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A Robust Numerical Method for the Singularly Perturbed Integro-Differential Equations

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Highlights

- This paper focuses on two different schemes.
- We construct this exponentially difference scheme on a uniform mesh using the FDM.
- We show that the method under study is convergent in the first order.

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Singularly Perturbed Problem(SPP), Fredholm Integro-Differential Equation(FIDE), Numerical Integration Method(NIM), Finite difference Method(FDM), Thomas Algorithm(TA)

Abstract

In the present study, NIM is given to approximate the solution of SPFIDEs. As a first step, the FDM on a uniform mesh is used, followed by the TM for integrals. After these calculations, the difference equation is obtained in the form of a system of equations. This system of equations is solved with the TA. Finally, example applications are made, showing the accuracy and economics of the presented method.

1. INTRODUCTION

Integro-differential equations contain many mathematical formulas in the natural sciences. In particular, the integro-differential equations with fixed integral bounds are called Fredholm integro-differential equations. They are found in various application areas of science and engineering. For example, economics, electrostatics, astronomy, physics, chemistry, mechanics, biology, fluid dynamics, electromagnetic theory, atomic physics, astronomy, fluid dynamics, etc. [1-6]. In most cases, it is difficult to find exact solutions to such problems using most analytical methods. Thus, these problems should be solved by appropriate approximate methods. Also, there are different approaches in the literature for solving Fredholm integro-differential equations. For example, Adomian decomposition method [7], reproducing kernel Hilbert space method [8], Galerkin method [9], Legendre–Galerkin method [10], exponential spline method [11], hybrid Taylor and block-pulse functions method [12], finite difference method [5-13]. Notice that a significant part of the studies related to the numerical integration method in the following works [14-21]. Also, the singularly perturbed problem [22-24] has a positive, minimal ε parameter in the coefficient of the highest order derivative. This parameter creates boundary multiples in the problem. In these parts, the behaviour of the solution changes. This state creates boundless derivatives in the solution of singular perturbation problems. In addition, the problems we deal with in the study include both ε parameter and integral terms,

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making it more difficult to reach analytical solutions. Many classical analytical and numerical methods that have been and are being applied so far cannot eliminate this problem. For this reason, the numerical integration method, which is one of the approximate methods that gives uniform convergence according to ε, is used in the study.

We will solve the singularly perturbed Fredholm integro-differential equation with boundary conditions by applying the numerical integration method as follows:

$$-\varepsilon y''(x) + b(x)y'(x) + \gamma \int_0^1 M(x,s)y(s)ds = g(x), x \in K = [0,1], \tag{1}$$

$$y(0) = \mu_1, y(1) = \mu_2, \tag{2}$$

μ_1 and μ_2 are finite constants; $\varepsilon \in (0,1); b(x) \in K,\ g(x) \in K$ ve $M(x,s) \in K \times K$ are continuous functions.

The obtained results can be organized in the following order:

The study aims is to obtain an economical and reliable approach for the approximate solution of the boundary value problems of the Fredholm integro-differential equation on a uniform mesh. Firstly, the problem has been introduced. Secondly, the numerical integration method is given. Thirdly, an application is made for an example problem of the proposed method. The obtained results are presented in tables and graphs.

2. NUMERICAL INTEGRATION METHOD(NIM) AND APPLICATION

The boundary value problems of Fredholm integro-differential equations, whose approximate solution is examined in the study, have a the right boundary layer. Accordingly, the solution steps of the numerical integration method will be given below. It will be seen that the presented method satisfies the stability conditions.

First, the integral of the equation whose numerical solution is being investigated is taken over the interval $[x_{i-1}, x_i]$. Then, the finite difference derivatives corresponding to the first two terms are obtained. The TM is also used for integrals. TA is applied, and an approximate solution is reached for the equation system obtained here. It is seen in the literature that the NIM is applied to many equations [25-27].

The problem (1)-(2) has the right boundary layer for $b(x) < \alpha < 0$, where α is constant. That is, $x = 1$ is the right boundary layer. The interval $[0,1]$ is divided into h segments of length N, so the grid is smooth.

The mesh points here are in the form of $x_i = x_0 + ih,\quad i = 0,\dots,N-1$. In Equation (1), each term is integrated from x_{i-1} to x_i.

$$\int_{x_{i-1}}^{x_i} [\varepsilon y'' + b(x)y']dx + \int_{x_{i-1}}^{x_i} \int_0^1 M(x,s)y(s)dsdx = \int_{x_{i-1}}^{x_i} g(x)dx.$$

Here, the TM is used for the second and third integrals, and the following equation is obtained:

$$\begin{aligned}&\varepsilon[y'(x_i) - y'(x_{i-1})] + b_i y(x_i) - b_{i-1} y(x_{i-1})\\&+\frac{\gamma h^2}{2}\left[M(x_0,x_0)y(x_0) + 2\sum_{k=1}^{i-1} M(x_0,x_k)y(x_k) + M(x_0,x_i)y(x_i)\right] = \frac{h}{2}[g_i + g_{i-1}].\end{aligned} \tag{3}$$

