where
$E(s) = -A'(B^2 + C^2 + D^2) - A(BB' + CC' + DD').$
By using (54) and (56), we obtain

$$k_{2\beta_{TN_1N_2}} = -\langle N'_{2\beta_{TN_1N_2}}, N_{1\beta_{TN_1N_2}} \rangle = \frac{-\sqrt{3}}{2A^2(B^2+C^2+D^2)}\begin{bmatrix} k_1\big(D'(B + C) - D(B' + C')\big) \\ +k_2\big(-B(C' + D') + B'(C + D)\big) \\ +k_1^2 D(C - B) + k_2^2 B(D - C) \\ +k_1 k_2(-B^2 + BC - DC + D^2) \end{bmatrix}. \tag{57}$$

**Theorem 13:** Let $\beta_{TN_1N_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. If $\beta_{TN_1N_2}$ Smarandache curve is a Tz-curve then, the equation must be

$$\frac{k_{2\beta_{TN_1N_2}}}{d_{osc}^2} = \frac{-3\sqrt{3}}{\big(D(2k_1 - k_2) - C(k_1 + k_2) + B(2k_2 - k_1)\big)^2}\begin{Bmatrix} k_1\big(D'(B + C) - D(B' + C')\big) \\ +k_2\big(-B(C' + D') + B'(C + D)\big) \\ +k_1^2 D(C - B) + k_2^2 B(D - C) \\ +k_1 k_2(-B^2 + BC - DC + D^2) \end{Bmatrix} \neq 0\ (constant)\ [31]. \tag{58}$$

**Proof:** By using (9) and (55), we get

$$d_{osc} = \langle \beta_{TN_1N_2}, N_{2\beta_{TN_1N_2}} \rangle = \frac{1}{\sqrt{6}A\sqrt{B^2+C^2+D^2}}\big(D(2k_1 - k_2) - C(k_1 + k_2) + B(2k_2 - k_1)\big). \tag{59}$$

Then, using equations (57) and (59), the expression (58) is obtained.
**Corollary 14:** Let $\beta_{TN_1N_2}$ curve be the Smarandache curve of the unit speed curve α(s), where $k_1, k_2 \neq 0$ are non-constant curvatures. If $k_1 = ck_2\ (c \neq 0\ constant)$, $\beta_{TN_1N_2}$ Smarandache curve becomes a planar curve.

**Theorem 15:** Let $\beta_{TN_1N_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$, where $k_1, k_2 \neq 0$ are non-constant curvatures. If $k_1 = ck_2$ $(c \neq 0\ constant)$, $\beta_{TN_1N_2}$ Smarandache curve becomes a planar Tz-curve [31].

**Proof:** By using (9) and (54), we obtain

$$d_{osc} = \langle \beta_{TN_1N_2}, N_{1\beta_{TN_1N_2}} \rangle = \frac{1}{\sqrt{3}\sqrt{B^2+C^2+D^2}}(B + C + D). \tag{60}$$

Using the expressions (4), (53) and (60), we get

$$\frac{k_{1\beta_{TN_1N_2}}}{d_{osc}^2} = \frac{3\sqrt{3}(B^2+C^2+D^2)^{\frac{3}{2}}}{4A^4(B+C+D)^2}. \tag{61}$$

If $k_1 = ck_2$ is used in (61), we obtain

$$\frac{k_{1\beta_{TN_1N_2}}}{d_{osc}^2} = \frac{3\sqrt{3}}{2\sqrt{2}}\left(\frac{c^2+1}{c^2-c+1}\right)^{\frac{3}{2}} = \text{constant}.$$

Therefore, $\beta_{TN_1N_2}$ Smarandache curve is a planar Tz-curve.

**Theorem 16:** Let $\beta_{TN_1N_2}$ curve be the Smarandache curve of the unit speed curve $\alpha(s)$. If $\alpha(s)$ is W-curve ($i.e.\ k_1, k_2 \neq 0\ constant$) then, $\beta_{TN_1N_2}$ Smarandache curve is a planar Tz-curve [31].

**Proof:** $k_1, k_2 \neq 0\ (constant)$, from equation (56), we obtain $(N_{2\beta_{TN_1N_2}})' = 0$. Then from equation (57), we get $k_{2\beta_{TN_1N_2}} = 0$. This means that the $\beta_{TN_1N_2}$ Smarandache curve is a planar curve. Then substituting $k_1, k_2 \neq 0\ (constant)$ into (61), we obtain

$$\frac{k_{1\beta_{TN_1N_2}}}{d_{osc}^2} = \frac{3\sqrt{3}(k_1^2+k_2^2)^{\frac{3}{2}}}{2\sqrt{2}(k_1^2+k_2^2-k_1k_2)^{\frac{3}{2}}} \neq 0\ (constant).$$

Therefore, $\beta_{TN_1N_2}$ Smarandache curve is a planar Tz-curve.

