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ON GENERALIZED CONFORMABLE FRACTIONAL OPERATORS

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Abstract. In this paper, we introduce the concepts of left and right generalized conformable fractional integrals, alongside the corresponding derivatives. Additionally, we extend our investigation to derive the generalized conformable derivatives for functions within specific spaces, elucidating their inherent properties.

1. Introduction

Fractional calculus, with its roots dating back to 1695, has evolved significantly over the years and garnered increasing significance, particularly in applied sciences. Its applications span various fields including physics, mechanics, electronics, chemistry, biology, and engineering $[2-7]$, [16]. Two commonly used approaches in fractional calculus are the Caputo and Riemann-Liouville derivatives.

The Riemann-Liouville approach entails iteratively applying the integral operator $n-$times, resulting in fractional integrals of non-integer order. This method has found widespread use due to its versatility across different disciplines. However, the standard fractional calculus framework may not always be sufficient for certain applications, necessitating the introduction of specialized kernels for a more unified approach to fractional derivatives.

The differentiation operator serves as a fundamental starting point for the iteration method in fractional calculus. By incorporating the required kernels, researchers aim to achieve a more comprehensive understanding and application of fractional derivatives across various scientific and engineering contexts $[8-10]$, $[19-20]$. In the present case, Abdeljawad defined as the following the left and right conformable derivatives, respectively[18],

$$\begin{aligned} &{}_{\phi}T^{\alpha}f(\tau)=(\tau-\phi)^{1-\alpha}f'(\tau),\\ &T_{\delta}^{\alpha}f(\tau)=(\delta-\tau)^{1-\alpha}f'(\tau). \end{aligned} \tag{1.1}$$

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In this context, assuming f is a differentiable function, we possess left and right conformable integrals as the following forms, respectively [1]

$$ {}^{\beta}_{\phi}J^{\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left(\frac{(\tau-\phi)^{\alpha}-(\theta-\phi)^{\alpha}}{\alpha}\right)^{\beta-1}f(\theta)\frac{d\theta}{(\theta-\phi)^{1-\alpha}} \tag{1.2}$$

and

$$ {}^{\beta}J^{\alpha}_{\delta}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{\tau}^{\delta}\left(\frac{(\delta-\tau)^{\alpha}-(\delta-\theta)^{\alpha}}{\alpha}\right)^{\beta-1}f(\theta)\frac{d\theta}{(\delta-\theta)^{1-\alpha}}. \tag{1.3}$$

In [1], authors introduced novel fractional operators characterized by two parameters, each with kernels distinct from conventional ones. Our paper closely examines the findings of [1], focusing on their implications and further developments. We extend upon their work by deriving new generalized fractional integrals and derivatives using the newly defined fractional operators.

Moreover, we provide a thorough exposition of basic definitions and tools essential to classical fractional calculus. These foundational concepts serve as the groundwork for our subsequent discussions and advancements.

Definition 1.1. ([17] , [21]) Let $\gamma(\tau)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The space $X^{d}_{\gamma}(0,\infty)$ is the following form for $(1\leq d<\infty)$,

$$\|f\|_{X^{d}_{\gamma}}=\left(\int_{0}^{\infty}|f(\theta)|^{d}\gamma'(\tau)\,d\theta\right)^{\frac{1}{d}}<\infty \tag{1.4}$$

and if we choose $d=\infty$,

$$\|f\|_{X^{\infty}_{\gamma}}=ess\sup_{1\leq\theta<\infty}\left[f(\theta)\gamma'(\tau)\right]. \tag{1.5}$$

Additionally, if we take $\gamma(\tau)=\tau$ $(1\leq d<\infty)$ the space $X^{d}_{\gamma}(0,\infty)$, then we have the $L_{d}[0,\infty)-$space. Moreover, if we take $\gamma(\tau)=\frac{\tau^{k+1}}{k+1}$ $(1\leq d<\infty,\ k\geq 0)$ the space $X^{d}_{\gamma}(0,\infty)$, then we have the $L_{d,k}[0,\infty)-$space [17].

The authors derived the generalized left and right fractional integrals for β belonging to the complex numbers $(\beta\in\mathbb{C})$ with $Re(\beta)>0$ in [8] ,

$$\left({}_{\phi}I^{\beta,\alpha}f\right)(\theta)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\theta}\left(\frac{\theta^{\alpha}-y^{\alpha}}{\alpha}\right)^{\beta-1}f(y)\frac{dy}{y^{1-\alpha}} \tag{1.6}$$

and

$$\left(I^{\beta,\alpha}_{\delta}f\right)(\theta)=\frac{1}{\Gamma(\beta)}\int_{\theta}^{\delta}\left(\frac{y^{\alpha}-\theta^{\alpha}}{\alpha}\right)^{\beta-1}f(y)\frac{dy}{y^{1-\alpha}}, \tag{1.7}$$

respectively.

The authors obtained left and right generalized fractional derivatives for β belonging to the complex numbers $(\beta\in\mathbb{C})$ with $Re(\beta)\geq 0$ in [9] ,

$$\begin{aligned}\left({}_{\phi}D^{\beta,\alpha}f\right)(\theta)&=\zeta^{n}\left({}_{\phi}I^{n-\beta,\alpha}f\right)(\theta)\\&=\frac{\zeta^{n}}{\Gamma(n-\beta)}\int_{\phi}^{\theta}\left(\frac{\theta^{\alpha}-y^{\alpha}}{\alpha}\right)^{n-\beta-1}f(y)\frac{dy}{y^{1-\alpha}}\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\left(D^{\beta,\alpha}_{\delta}f\right)(\theta)&=(-\zeta)^{n}\left({}_{\phi}I^{n-\beta,\alpha}f\right)(\theta)\\&=\frac{(-\zeta)^{n}}{\Gamma(n-\beta)}\int_{\theta}^{\delta}\left(\frac{y^{\alpha}-\theta^{\alpha}}{\alpha}\right)^{n-\beta-1}f(y)\frac{dy}{y^{1-\alpha}}\end{aligned} \tag{1.9}$$

respectively, where $\alpha > 0$ and where $\zeta = \theta^{1-\alpha}\frac{d}{d\theta}$.