The finite difference derivatives $y_i' = \frac{y(x_{i+1}) - y(x_i)}{h}$ and $y_{i-1}' = \frac{y(x_i) - y(x_{i-1})}{h}$ for $y'(x_i)$ ve $y'(x_{i-1})$in Equation (3) are written in their place, and three diagonal system is found as

$$\begin{aligned}&\varepsilon\left[\frac{y(x_{i+1}) - y(x_i)}{h} - \frac{y(x_i) - y(x_{i-1})}{h}\right] + b_i y_i - b_{i-1} y_{i-1}\\&+\frac{\gamma h^2}{2}\left[M(x_0,x_0)y(x_0) + 2\sum_{k=1}^{i-1} M(x_0,x_k)y(x_k) + M(x_0,x_i)y(x_i)\right] = \frac{h}{2}[g_i + g_{i-1}],\end{aligned} \tag{4}$$

$$y_{i-1}\left[-\frac{\varepsilon}{h}-b_{i-1}\right]-y_i\left[-\frac{2\varepsilon}{h}+b_i-\frac{\gamma h^2}{2}K(x_0,x_i)\right]+y_{i+1}\left[-\frac{\varepsilon}{h}\right]=\frac{h}{2}[g_i+g_{i-1}]$$
$$-\frac{\gamma h^2}{2}\left[M(x_0,x_0)y(x_0)+2\sum_{k=1}^{i-1}M(x_0,x_k)y(x_k)\right], \tag{5}$$
$$y_0=\mu_1, y_N=\mu_2,$$

the finite difference problem arises. The solution to this difference problem is reached by applying the TA given below by the problem.

$$A_i=-\frac{\varepsilon}{h}-b_{i-1}, B_i=-\frac{\varepsilon}{h}, C_i=-\frac{2\varepsilon}{h}+b_i-\frac{\gamma h^2}{2}K(x_0,x_i),$$

$$F_i=\frac{\gamma h^2}{2}\left[M(x_0,x_0)y(x_0)+2\sum_{k=1}^{i-1}M(x_0,x_k)y(x_k)\right]-\frac{h}{2}[g_i+g_{i-1}],$$
$$\alpha_{i+1}=\frac{B_i}{C_i-\alpha_i A_i},\quad \beta_{i+1}=\frac{F_i+\beta_i A_i}{C_i-\alpha_i A_i},\quad i=1,2,\ldots,N-1, \alpha_1=0, \beta_1=\mu_1,$$
$$u_i=\alpha_{i+1}y_{i+1}+\beta_{i+1},\quad i=N-1,\ldots,2,1.$$

By following all these steps, an approximate solution is found.

$$A_i>0,\ B_i>0 \quad \text{ve} \quad C_i>A_i+B_i>0,$$
$$|\alpha_i|<1,\quad i=0,1,\ldots,N-1.$$

According to the conditions, the Thomas algorithm is stable, and under these conditions (5) the difference problem has unique solution [28].

3. NUMERICAL METHOD AND ITS IMPLEMENTATION

In this section, the boundary value problem of a singularly perturbed Fredholm integro-differential equation with the following left boundary layer will be approximately solved to demonstrate the efficiency, reliability, and time economy of the numerical integration method:

$$-\varepsilon y''(x)+(2-e^{-x})y'(x)+\frac{1}{2}\int_0^1 e^{x\cos(\pi s)}y(s)ds=\frac{1}{1+x},\quad 0<x<1, \tag{6}$$
$$y(0)=1, y(1)=0.$$

The exact solution to the problem (6) is unknown. Therefore, the double mesh is used [28]. The maximum errors obtained are shown as follows:

$$e_\varepsilon^N=\max_i\left|y_i^{\varepsilon,N}-\tilde{y}_{2i}^{\varepsilon,2N}\right|.$$

Let's apply the numerical integration method to the problem (6). First, in the problem (6), the integral of each term of the Fredholm integro-differential equation is taken, and necessary adjustments are made.

$$\int_{x_{i-1}}^{x_i}[-\varepsilon y''(x)+(2-e^{-x})y'(x)]dx+\frac{1}{2}\int_{x_{i-1}}^{x_i}\int_0^1 e^{x\cos(\pi s)}y(s)dsdx=\int_{x_{i-1}}^{x_i}\frac{1}{1+x}dx, \tag{7}$$
$$0<x<1, y(0)=1, y(1)=0.$$

After taking the first integral in the equation, $y'(x_i)$ and $y'(x_{i-1})$ difference approximations are used for the $y_i'=\frac{y_{i+1}-y_i}{h}$, $y_{i-1}'=\frac{y_i-y_{i-1}}{h}$ derivatives. If the trapezoidal method is applied and edited in response to other integrals,

$$y_{i-1}\left[-\frac{\varepsilon}{h}-(2-e^{-x_{i-1}})\right]-y_i\left[-\frac{2\varepsilon}{h}+(2-e^{-x})-\frac{\gamma h^2}{2}e^{x_0\cos(\pi x_i)}\right]+y_{i+1}\left[-\frac{\varepsilon}{h}\right]$$
$$=\frac{h}{2}\left[\frac{1}{1+x_i}+\frac{1}{1+x_{i-1}}\right]-\frac{\gamma h^2}{2}\left[e^{x_0\cos(\pi x_0)}y(x_0)+2\sum_{k=1}^{i-1}e^{x_0\cos(\pi x_k)}y(x_k)\right],$$
$$y_0=1, y_N=0.$$

finite difference problem is obtained. Thomas algorithm is used to solve this system.