**Example 17:** Let $\alpha_1(s)$ be a unit speed helix curve (W-curve) given with the parametrization

$$\alpha_1(s) = \left(cos\left(\frac{s}{\sqrt{2}}\right), sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$$

(Fig. 1). Frenet vectors, curvature and torsion of $\alpha_1$ curve can be given by

$$T_{\alpha_1}(s) = \left(\frac{-1}{\sqrt{2}}sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$$

$$N_{1\alpha_1}(s) = \left(-cos\left(\frac{s}{\sqrt{2}}\right), -sin\left(\frac{s}{\sqrt{2}}\right), 0\right)$$

$$N_{2\alpha_1}(s) = \left(\frac{1}{\sqrt{2}}sin\left(\frac{s}{\sqrt{2}}\right), \frac{-1}{\sqrt{2}}cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$$

$k_{1\alpha_1}(s) = \frac{1}{2}$ and $k_{2\alpha_1}(s) = \frac{1}{2}$.

**a) $TN_1$ -Smarandache curve of $\alpha_1(s)$ curve**

$$\beta_{TN_1} = \frac{1}{\sqrt{2}}\left(T_{\alpha_1} + N_{1\alpha_1}\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}sin\left(\frac{s}{\sqrt{2}}\right) - cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}cos\left(\frac{s}{\sqrt{2}}\right) - sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$$

(Fig. 2). Frenet vectors, curvature and torsion of $\beta_{TN_1}$ curve using equations (11), (12), (14), (15), (16) and (18), we obtain

$$T_{\beta_{TN_1}}(s_\beta) = \frac{1}{\sqrt{3}}\left(-cos\left(\frac{2}{\sqrt{3}}s_\beta\right) + \sqrt{2}sin\left(\frac{2}{\sqrt{3}}s_\beta\right), -sin\left(\frac{2}{\sqrt{3}}s_\beta\right) - \sqrt{2}cos\left(\frac{2}{\sqrt{3}}s_\beta\right), 0\right)$$

$$N_{1\beta_{TN_1}}(s_\beta) = \frac{1}{\sqrt{3}}\left(sin\left(\frac{2}{\sqrt{3}}s_\beta\right) + \sqrt{2}\,cos\left(\frac{2}{\sqrt{3}}s_\beta\right), -cos\left(\frac{2}{\sqrt{3}}s_\beta\right) + \sqrt{2}sin\left(\frac{2}{\sqrt{3}}s_\beta\right), 0\right)$$

$$N_{2\beta_{TN_1}}(s_\beta) = (0,0,1)$$

$k_{1\beta_{TN_1}}(s_\beta) = \frac{2}{\sqrt{3}}$ and $k_{2\beta_{TN_1}}(s_\beta) = 0$.

In this case, since $k_{2\beta_{TN_1}}(s_\beta)=0$, the $\beta_{TN_1}$ curve becomes a planar curve. By using (21), we get
$d_{osc}=\langle \beta_{TN_1}(s_\beta), N_{1\beta_{TN_1}}(s_\beta)\rangle = \frac{-\sqrt{3}}{2}$.
By using (22), we obtain
$\frac{k_{1\beta_{TN_1}}}{d_{osc}^2}=\frac{8}{3\sqrt{3}}\neq 0\ (constant)$ .

Therefore, $\beta_{TN_1}$ curve becomes a planar Tz-curve.

**b) $N_1N_2$ -Smarandache curve of $\alpha_1(s)$ curve**

$$\beta_{N_1N_2}=\frac{1}{\sqrt{2}}\left(N_{1\alpha_1}+N_{2\alpha_1}\right)=\frac{1}{\sqrt{2}}\left(-\cos\left(\frac{s}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right),-\sin\left(\frac{s}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right),\frac{1}{\sqrt{2}}\right)$$

(Fig. 3). Frenet vectors, curvature and torsion of $\beta_{N_1N_2}$ curve using equations (37), (38), (40), (41), (42) and (44), we obtain

$$T_{\beta_{N_1N_2}}(s_\beta)=\frac{1}{\sqrt{3}}\left(\sqrt{2}\sin\left(\frac{2}{\sqrt{3}}s_\beta\right)+\cos\left(\frac{2}{\sqrt{3}}s_\beta\right),-\sqrt{2}\cos\left(\frac{2}{\sqrt{3}}s_\beta\right)+\sin\left(\frac{2}{\sqrt{3}}s_\beta\right),0\right)$$

$$N_{1\beta_{N_1N_2}}(s_\beta)=\frac{1}{\sqrt{3}}\left(\sqrt{2}\cos\left(\frac{2}{\sqrt{3}}s_\beta\right)-\sin\left(\frac{2}{\sqrt{3}}s_\beta\right),\sqrt{2}\sin\left(\frac{2}{\sqrt{3}}s_\beta\right)+\cos\left(\frac{2}{\sqrt{3}}s_\beta\right),0\right)$$

$$N_{2\beta_{N_1N_2}}(s_\beta)=(0,0,1)$$

$k_{1\beta_{N_1N_2}}(s_\beta)=\frac{2}{\sqrt{3}}$ and $k_{2\beta_{N_1N_2}}(s_\beta)=0$.