The left and right generalized Caputo fractional derivatives, as defined by the authors in [15] through the utilization of [9], are expressed in the following forms, respectively,

$$\begin{aligned} \left({}_{\phi}^{C}D^{\beta,\alpha}f\right)(\theta) &= \left({}_{\phi}I^{n-\beta,\alpha}\left(\zeta\right)^{n}f\right)(\theta) \\ &= \frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\theta}\left(\frac{\theta^{\alpha}-u^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{\zeta^{n}f(y)dy}{y^{1-\alpha}} \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \left({}^{C}D_{\delta}^{\beta,\alpha}f\right)(\theta) &= \left({}_{\phi}I^{n-\beta,\alpha}\left(-\zeta\right)^{n}f\right)(\theta) \\ &= \frac{1}{\Gamma(n-\beta)}\int_{\theta}^{\delta}\left(\frac{y^{\alpha}-\theta^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{(-\zeta)^{n}f(y)dy}{y^{1-\alpha}}. \end{aligned} \tag{1.11}$$

After introducing the generalized fractional conformable integral and derivative operators, we will highlight their significant implications and key characteristics. Additionally, we will delve into the properties of the defined generalized conformable derivative and extend our analysis to include the generalized conformable fractional derivatives within the Caputo framework. As a result, we will consolidate our findings and build upon the previously established consequences for both the generalized conformable derivatives and integrals.

2. THE GENERALIZED CONFORMABLE OPERATORS

In light of Abdeljawad's work on conformable integrals, which were extended to higher orders in reference [10], and Jarad et al.'s definition of fractional integrals in [1], we aim to establish a generalized conformable derivative. To achieve this, we'll consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. With these conditions in mind, our objective is to formulate the generalized conformable derivative based on the existing definitions of the conformable derivative

$${}_{\phi}^{\gamma}T^{\alpha}f(\theta) = \lim_{\varepsilon\to 0}\frac{f\left(\theta+\varepsilon\frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}\right)-f(\theta)}{\varepsilon}. \tag{2.1}$$

By taking into account equation (2.1). In here,

$$\Delta t = \varepsilon\frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)} \Rightarrow \varepsilon = \frac{\Delta t.\gamma'(\theta)}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \tag{2.2}$$

we select Δt in the form. Then,

$${}_{\phi}^{\gamma}T^{\alpha}f(\theta) = \frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}f'(\theta). \tag{2.3}$$

We can assert formula of generalized conformable derivative, respectively,

$$\begin{aligned} {}_{\phi}^{\gamma}T^{\alpha}f(\theta) &= \frac{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}{\gamma'(\theta)}f'(\theta) \\ {}^{\gamma}T_{\delta}^{\alpha}f(\theta) &= \frac{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}}{\gamma'(\theta)}f'(\theta). \end{aligned} \tag{2.4}$$

Additionally, we acquire generalized conformable integral operator. For this reason,

$$\int_{\phi}^{\tau}\frac{\gamma'(\theta_1)d\theta_1}{(\gamma(\theta_1)-\gamma(\phi))^{1-\alpha}}\int_{\phi}^{\theta_1}\frac{\gamma'(\theta_2)d\theta_2}{(\gamma(\theta_2)-\gamma(\phi))^{1-\alpha}}\cdots\int_{\phi}^{\theta_{n-1}}\frac{\gamma'(\theta_n)f(\theta_n)d\theta_n}{(\gamma(\theta_n)-\gamma(\phi))^{1-\alpha}}, \tag{2.5}$$

we should take $n-$times repeated integrals of the forms. Furthermore, If we employ a method akin to classical fractional integral techniques, then we write the equality

$$ {}_{\phi}^{\gamma}J^{n,\alpha}f(\tau) = \frac{1}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}. \tag{2.6}$$

Furthermore, we can acquire definition of the following for generalized conformable integrals drawing upon the equality presented in reference [2].

Definition 2.1. Let $f \in X_{\gamma}(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0) = 0$. The left and right generalized conformable fractional integrals of order $n \in \mathbb{C}$, $Re(n) \geq 0$ and $\alpha > 0$, respectively,

$$ {}_{\phi}^{\gamma}J^{n,\alpha}f(\tau) = \frac{1}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \tag{2.7}$$

and

$$ {}^{\gamma}J_{\delta}^{n,\alpha}f(\tau) = \frac{1}{\Gamma(n)}\int_{\tau}^{\delta}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}-(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right]^{n-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}}. \tag{2.8}$$

Within this context, we introduce the subsequent definition, leveraging the framework provided by the generalized conformable derivative and integral operators.

Example 2.2. Let's calculate the result of the generalized conformable fractional integral ${}_{0}^{\gamma}J^{\frac{1}{2},1}f(\tau)$ for $f(\tau) = 4\tau^{3}$.

Proof. If we choose $\gamma(x) = x$, $f(\theta) = 4\theta^{3}$, $\alpha = 1$, $n = \frac{1}{2}$ and $\phi = 0$ in (2.7), we have

$$ {}_{0}^{\gamma}J^{\frac{1}{2},1}f(\tau) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\tau}(\tau-\theta)^{-\frac{1}{2}}\,4\theta^{3}d\theta.$$

Furhermore, by using variable change $\theta = \tau u$ and $d\theta = \tau du$, we acquire

$$\begin{aligned} {}_{0}^{\gamma}J^{\frac{1}{2},1}f(\tau) &= \frac{4}{\Gamma\left(\frac{1}{2}\right)}\theta^{\frac{7}{2}}\int_{0}^{1}(1-u)^{-\frac{1}{2}}u^{3}du \\ &= \frac{4.\theta^{\frac{7}{2}}}{\Gamma\left(\frac{1}{2}\right)}B\left(\frac{1}{2},4\right) \\ &= \frac{4.\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}. \end{aligned}$$

The proof is done with Beta function and property of Beta function . □

Example 2.3. Let's calculate the result of the generalized conformable fractional integral ${}_{0}^{\gamma}J^{\frac{1}{2},1}\left({}_{0}^{\gamma}J^{\frac{1}{2},1}\left(4\tau^{3}\right)\right)$.