$$A_i = -\frac{\varepsilon}{h} - (2 - e^{-x_{i-1}}), \quad B_i = -\frac{\varepsilon}{h}, \quad C_i = -\frac{2\varepsilon}{h} + (2 - e^{-x}) - \frac{\gamma h^2}{2} e^{x_0 \cos(\pi x_i)},$$
$$F_i = \frac{\gamma h^2}{2}\left[e^{x_0 \cos(\pi x_0)} y(x_0) + 2\sum_{k=1}^{i-1} e^{x_0 \cos(\pi x_k)} y(x_k)\right] - \frac{h}{2}\left[\frac{1}{1+x_i} + \frac{1}{1+x_{i-1}}\right],$$
$$\alpha_1 = 1, \beta_1 = 0.$$

Approximate solution results of the boundary value problem of the singularly perturbed Fredholm integro-differential equation are obtained by using a suitable mathematical program with the flow of the Thomas algorithm given above. For N and ε values, approximate solution curves are compared, and error graphs are drawn. Maximum errors are found. Thus, it is revealed that the proposed method is suitable for boundary value problems of the singularly perturbed Fredholm integro-differential equation.

***Table 1.** Maximum errors for the value of ε and N*

N	$\varepsilon = 2^{-1}$	$\varepsilon = 2^{-2}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-4}$
16	0.0148915991	0.0317072199	0.0528845026	0.0829121967
32	0.0080281049	0.0178568290	0.0326101135	0.0416651252
64	0.0041765595	0.0094965848	0.0181430513	0.0221928854
128	0.0021289622	0.0049035729	0.0096175579	0.0118161927

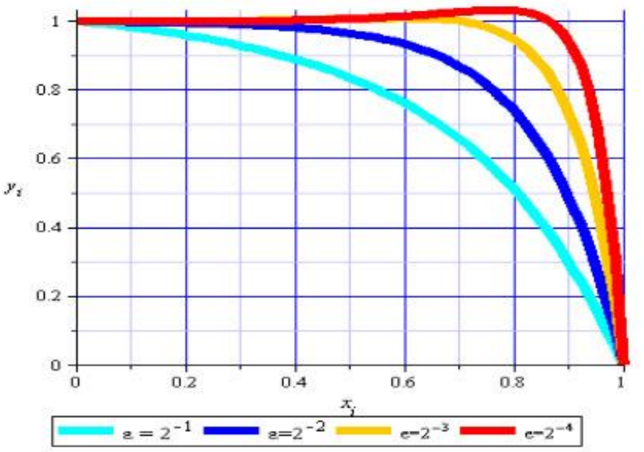

***Figure 1.** Approximate solutions curves for $N = 64$*

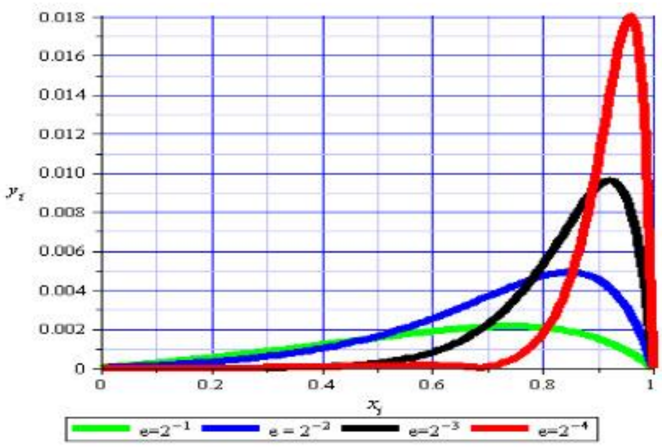

***Figure 2.** Maximum error curves for $N = 128$*

4. CONCLUSION

The boundary value problem of the singularly perturbed Fredholm integro-differential equation with a left boundary layer is solved numerically by the numerical integration method. Numerical solution and exact solution values are very close to each other. We see this situation as the maximum error values given in Table 1. In addition, According to Figure 1, as the value of ε decreases, the approximate solution curves lean towards the axes around $x = 1$. This means that the numerical method is reliable and convenient. As seen in Figure 2, the error results in the boundary layer are the maximum due to the sudden and rapid change of the solution. In addition, the stability conditions of the Thomas algorithm are provided, that is, $|\alpha_i| < 1,\ \ i = 0,1,\dots,N-1$. Since the maximum error values are reduced by half, the proposed method is convergent in the first order $\left(O(h)\right)$ on the uniform mesh. All these results show that the proposed method works well enough. Based on the results of this study, it can be said that it can be suitable for Volterra, Volterra-Fredholm integro-differential equations and their delayed types.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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