In this case, since $k_{2\beta_{N_1N_2}}(s_\beta)=0$, the $\beta_{N_1N_2}$ curve becomes a planar curve. By using (47), we get
$d_{osc}=\langle \beta_{N_1N_2}(s_\beta), N_{1\beta_{N_1N_2}}(s_\beta)\rangle=\frac{-\sqrt{3}}{2}$.
By using equation (48)
$\frac{k_{1\beta_{N_1N_2}}}{d_{osc}^2}=\frac{8}{3\sqrt{3}}\neq 0\ (constant)$.
Therefore, $\beta_{N_1N_2}$ curve becomes a planar Tz-curve.

**c) $TN_1N_2$ -Smarandache curve of $\alpha_1(s)$ curve**

$$\beta_{TN_1N_2}=\frac{1}{\sqrt{3}}\left(T_{\alpha_1}+N_{1\alpha_1}+N_{2\alpha_1})\right)=\frac{1}{\sqrt{3}}\left(-\cos\left(\frac{s}{\sqrt{2}}\right),-\sin\left(\frac{s}{\sqrt{2}}\right),\sqrt{2}\right)$$

(Fig. 4). Frenet vectors, curvature and torsion of curve $\beta_{TN_1N_2}$ using equations (50), (51), (53), (54), (55) and (57), we obtain

$$T_{\beta_{TN_1N_2}}(s_\beta)=\left(\sin(\sqrt{3}\,s_\beta),-\cos(\sqrt{3}\,s_\beta),0\right)$$

$$N_{1\beta_{TN_1N_2}}(s_\beta)=\left(-\cos(\sqrt{3}\,s_\beta),-\sin(\sqrt{3}\,s_\beta),0\right)$$

$$N_{2\beta_{TN_1N_2}}(s_\beta)=(0,0,1)$$

$k_{1\beta_{TN_1N_2}}(s_\beta)=\sqrt{3}$ and $k_{2\beta_{TN_1N_2}}(s_\beta)=0$.

In this case, since $k_{2\beta_{TN_1N_2}}(s_\beta)=0$, the $\beta_{TN_1N_2}$ curve becomes a planar curve. By using (60), we get
$d_{osc}=\langle \beta_{TN_1N_2}(s_\beta), N_{1\beta_{TN_1N_2}}(s_\beta)\rangle=\frac{-1}{\sqrt{3}}$.
By using (61), we obtain
$\frac{k_{1\beta_{TN_1N_2}}}{d_{osc}^2}=3\sqrt{3}\neq 0\ (constant)$.
Therefore, $\beta_{TN_1N_2}$ curve becomes a planar Tz-curve