Proof. If we choose $\gamma(x) = x$, $f(\theta) = \frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}$, $\alpha = 1$, $n = \frac{1}{2}$ and $\phi = 0$ in (2.7), we get

$$ {}_{0}^{\gamma}J^{\frac{1}{2},1}\left({}_{0}^{\gamma}J^{\frac{1}{2},1}\left(4\tau^{3}\right)\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\tau}(\tau-\theta)^{-\frac{1}{2}}\frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}d\theta.$$

Moreover, by utilizing variable change $\theta = \tau u$ and $d\theta = \tau du$, we take

$$\begin{aligned} {}_{0}^{\gamma}J^{\frac{1}{2},1}\left(\frac{4\theta^{\frac{7}{2}}\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}.\frac{4\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\theta^{4}\int_{0}^{1}(1-u)^{-\frac{1}{2}}u^{\frac{7}{2}}du \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}.\frac{4\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)}\theta^{4}B\left(\frac{9}{2},\frac{1}{2}\right) \\ &= \theta^{4}. \end{aligned}$$

The proof is done with Beta function and property of Beta function . □

Definition 2.4. Let $f \in X_\gamma(0,\infty)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The left and right generalized conformable fractional derivatives of order $\beta \in \mathbb{C}$ and $Re(\beta) \geq 0$,

(2.9)
$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\right) f(\tau) \\ &= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \int_{\phi}^{\tau} \left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\theta)-\gamma(\phi))^\alpha}{\alpha}\right]^{n-\beta-1} \frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \end{aligned}$$

and

(2.10)
$$\begin{aligned} {}^{\gamma}D_{\delta}^{\beta,\alpha} f(\tau) &= {}^{\gamma}T_{\delta}^{n,\alpha}\left({}^{\gamma}J_{\delta}^{n-\beta,\alpha}\right) f(\tau) \\ &= \frac{{}^{\gamma}T_{\delta}^{n,\alpha}(-1)^n}{\Gamma(n-\beta)} \int_{\tau}^{\delta} \left[\frac{(\gamma(\delta)-\gamma(\tau))^\alpha-(\gamma(\delta)-\gamma(\theta))^\alpha}{\alpha}\right]^{n-\beta-1} \frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}}, \end{aligned}$$

where $n=[Re(\beta)]+1$,

(2.11)
$$\begin{aligned} {}_{\phi}^{\gamma}T^{n,\alpha} &= \underbrace{{}_{\phi}^{\gamma}T^{\alpha}\;\;{}_{\phi}^{\gamma}T^{\alpha}\cdots{}_{\phi}^{\gamma}T^{\alpha}}_{n-times}, \\ {}^{\gamma}T_{\delta}^{n,\alpha} &= \underbrace{{}^{\gamma}T_{\delta}^{\alpha}\;\;{}^{\gamma}T_{\delta}^{\alpha}\cdots{}^{\gamma}T_{\delta}^{\alpha}}_{n-times}, \end{aligned}$$

${}_{\phi}^{\gamma}T^{\alpha}$ and ${}^{\gamma}T_{\delta}^{\alpha}$ are the left and right generalized conformable differential operators.

Example 2.5. Let's calculate the result of the generalized conformable fractional derivative ${}_{0}^{\gamma}D^{\frac{1}{2},1}f(\tau)$ for $f(\tau)=\tau^4$.

Proof. If we choose $\gamma(x)=x$, $f(\theta)=\theta^4$, $\alpha=1$, $n=1$, $\beta=\frac{1}{2}$ and $\phi=0$ in (2.9), we get

$$ {}_{0}^{\gamma}D^{\frac{1}{2},1}f(\tau) = \frac{{}_{0}^{\gamma}T^{1,1}}{\Gamma\left(\frac{1}{2}\right)} \int_0^{\tau} (\tau-\theta)^{-\frac{1}{2}}\,\theta^4 d\theta. $$

Furhermore, by using variable change $\theta=\tau u$ and $d\theta=\tau du$, we have

$$\begin{aligned} {}_{0}^{\gamma}D^{\frac{1}{2},1}f(\tau) &= \frac{\frac{d}{d\theta}}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 (1-u)^{-\frac{1}{2}}\, u^4\theta^{\frac{9}{2}} du \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{d\theta}\left(\theta^{\frac{9}{2}}\right) \int_0^1 u^4 (1-u)^{-\frac{1}{2}} du \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{9}{2}\theta^{\frac{7}{2}} \frac{\Gamma(5)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} \\ &= \frac{\Gamma(5)\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}. \end{aligned}$$

The proof is done with Beta function and property of Beta function . □

Example 2.6. Let's calculate the result of the generalized conformable fractional derivative ${}_{0}^{\gamma}D^{\frac{1}{2},1}\left({}_{0}^{\gamma}D^{\frac{1}{2},1}f(\tau)\right)$ for $f(\tau)=\frac{\Gamma(5)\tau^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}$.