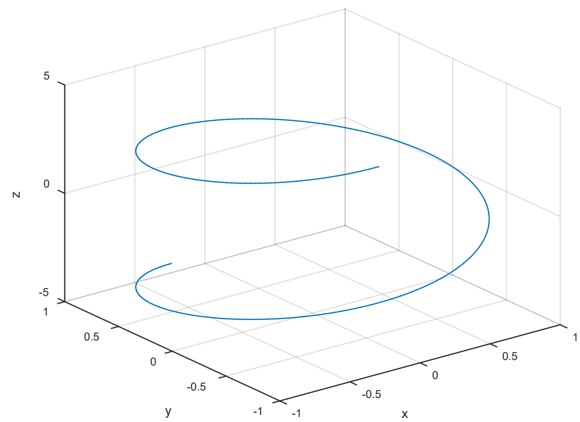


Fig. 1. $\alpha_1(s)$ curve

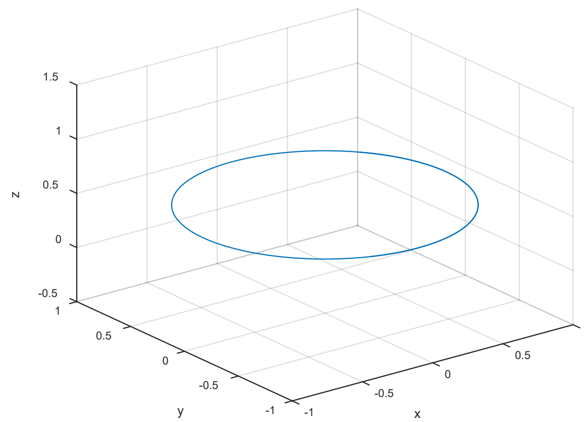


Fig. 2. $\beta_{TN_1}$ curve of $\alpha_1(s)$ curve

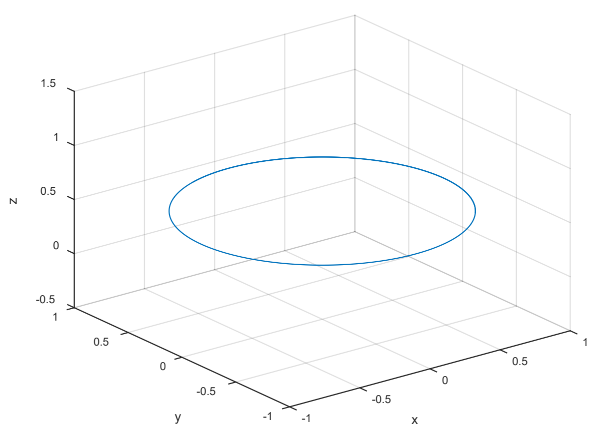


Fig. 3. $\beta_{N_1N_2}$ curve of $\alpha_1(s)$ curve

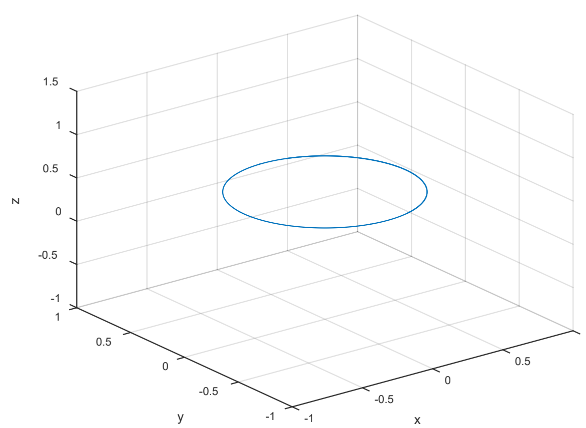


Fig. 4. $\beta_{TN_1N_2}$ curve of $\alpha_1(s)$ curve

**Example 18:** Let $\alpha_2(s)$ be a unit speed cylindrical helix curve given with the parametrization

$$\alpha_2(s) = \frac{1}{\sqrt{5}}\left(\sqrt{1+s^2}, 2s, \ln\left(s+\sqrt{1+s^2}\right)\right)$$

(Fig. 5). Frenet vectors, curvature and torsion of $\alpha_2$ curve can be given by

$$T_{\alpha_2}(s) = \frac{1}{\sqrt{5}}\left(\frac{s}{\sqrt{1+s^2}}, 2, \frac{1}{\sqrt{1+s^2}}\right)$$

$$N_{1\alpha_2}(s) = \left(\frac{1}{\sqrt{1+s^2}}, 0, \frac{-s}{\sqrt{1+s^2}}\right)$$

$$N_{2\alpha_2}(s) = \frac{1}{\sqrt{5}}\left(\frac{-2s}{\sqrt{1+s^2}}, 1, \frac{-2}{\sqrt{1+s^2}}\right)$$

$$k_{1\alpha_2}(s) = \frac{1}{\sqrt{5}(1+s^2)} \text{ and } k_{2\alpha_2}(s) = \frac{2}{\sqrt{5}(1+s^2)}.$$

**a) $TN_1$ -Smarandache curve of $\alpha_2(s)$ curve**

$\beta_{TN_1}=\frac{1}{\sqrt{2}}\left(T_{\alpha_2}+N_{1\alpha_2}\right)=\frac{1}{\sqrt{10}}\left(\frac{s+\sqrt{5}}{\sqrt{1+s^2}},2,\frac{1-\sqrt{5}s}{\sqrt{1+s^2}}\right)$

(Fig. 6). Frenet vectors, curvature and torsion of $\beta_{TN_1}$ curve using equations (11), (12), (14), (15), (16) and (18), we obtain

$T_{\beta_{TN_1}}(s_\beta)=\frac{1}{\sqrt{6}}\left(cos\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)-\sqrt{5}sin\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right),0,-\sqrt{5}cos\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)-sin\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)\right)$

$N_{1\beta_{TN_1}}(s_\beta)=\frac{1}{\sqrt{6}}\left(-sin\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)-\sqrt{5}cos\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right),0,\sqrt{5}sin\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)-cos\left(\frac{\sqrt{5}}{\sqrt{3}}s_\beta\right)\right)$

$N_{2\beta_{TN_1}}(s_\beta)=(0,1,0)$

$k_{1\beta_{TN_1}}(s_\beta)=\frac{\sqrt{5}}{\sqrt{3}}$ and $k_{2\beta_{TN_1}}(s_\beta)=0$.