Proof. If we choose $\gamma(x)=x$, $f(\theta)=\frac{\Gamma(5)\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}$, $\alpha=1$, $n=1$, $\beta=\frac{1}{2}$ and $\phi=0$ in (2.9), we get

$$ {}_{0}^{\gamma}D^{\frac{1}{2},1}f(\tau) = \frac{{}_{0}^{\gamma}T^{1,1}}{\Gamma\left(\frac{1}{2}\right)} \int_0^{\tau} (\tau-\theta)^{-\frac{1}{2}} \frac{\Gamma(5)\,\theta^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)} d\theta. $$

Furhermore, by using variable change $\theta=\tau u$ and $d\theta=\tau du$, we have

$$\begin{aligned}
{}_{0}^{\gamma}D^{\frac{1}{2},1}\left(\frac{\Gamma(5)\tau^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}\right) &= \frac{\frac{d}{d\theta}}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}\int_0^1 (1-u)^{-\frac{1}{2}}u^{\frac{7}{2}}\theta^4 du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}\frac{d}{d\theta}\left(\theta^4\right)\int_0^1 u^{\frac{7}{2}}(1-u)^{-\frac{1}{2}}du \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}4\theta^3 B\left(\frac{9}{2},\frac{1}{2}\right) \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}4\theta^3\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)} \\
&= 4\theta^3.
\end{aligned}$$

The proof is done with Beta function and property of Beta function . □

Theorem 2.7. *Let $f\in X_\gamma(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, we have fractional integrals for $Re(\beta)>0$ and $Re(\varsigma)>0$,*

$$\begin{aligned}
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\varsigma,\alpha}\right)f(\tau) &= {}_{\phi}^{\gamma}J^{(\beta+\varsigma),\alpha}f(\tau), \\
{}^{\gamma}J_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\varsigma,\alpha}\right)f(\tau) &= {}^{\gamma}J_{\delta}^{(\beta+\varsigma),\alpha}f(\tau).
\end{aligned} \tag{2.12}$$

Proof. With the assistance of equation (2.7) , we obtain

(2.13)

$$\begin{aligned}
&{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\varsigma,\alpha}\right)f(\tau) \\
&= \frac{1}{\Gamma(\beta)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\theta)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta-1}\frac{\left({}_{\phi}^{\gamma}J^{\varsigma,\alpha}\right)\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(\theta)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta-1} \\
&\times\left(\int_\phi^\theta\left[\frac{(\gamma(\theta)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\varsigma-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\right)\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta+\varsigma-1}\left(\int_0^1(1-z)^{\beta-1}z^{\varsigma+1}dz\right)\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}} \\
&= \frac{1}{\Gamma(\beta+\varsigma)}\int_\phi^\tau\left[\frac{(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha}{\alpha}\right]^{\beta+\varsigma-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}} \\
&= {}_{\phi}^{\gamma}J^{(\beta+\varsigma),\alpha}f(\tau).
\end{aligned}$$

In here, we employed the change of variable,

(2.14)

$$(\gamma(\theta)-\gamma(\phi))^\alpha=(\gamma(u)-\gamma(\phi))^\alpha+z\left[(\gamma(\tau)-\gamma(\phi))^\alpha-(\gamma(u)-\gamma(\phi))^\alpha\right].$$

The proof of the second formula can similarly be illustrated using the similar approach. □

Lemma 2.8. *Let $f\in X_\gamma(0,\infty)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. We possess for $Re(r)>0$,*

$$\begin{aligned}
{}_{\phi}^{\gamma}J^{\beta,\alpha}(\gamma(\theta)-\gamma(\phi))^{\alpha(r-1)}(\tau) &= \frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{\left[(\gamma(\tau)-\gamma(\phi))^\alpha\right]^{\beta+r-1}}{\alpha^\beta}, \\
{}^{\gamma}J_{\delta}^{\beta,\alpha}(\gamma(\delta)-\gamma(\theta))^{\alpha(r-1)}(\tau) &= \frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{\left[(\gamma(\delta)-\gamma(\tau))^\alpha\right]^{\beta+r-1}}{\alpha^\beta}.
\end{aligned} \tag{2.15}$$

Proof. With the assistance of (2.7), we hold

$$\begin{aligned} & {}_{\phi}^{\gamma}J^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\left(\tau\right) \\ & =\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{[(\gamma(\theta)-\gamma(\phi))^{\alpha}]^{r-1}\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ & =\frac{[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{\beta+r-1}}{\Gamma(\beta)\alpha^{\beta-1}}\int_{0}^{1}\left(1-z\right)^{\beta-1}z^{r-1}dz \\ & =\frac{\Gamma(r)}{\Gamma(\beta+r)}\frac{[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{\beta+r-1}}{\alpha^{\beta}}. \end{aligned} \tag{2.16}$$

Moreover, we employed the change of variable,

$$\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}=z\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}. \tag{2.17}$$

The proof of the second formula can similarly be illustrated using the similar approach. □

Lemma 2.9. *Let $f\in X_{\gamma}(0,\infty)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right)=0$. We possess for $Re\left(n-\alpha\right)>0$,*

$$\begin{aligned} \left[{}_{\phi}^{\gamma}D^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\right]\left(\tau\right) &= \frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{r-\beta-1}, \\ \left[{}^{\gamma}D_{\delta}^{\beta,\alpha}\left(\gamma\left(\delta\right)-\gamma\left(\theta\right)\right)^{\alpha(r-1)}\right]\left(\tau\right) &= \frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\delta\right)-\gamma\left(\tau\right)\right)^{\alpha}\right]^{r-\beta-1}. \end{aligned} \tag{2.18}$$

Proof. With the assistance of (2.9), we hold

$$\begin{aligned} & \left[{}_{\phi}^{\gamma}D^{\beta,\alpha}\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha(r-1)}\right]\left(\tau\right) \\ & =\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{[(\gamma(\theta)-\gamma(\phi))^{\alpha}]^{r-1}.\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ & =\frac{{}_{\phi}^{\gamma}T^{n,\alpha}[(\gamma(\tau)-\gamma(\phi))^{\alpha}]^{n+r-\beta-1}}{\Gamma(n-\beta)\alpha^{n-\beta}}\int_{0}^{1}\left(1-z\right)^{n-\beta-1}z^{r-1}dz \\ & =\frac{\alpha^{\beta}\Gamma(r)}{\Gamma(r-\beta)}\left[\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}\right]^{r-\beta-1}. \end{aligned} \tag{2.19}$$

In here, we employed the change of variable,

$$\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}=z\left(\gamma\left(\tau\right)-\gamma\left(\phi\right)\right)^{\alpha}. \tag{2.20}$$

The proof of the second formula can similarly be illustrated using the same approach. □

Remark 2.10. It can be illustrated that

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha}f &= {}_{\phi}^{\gamma}J^{\beta,-\alpha} \\ {}^{\gamma}D_{\delta}^{\beta,\alpha}f &= {}^{\gamma}J_{\delta}^{\beta,-\alpha}. \end{aligned} \tag{2.21}$$

3. GENERALIZED CONFORMABLE FRACTIONAL DERIVATIVES ON THE SPECIFIC SPACES

In this section, we will introduce several definitions pertaining to lemmas and theorems. Furthermore, we will showcase the significant outcomes of the generalized conformable fractional derivatives within the spaces $C_{\alpha,\phi}^{n}$ and $C_{\alpha,\delta}^{n}$.

Definition 3.1. [18] For $0<\alpha\leq 1$ and an interval $[\phi,\delta]$, we describe

$$\begin{aligned} {}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)= & \Big\{f:[\phi,\delta]\rightarrow\mathbb{R}:f\left(\tau\right)=\left({}_{\phi}^{\gamma}I^{\beta,\alpha}\varphi\right)\left(\tau\right)+f\left(\phi\right) \\ & \qquad \text{for some } \varphi\in {}^{\gamma}L_{\alpha}\left(\phi\right)\Big\} \end{aligned} \tag{3.1}$$

and

$$ {}_{\alpha}^{\gamma}I\left([\phi,\delta]\right)= \left\{g:[\phi,\delta]\to\mathbb{R}:g\left(\tau\right)=\left({}^{\gamma}I_{\delta}^{\beta,\alpha}\varphi\right)\left(\tau\right)+g\left(\delta\right) \text{ for some } \varphi\in {}^{\gamma}L_{\alpha}\left(\delta\right)\right\}. \tag{3.2}$$

Where

$$ {}^{\gamma}L_{\alpha}\left(\phi\right)=\left\{\varphi:[\phi,\delta]\to\mathbb{R},\ \left({}_{\phi}^{\gamma}I^{\alpha,\beta}\varphi\right)\left(\tau\right)\ \text{exists}\ \forall\ \tau\in[\phi,\delta]\right\} \tag{3.3}$$

and

$$ {}^{\gamma}L_{\alpha}\left(\delta\right)=\left\{\varphi:[\phi,\delta]\to\mathbb{R},\ \left({}^{\gamma}I_{\delta}^{\alpha,\beta}\varphi\right)\left(\tau\right)\ \ \text{exists}\ \forall\ \tau\in[\phi,\delta]\right\}. \tag{3.4}$$

Definition 3.2. We can define for $\alpha\in(0,1]$ and $n=1,2,3,...,$

$$\begin{aligned} C_{\alpha,\phi}^{n}\left([\phi,\delta]\right)&=\left\{f:[\phi,\delta]\to\mathbb{R}\ such\ \ that\ {}_{\phi}^{\gamma}T^{n-1,\alpha}f\in\ {}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)\right\},\\ C_{\alpha,\delta}^{n}\left([\phi,\delta]\right)&=\left\{f:[\phi,\delta]\to\mathbb{R}\ suc\gamma\ \ that\ {}^{\gamma}T_{\delta}^{n-1,\alpha}f\in\ {}_{\alpha}^{\gamma}I\left([\phi,\delta]\right)\right\}.\end{aligned} \tag{3.5}$$

Lemma 3.3. *Let $f\in C_{\alpha,\phi}^{n}\left([\phi,\delta]\right)$ for $\alpha>0$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right)=0$. Then f is expressed as the following form,*

$$\begin{aligned} f\left(\tau\right)=\tfrac{1}{(n-1)!}\int_{\phi}^{\tau}&\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\tfrac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\ &+\textstyle\sum_{s=0}^{n-1}\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{s}\tfrac{1}{s!}{}_{\phi}^{\gamma}T^{s,\alpha}f\left(\phi\right).\end{aligned} \tag{3.6}$$

In this place is $\varphi\left(\theta\right)=\left({}_{\phi}^{\gamma}T^{s,\alpha}f\right)\left(\theta\right).$

Proof. If we take $f\in C_{\alpha,\phi}^{n}\left([\phi,\delta]\right)$, ${}_{\phi}^{\gamma}T^{n-1,\alpha}f\in{}^{\gamma}I_{\alpha}\left([\phi,\delta]\right)$ and φ is continuous function, then we acquire,

$$\begin{aligned} {}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\tau\right)&=\int_{\phi}^{\tau}\tfrac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}+{}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\phi\right)\\ \tfrac{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}{\gamma^{'}(\tau)}\tfrac{d}{d\tau}\left({}_{\phi}^{\gamma}T^{n-2,\alpha}f\left(\tau\right)\right)&=\int_{\phi}^{\tau}\tfrac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}+{}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\phi\right)\\ \tfrac{d}{d\tau}\left({}_{\phi}^{\gamma}T^{n-2,\alpha}f\left(\tau\right)\right)&=\tfrac{\gamma^{'}(\tau)}{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}\left[\int_{\phi}^{\tau}\tfrac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right.\\ &\quad\left.+\tfrac{\gamma^{'}(\tau)}{(\gamma(\tau)-\gamma(\phi))^{1-\alpha}}.{}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\phi\right)\right].\end{aligned} \tag{3.7}$$

If we integrate both of parties (3.7) from ϕ to τ, substituting τ with θ and θ with s on the both side of the equation, then we have

$$\begin{aligned} {}_{\phi}^{\gamma}T^{n-2,\alpha}f\left(\tau\right)&=\int_{\phi}^{\tau}\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]\tfrac{\varphi(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\ &\quad+\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}.{}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\phi\right)+{}_{\phi}^{\gamma}T^{n-2,\alpha}f\left(\phi\right).\end{aligned} \tag{3.8}$$