In this case, since $k_{2\beta_{TN_1}}(s_\beta)=0$, the $\beta_{TN_1}$ curve becomes a planar curve. By using (21), we get

$d_{osc}=\langle\beta_{TN_1}(s_\beta),N_{1\beta_{TN_1}}(s_\beta)\rangle=\frac{-\sqrt{3}}{\sqrt{5}}$.

By using (22), we obtain

$\frac{k_{1\beta_{TN_1}}}{d_{osc}^2}=\frac{5\sqrt{5}}{3\sqrt{3}}\neq 0\ (constant)$ .

Therefore, $\beta_{TN_1}$ curve becomes a planar Tz-curve.

**b) $TN_2$ -Smarandache curve of $\alpha_2(s)$ curve**

$\beta_{TN_2}=\frac{1}{\sqrt{2}}\left(T_{\alpha_2}+N_{2\alpha_2}\right)=\frac{1}{\sqrt{2}}\left(\frac{-s}{\sqrt{5}\sqrt{1+s^2}},\frac{3}{\sqrt{5}},\frac{-1}{\sqrt{5}\sqrt{1+s^2}}\right)$

(Fig. 7). Frenet vectors, curvature and torsion of $\beta_{TN_2}$ curve using equations (24), (25), (27), (28), (29) and (31), we obtain

$T_{\beta_{TN_2}}(s_\beta)=\left(-cos\left(\sqrt{10}s_\beta\right),0,sin\left(\sqrt{10}s_\beta\right)\right)$

$N_{1\beta_{TN_2}}(s_\beta)=\left(sin\left(\sqrt{10}s_\beta\right),0,cos\left(\sqrt{10}s_\beta\right)\right)$

$N_{2\beta_{TN_2}}(s_\beta)=(0,1,0)$

$k_{1\beta_{TN_2}}(s_\beta)=\sqrt{10}$ and $k_{2\beta_{TN_2}}(s_\beta)=0$.

In this case, since $k_{2\beta_{TN_2}}(s_\beta)=0$, the $\beta_{TN_2}$ curve becomes a planar curve. By using (34), we get

$d_{osc}=\langle\beta_{TN_2}(s_\beta),N_{1\beta_{TN_2}}(s_\beta)\rangle=\frac{-1}{\sqrt{10}}$.

By using (35), we obtain

$\frac{k_{1\beta_{TN_2}}}{d_{osc}^2}=10\sqrt{10}\neq 0\ (constant)$.

Therefore, $\beta_{TN_2}$ curve becomes a planar Tz-curve.

**c) $N_1N_2$ -Smarandache curve of $\alpha_2(s)$ curve**

$\beta_{N_1N_2}=\frac{1}{\sqrt{2}}\left(N_{1\alpha_2}+N_{2\alpha_2}\right)=\frac{1}{\sqrt{10}}\left(\frac{\sqrt{5}-2s}{\sqrt{1+s^2}},1,\frac{-2-\sqrt{5}s}{\sqrt{1+s^2}}\right)$

(Fig. 8). Frenet vectors, curvature and torsion of $\beta_{N_1N_2}$ curve using equations (37), (38), (40), (41), (42) and (44), we obtain

$T_{\beta_{N_1N_2}}(s_\beta)=\frac{1}{3}\left(-\sqrt{5}sin\left(\frac{\sqrt{10}}{3}s_\beta\right)-2cos\left(\frac{\sqrt{10}}{3}s_\beta\right),0,-\sqrt{5}cos\left(\frac{\sqrt{10}}{3}s_\beta\right)+2sin\left(\frac{\sqrt{10}}{3}s_\beta\right)\right)$

$N_{1\beta_{N_1N_2}}(s_\beta)=\frac{1}{3}\left(-\sqrt{5}cos\left(\frac{\sqrt{10}}{3}s_\beta\right)+2sin\left(\frac{\sqrt{10}}{3}s_\beta\right),0,\sqrt{5}sin\left(\frac{\sqrt{10}}{3}s_\beta\right)+2cos\left(\frac{\sqrt{10}}{3}s_\beta\right)\right)$

$N_{2\beta_{N_1N_2}}(s_\beta) = (0,1,0)$

$k_{1\beta_{N_1N_2}}(s_\beta) = \frac{\sqrt{10}}{3}$ and $k_{2\beta_{N_1N_2}}(s_\beta) = 0.$

In this case, since $k_{2\beta_{N_1N_2}}(s_\beta) = 0$, the $\beta_{N_1N_2}$ curve becomes a planar curve. By using (47), we get $d_{osc} = \langle \beta_{N_1N_2}(s_\beta), N_{1\beta_{N_1N_2}}(s_\beta)\rangle = \frac{-3}{\sqrt{10}}.$

By using equation (48)

$\frac{k_{1\beta_{N_1N_2}}}{d_{osc}^2} = \frac{10\sqrt{10}}{27} \neq 0\ (constant).$

Therefore, $\beta_{N_1N_2}$ curve becomes a planar Tz-curve.

**d) $TN_1N_2$ -Smarandache curve of $\alpha_2(s)$ curve**

$\beta_{TN_1N_2} = \frac{1}{\sqrt{3}}\left(T_{\alpha_2} + N_{1\alpha_2} + N_{2\alpha_2}\right) = \frac{1}{\sqrt{15}}\left(\frac{\sqrt{5}-s}{\sqrt{1+s^2}}, 3, \frac{-1-\sqrt{5}s}{\sqrt{1+s^2}}\right)$

(Fig. 9). Frenet vectors, curvature and torsion of curve $\beta_{TN_1N_2}$ using equations (50), (51), (53), (54), (55) and (57), we obtain

$T_{\beta_{TN_1N_2}}(s_\beta) = \frac{1}{\sqrt{6}}\left(-cos\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right) - \sqrt{5}sin\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right), 0, -\sqrt{5}cos\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right) + sin\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right)\right)$

$N_{1\beta_{TN_1N_2}}(s_\beta) = \frac{1}{\sqrt{6}}\left(sin\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right) - \sqrt{5}cos\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right), 0, \sqrt{5}sin\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right) + cos\left(\frac{\sqrt{5}}{\sqrt{2}}s_\beta\right)\right)$

$N_{2\beta_{TN_1N_2}}(s_\beta) = (0,1,0)$

$k_{1\beta_{TN_1N_2}}(s_\beta) = \frac{\sqrt{5}}{\sqrt{2}}$ and $k_{2\beta_{TN_1N_2}}(s_\beta) = 0.$

In this case, since $k_{2\beta_{TN_1N_2}}(s_\beta) = 0$, the $\beta_{TN_1N_2}$ curve becomes a planar curve. By using (60), we get $d_{osc} = \langle \beta_{TN_1N_2}(s_\beta), N_{1\beta_{TN_1N_2}}(s_\beta)\rangle = \frac{-\sqrt{2}}{\sqrt{5}}.$

By using (61), we obtain

$\frac{k_{1\beta_{TN_1N_2}}}{d_{osc}^2} = \frac{5\sqrt{5}}{2\sqrt{2}} \neq 0\ (constant).$