By applying the equation (3.8) again same method, we get

$$\begin{aligned} {}_{\phi}^{\gamma}T^{n-3,\alpha}f\left(\tau\right)&=\int_{\phi}^{\tau}\tfrac{1}{2}\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2}\tfrac{\varphi(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\ &\quad+\tfrac{1}{2}\left[\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2}.{}_{\phi}^{\gamma}T^{n-1,\alpha}f\left(\phi\right)\\ &\quad+\tfrac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}.{}_{\phi}^{\gamma}T^{n-2,\alpha}f\left(\phi\right)+{}_{\phi}^{\gamma}T^{n-3,\alpha}f\left(\phi\right).\end{aligned} \tag{3.9}$$

By applying the same method iteratively $n-3$ times, then we have,

$$\begin{aligned} f(\tau) = & \frac{1}{(n-1)!}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ & +\sum_{s=0}^{n-1}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{s}\frac{1}{s!}\cdot{}_{\phi}^{\gamma}T^{s,\alpha}f(\phi). \end{aligned} \tag{3.10}$$

For $\varphi(\theta) =_{\phi}^{\gamma} T^{n,\alpha} f(\theta)$. It is evident that an analogous lemma holds for right generalized conformable fractional derivatives. □

Lemma 3.4. *Let $f\in C_{\alpha,\phi}^{n}([\phi,\delta])$ for $\alpha>0$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, f is expressed in form,*

$$\begin{aligned} f(\tau) & = \frac{1}{(n-1)!}\int_{\tau}^{\delta}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}-(\gamma(\delta)-\gamma(\theta))^{\alpha}}{\alpha}\right]^{n-1}\frac{\varphi(\theta)\gamma^{'}(\theta)d\theta}{(\gamma(\delta)-\gamma(\theta))^{1-\alpha}} \\ & +\sum_{s=0}^{n-1}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{s}\frac{(-1)^{s}\cdot{}^{\gamma}T_{\delta}^{s,\alpha}f(\delta)}{s!}. \end{aligned} \tag{3.11}$$

In this place is $\varphi(\theta)=({}^{\gamma}T_{\delta}^{s,\alpha}f)(\theta)$.

Proof. The proof follows a similar structure to *lemma* 3. □

In the *Theorem* 2, we will establish the generalized conformable fractional derivatives within the spaces $C_{\alpha,\phi}^{n}$ and $C_{\alpha,\delta}^{n}$.

Theorem 3.5. *Let $\beta\in\mathbb{C}$, $Re(\beta)>0$ and $n=[\beta]+1$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. The left and right generalized conformable fractional derivative are illustrated in the form for $f\in C_{\alpha,\phi}^{n}$ and $f\in C_{\alpha,\delta}^{n}$. Then, we write*

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) = & \left({}_{\phi}^{\gamma}J^{n-\beta}\left({}_{\phi}^{\gamma}T^{n,\alpha}f\right)\right)(\tau) \\ & +\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} {}^{\gamma}D_{\delta}^{\beta,\alpha}f(\tau) = & \left({}^{\gamma}J_{\delta}^{n-\beta}\left({}^{\gamma}T_{\delta}^{n,\alpha}f\right)\right)(\tau) \\ & +\sum_{m=0}^{n-1}\frac{(-1)^{n}\cdot{}^{\gamma}T_{\delta}^{n,\alpha}f(\delta)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{m-\beta}. \end{aligned} \tag{3.13}$$

Proof. By using $f\in C_{\alpha,\phi}^{n}([\phi,\delta])$, we should select $f(\tau)$ in the *Lemma* 3, substituting τ with θ and θ with s that is as following form

$$\begin{aligned} f(\theta) & = \frac{1}{(n-1)!}\int_{\phi}^{\theta}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}} \\ & +\sum_{m=0}^{n-1}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}\frac{1}{m!}\cdot{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi). \end{aligned} \tag{3.14}$$

In here, we can state the following equality by using (2.9) for (3.14),

(3.15)
$$\begin{aligned}
&{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{\gamma^{'}(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\
&\times\left(\frac{1}{(n-1)!}\int_{\phi}^{\theta}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\right)\frac{\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&+\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\
&\times\left(\sum_{m=0}^{n-1}\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}\frac{1}{m!}.{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)\right)\frac{\gamma^{'}(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.
\end{aligned}$$

By employing techniques such as changing the order of integration, the gamma function, and the beta function, along with the utilization of the following equations

(3.16)
$$(\gamma(\theta)-\gamma(\phi))^{\alpha}=(\gamma(s)-\gamma(\phi))^{\alpha}+z\left[(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}\right]$$

and

$$(\gamma(\theta)-\gamma(\phi))^{\alpha}=u(\gamma(\tau)-\gamma(\phi))^{\alpha}.$$

Then we obtain following form

$$\begin{aligned}
&{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)(n-1)!}\int_{\phi}^{\tau}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(s)\gamma^{'}(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\
&\times\left(\int_{0}^{1}(1-z)^{n-\beta-1}(z)^{n-1}dz\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{2n-\beta-1}\\
&+\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}.{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(n-\beta).m!}\\
&\times\left(\int_{0}^{1}(1-u)^{n-\beta-1}(u)^{m}du\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta+m}.
\end{aligned}\tag{3.17}$$

In here, we obtain by means of the operator ${}_{\phi}^{\gamma}T^{n,\alpha}$,

(3.18)
$$\begin{aligned}
{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)&=\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(s)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{\gamma^{'}(s){}_{\phi}^{n}T^{\alpha}f(s)ds}{(\gamma(s)-\gamma(\phi))^{1-\alpha}}\\
&\quad+\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}.
\end{aligned}$$