Therefore, $\beta_{TN_1N_2}$ curve becomes a planar Tz-curve

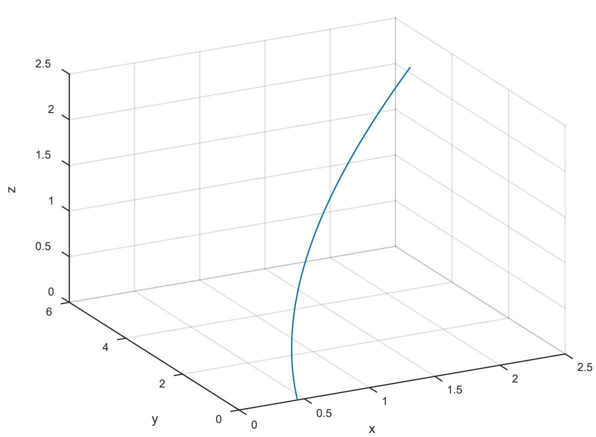


Fig. 5. $\alpha_2(s)$ curve

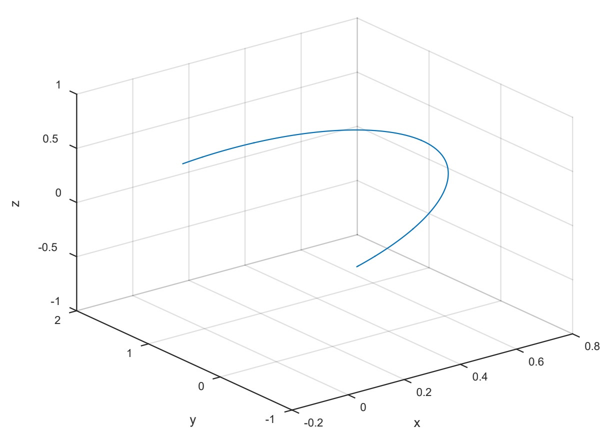


Fig. 6. $\beta_{TN_1}$ curve of $\alpha_2(s)$ curve

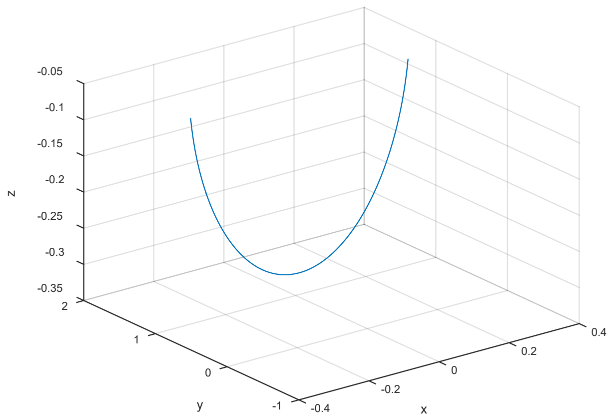


Fig. 7. $\beta_{TN_2}$ curve of $\alpha_2(s)$ curve

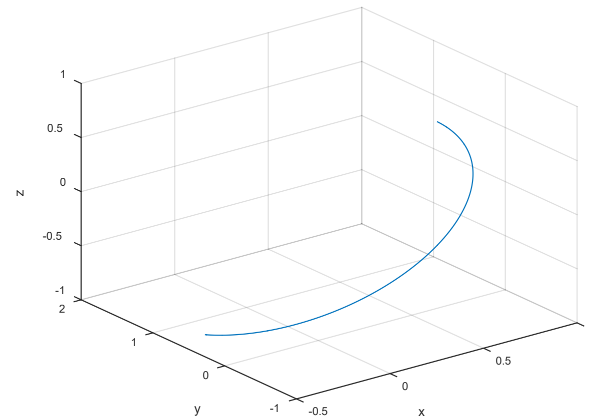


Fig. 8. $\beta_{N_1N_2}$ curve of $\alpha_2(s)$ curve

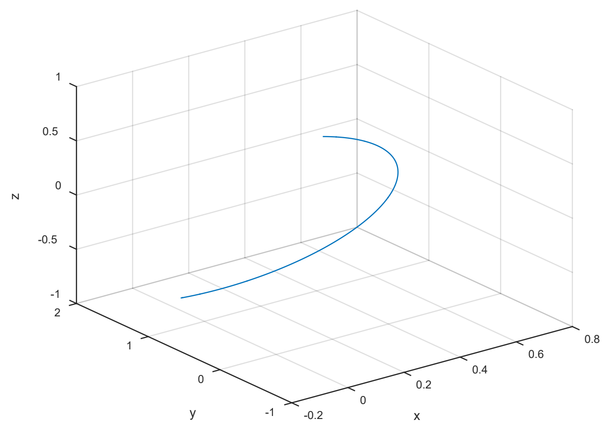


Fig. 9. $\beta_{TN_1N_2}$ curve of $\alpha_2(s)$ curve

## REFERENCES


1. M. Turgut, S. Yilmaz, "Smarandache curves in Minkowski space-time", *International J. Math. Combin.,* vol. 3, pp. 51-55, 2008.
2. A. T. Ali, "Special Smarandache curve in the Euclidean space", *International J. Math. Combin.,* vol. 2, pp. 30-36, 2010.
3. S. Şenyurt, S. Sivas, "An application of Smarandache curve", *Ordu Univ. J. Sci. Tech.,* vol. 3, no. 1, pp. 46-60, 2013.
4. V. Bulut, A. Caliskan, "Spherical images of special Smarandache curves in $\mathbb{E}^3$", *International J. Math. Combin.,* vol. 3, pp. 43-54, 2015.
5. H. S. Abdel-Aziz, M. Khaalifa Saad, "Computation of Smarandache curves according to Darboux frame in Minkowski 3-space", *Journal of the Egyptian Mathematical Society,* vol. 25, pp. 382-390, 2017.
6. M. Elzawy, "Smarandache curves in Euclidean 4-space $\mathbb{E}^4$ ", *Journal of the Egyptian Mathematical Society,* vol. 25, pp. 268-271, 2017.
7. S. Şenyurt, B. Öztürk, "Smarandache curves of Anti-Salkowski curve according to Frenet frame", *Proceedings of The International Conference on Mathematical Studies and Application,* Karamanoğlu Mehmetbey University, Karaman, Turkey, 4-6 October 2018.