We have successfully concluded the proof. The proof of the right generalized conformable fractional derivative can be conducted in a similar manner. □

Theorem 3.6. *We assume that is* $Re(\beta)>m>0$ *for* $m\in\mathbb{N}$*. Furthermore, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$*, with its derivative* γ' *being continuous and* $\gamma(0)=0$*. Then, we have*

$$\begin{aligned}
{}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)&={}_{\phi}^{\gamma}J^{\beta-m,\alpha}f(\tau),\\
{}^{\gamma}T_{\delta}^{m,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f(\tau)\right)&={}^{\gamma}J_{\delta}^{\beta-m,\alpha}f(\tau).
\end{aligned}\tag{3.19}$$

Proof. We have by using (2.7),

(3.20)
$${}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)={}_{\phi}^{\gamma}T^{m,\alpha}\left[\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{\gamma^{'}(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right].$$

By utilizing Leibniz rule for integrals, we get

$$
\begin{aligned}
&{}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)\\
&={}_{\phi}^{\gamma}T^{m-1,\alpha}\left[\frac{1}{\Gamma(\beta-1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-2}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&={}_{\phi}^{\gamma}T^{m-2,\alpha}\left[\frac{1}{\Gamma(\beta-2)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-3}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&\vdots\\
&=\left[\frac{1}{\Gamma(\beta-m)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-m-1}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right]\\
&={}_{\phi}^{\gamma}J^{\beta-m,\alpha}f\left(\tau\right).
\end{aligned}
\tag{3.21}
$$

The poof is successfully completed. The proof of the second formula can be similarly illustrated. □

Corollary 3.6.1. *We will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. If we take $Re(\varsigma)<Re(\beta)$, then we write*

$$
\begin{aligned}
{}_{\phi}^{\gamma}D^{\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)&={}_{\phi}^{\gamma}J^{\beta-\varsigma,\alpha}f\left(\tau\right),\\
{}^{\gamma}D_{\delta}^{\varsigma,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\left(\tau\right)\right)&={}^{\gamma}J_{\delta}^{\beta-\varsigma,\alpha}f\left(\tau\right).
\end{aligned}
\tag{3.22}
$$

Proof. By employing $Theorem$ 1 and $Theorem$ 3, we acquire

$$
\begin{aligned}
{}_{\phi}^{\gamma}D^{\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)&={}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{m-\varsigma,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)\right)\\
&={}_{\phi}^{\gamma}T^{m,\alpha}\left({}_{\phi}^{\gamma}J^{\beta+m-\varsigma,\alpha}f\left(\tau\right)\right)\\
&={}_{\phi}^{\gamma}J^{\beta-\varsigma,\alpha}f\left(\tau\right).
\end{aligned}
\tag{3.23}
$$

The poof is successfully completed. The proof of the second formula can be similarly illustrated. □

Theorem 3.7. *Let $\beta>0$ and $f\in C_{\alpha,\phi}^{n}\left[\phi,\delta\right]\left(f\in C_{\alpha,\delta}^{n}\left[\phi,\delta\right]\right)$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, we have*

$$
\begin{aligned}
{}_{\phi}^{\gamma}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f\left(\tau\right)\right)&=f\left(\tau\right),\\
{}^{\gamma}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\left(\tau\right)\right)&=f\left(\tau\right).
\end{aligned}
\tag{3.24}
$$

Proof. If we possess by using (2.7) and (2.9) , then we have

$$
\begin{aligned}
&{}_{\phi}^{\gamma}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)\right)\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{\phi}^{\tau}\int_{\phi}^{\theta}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1}\\
&\times\left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)[\alpha]^{n-2}}\int_{\phi}^{\tau}\int_{\phi}^{\theta}\left[(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}\right]^{n-\beta-1}\\
&\times\left[(\gamma(\theta)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}\right]^{\beta-1}\frac{\gamma'(u)f(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\frac{\gamma'(\theta)f(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{\phi}^{\tau}\frac{f(u)\gamma'(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\\
&\times\left(\int_{0}^{1}(1-y)^{n-\beta-1}(y)^{\beta-1}dy\right)\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\\
&=\frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\frac{\Gamma(n-\beta)\Gamma(\beta)}{\Gamma(n)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(u)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-1}\frac{f(u)\gamma'(u)du}{(\gamma(u)-\gamma(\phi))^{1-\alpha}}\\
&={}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n,\alpha}f\right)(\tau)\\
&=f(\tau).
\end{aligned}
\tag{3.25}
$$

We complete the proof . □

Theorem 3.8. *Let $Re(\beta)>0,\ n=[Re(\beta)],\ f\in X_{\gamma}$ and ${}_{\phi}^{\gamma}J^{\beta,\alpha}f\in C_{\alpha,\phi}^{n}[\phi,\delta]\left({}^{\gamma}J_{\delta}^{\beta,\alpha}f\in C_{\alpha,\delta}^{n}[\phi,\delta]\right)$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma(0)=0$. Then, we have*

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\right)=f(\tau)-\sum_{j=0}^{n}\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f(\phi)}{\Gamma(\beta-j+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j}
\tag{3.26}
$$

and

$$
{}^{\gamma}J_{\delta}^{\beta,\alpha}\left({}^{\gamma}D_{\delta}^{\beta,\alpha}f(\tau)\right)=f(\tau)-\sum_{j=0}^{n}\frac{(-1)^{n-j}.{}^{\gamma}D_{\delta}^{\beta-j,\alpha}f(\delta)}{\Gamma(\beta-j+1)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{\beta-j}.
\tag{3.27}
$$

Proof. We can write by using (2.7) and (2.9) ,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\right)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-1}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.
\tag{3.28}
$$

By using the integration by parts once, we get

$$
\begin{aligned}
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau)\right)&=\frac{{}_{\phi}^{\gamma}T^{1,\alpha}}{\Gamma(\beta+1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta}\frac{{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&\quad-\frac{1}{\Gamma(\beta+1)}.{}_{\phi}^{\gamma}T^{n,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right).\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta}.
\end{aligned}
\tag{3.29}
$$