8. S. Lal, “Smarandache curves in Euclidean space of parallel transport frame”, *Journal of Energing Technologies and Innovative Research,* vol. 6, no. 7, 2019.
9. M. Altın, A. Kazan and H. B. Karadağ, “Hypersurface families with Smarandache curves in Galilean 4-space”, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,* vol. 70, no. 2, pp. 744-761, 2021.
10. Ş. Alıç, B. Yılmaz, “Smarandache curves according to alternative frame in $\mathbb{E}^3$”, *Journal of Universal Mathematics,* vol. 4, no. 2, pp. 140-156, 2021.Guo, Q., Chen, Z., Liu, P., Li, Y., Hu, J., Gao, Y., Li, X. (2021). Influence of basalt fiber on mode I and II fracture properties of asphalt mixture at medium and low temperatures. *Theor. Appl. Fract. Mech., V.112,* 102884.
11. S. Kaya Nurkan, İ. A. Güven, “A New approach for Smarandache curve”, *Turk. J. Math. Comput. Sci.,* vol. 14, no. 1, pp. 155-165, 2022.
12. S. Şenyurt, D. Canlı, E. Çan and S. G. Mazlum, “Some special Smarandache Ruled surfaces by Frenet frame in $\mathbb{E}^3$-2”, *Honam Mathematical Journal,* vol. 44, no. 4, pp. 594-617, 2022.
13. S. Şenyurt, K. H. Ayvacı and D. Canlı, “Smarandache curves according to Flc-frame in Euclidean 3-space”, *Fundamentals of Contemporary Mathematical Sciences,* vol. 4, no. 1, pp. 16-30, 2023.
14. G. Tzitzeica, “Sur certaines courbes gauches”, *Annales scientifiques de l'École Normale Supérieure,* vol. 28, no. 3, pp. 9-32, 1911.
15. A. Bobe, W. G. Boskoff and M. G. Ciuca, “Tzitzeica type centro-affine invariants in Minkowski space”, *An. St. Univ. Ovidius Constanta,* vol. 20, no. 2, pp. 27-34, 2012.
16. M. Crasmareanu, “Cylindrical Tzitzeica curves implies forced harmonic oscillators”, *Balkan Journal of Geometry and Its Applications,* vol. 7, no. 1, pp. 37-42, 2002.
17. M. K. Karacan, B. Bükcü, “On the hyperbolic cylindrical Tzitzeica curves in Mikowski 3-space”, *BAÜ FBE Dergisi,* vol. 10, no. 1, pp. 46-51, 2009.
18. M. K. Karacan, B. Bükcü, “On the elliptic cylindrical Tzitzeica curves in Mikowski 3-space”, *Scientia Magna,* vol. 5, no. 3, pp. 44-48, 2009.
19. N. Bila, “Symmetry reductions for the Tzitzeica curve equation”, *Math and Computer Science Working Papers,* vol. 16, 2012.
20. B. Bayram, E. Tunç, K. Arslan, G. Öztürk, “On Tzitzeica curves in Euclidean 3-space $\mathbb{E}^3$”, *Facta Universitatis, Series Mathematics and Informatics,* vol. 33, no. 3, pp. 409-416, 2018.
21. B. Bayram, E. Tunç, “On tzitzeica surfaces in euclidean 3-space $\mathbb{E}^3$”, *Journal of Balikesir University Institute of Science and Technology,* vol. 23, no. 1, pp. 277-290, 2021.
22. B. Bayram, E. Tunç, “A New characterization of Tzitzeica curves in Euclidean 4-space”, *Fundamentals of Contemporary Mathematical Sciences,* vol. 4, no. 2, pp. 77-86, 2023
23. B. Bayram, E. Tunç, “A Note on Tzitzeica curve in Euclidean 4-spaces $\mathbb{E}^4$”, *International Theory, Research and Reviews in Science and Mathematics,* Chapter. 10, Serüven Yayınevi, pp. 141-166, 2023
24. G. Öztürk, S. Büyükkütük, İ. Kişi, “A Characterization of Curves in Galilean 4-Space”, *Bulletin of Irannian Mathematical Society*, vol. 43, no. 3, pp. 771-780, 2017.
25. H. Gluck, “Higher curvatures of curves in Euclidean space”, *The American Mathematical Monthly,* vol. 73, no. 7, pp. 243-245, 1966.
26. A. Gray, “Modern differential geometry of curves and surfaces”, *CRC Press,* 1993.
27. F. Klein, S. Lie, “Uber diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen”, *The American Mathematical Monthly,* vol. 4, pp. 50-84, 1871.
28. G. Öztürk, K. Arslan, H. Hacısalihoğlu, “A characterization of ccr-curves in $\mathbb{R}^n$”, *Proceedings of the Estonian Academy of Sciences,* vol. 57, pp. 217-224, 2008.
29. B. Y. Chen, “When does the position vector of a space curve always lies in its rectifying plane?”, *The American Mathematical Monthly,* vol. 110, pp. 147-152, 2003.

30. K. İlarslan, E. Nesovic, “Some characterizations of osculating curves in the Euclidean spaces”, *Demonstratio Mathematica,* vol. 16, no. 4, pp. 931-939, 2008.
31. O. Karacan, Tzitzeica Smarandache Curves in Three Dimensional Euclidean Space, M.S. thesis, Instutıte of Science, Balıkesir University, Balıkesir, Türkiye, 2024