By using the integration by parts $n-$times, we obtain

(3.30)

$$\begin{aligned}
&{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}D^{\beta,\alpha}f\left(\tau\right)\right)\\
&=\frac{{}_{\phi}^{\gamma}T^{1,\alpha}}{\Gamma(\beta-n+1)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-n}\frac{\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\theta)\right)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\\
&-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\\
&={}_{\phi}^{\gamma}T^{1,\alpha}\left[{}_{\phi}^{\gamma}J^{\beta-n+1,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f\left(\tau\right)\right)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\right]\\
&={}_{\phi}^{\gamma}T^{1,\alpha}\left[\left({}_{\phi}^{\gamma}J^{1,\alpha}f\left(\tau\right)\right)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}T^{n-j,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\phi)\right)}{\Gamma(\beta+2-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j+1}\right]\\
&=f\left(\tau\right)-\sum_{j=1}^{n}\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f(\phi)}{\Gamma(\beta+1-j)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j}.
\end{aligned}$$

The proof is successfully completed. The proof of the second formula can be illustrated in a similar fashion. □

4. GENERALIZED CONFORMABLE FRACTIONAL DERIVATIVES WITHIN CAPUTO FRAMEWORK

In this section, we will introduce several definitions relevant to the theorem, while also elucidating some properties of the generalized conformable derivative within the Caputo setting.

Definition 4.1. Let $\alpha > 0$, $Re\left(\beta\right) \geq 0$ and $n = [Re\left(\beta\right)] + 1$. Moreover, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right) = 0$. If we get $f \in C^{n}_{\alpha,\phi}\left(f \in C^{n}_{\alpha,\delta}\right)$, then, we acquire the left and right generalized Caputo conformable fractional derivatives, respectively.

(4.1)

$$\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f\left(\tau\right)\right)={}_{\phi}^{\gamma}D^{\beta,\alpha}\left[f\left(\theta\right)-\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f\left(\phi\right)}{m!}\left(\frac{\left(\gamma\left(\theta\right)-\gamma\left(\phi\right)\right)^{\alpha}}{\alpha}\right)^{m}\right]\left(\tau\right)$$

and

(4.2)

$$\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f\left(\tau\right)\right)={}^{\gamma}D_{\delta}^{\beta,\alpha}\left[f\left(\theta\right)-\sum_{m=0}^{n-1}\frac{\left(-1\right)^{m}.{}^{\gamma}T_{\delta}^{m,\alpha}f\left(\delta\right)}{m!}\left(\frac{\left(\gamma\left(\delta\right)-\gamma\left(\theta\right)\right)^{\alpha}}{\alpha}\right)^{m}\right]\left(\tau\right).$$

Theorem 4.2. *Let $Re\left(\beta\right) \geq 0$ and $n = [Re\left(\beta\right)] + 1$. Furthermore, we will consider γ as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative γ' being continuous and $\gamma\left(0\right) = 0$. If we take $f \in C^{n}_{\alpha,\phi}\left(f \in C^{n}_{\alpha,\delta}\right)$, then we obtain the left and right generalized Caputo fractional conformable derivatives in Caputo setting, respectively.*

$$\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f\left(\tau\right)\right)={}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f\left(\tau\right)\right)\tag{4.3}$$

and

$$\left({}^{\gamma,C}D_{\delta}^{\beta,\alpha}f\left(\tau\right)\right)={}^{\gamma}J_{\delta}^{n-\beta,\alpha}\left({}^{\gamma}T_{\delta}^{n,\alpha}f\left(\tau\right)\right).\tag{4.4}$$

Proof. By considering $Definition$ 5, we possess

$$\begin{aligned}(4.5)\quad &\left({}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\tau)\right)\\ &={}^{\gamma}_{\phi}D^{\beta,\alpha}\left[f(\theta)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{m!}\left(\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{m}\right](\tau)\\ &={}^{\gamma}_{\phi}D^{\beta,\alpha}f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{m!}\frac{{}^{\gamma}_{\phi}T^{n,\alpha}}{\Gamma(n-\beta)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta+m}\frac{\Gamma(n-\beta)\Gamma(m+1)}{\Gamma(n-\beta+m+1)}\\ &={}^{\gamma}_{\phi}D^{\beta,\alpha}f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}_{\phi}T^{m,\alpha}f(\phi)}{\Gamma(m-\beta+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}.\end{aligned}$$

The proof is done. □

Lemma 4.3. *Let* $\alpha>0$, $Re(\beta)\geq 0$, $n=[Re(\beta)]+1$ *and* $Re(\beta)\notin\mathbb{N}$. *Moreover, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *If* $f\in C^{n}_{\alpha,\phi}([\phi,\delta])\,(f\in C^{n}_{\alpha,\delta}([\phi,\delta]))$ *then we have*

$$(4.6)\qquad \left.\begin{aligned}&{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\phi)=0,\\ &{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\delta)=0\end{aligned}\right\}\textit{for } s=0,1,...,n-1.$$

Proof. We hold

$$\begin{aligned}(4.7)\quad {}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\tau)&={}^{\gamma}_{\phi}D^{s,\alpha}\left({}^{\gamma}_{\phi}J^{\beta,\alpha}f(\tau)\right)\\ &=\frac{1}{\Gamma(\beta-s)}\int_{\phi}^{\tau}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-s-1}\frac{f(\theta)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}.\end{aligned}$$

In here, we can express through Hölder's inequality,

$$\begin{aligned}(4.8)\quad &\left|{}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\tau)\right|\\ &\leq\frac{1}{\Gamma(\beta-s)}\left(\int_{\phi}^{\tau}|f(\theta)|^{p}\gamma'(\theta)\right)^{\frac{1}{p}}\left(\int_{\phi}^{\tau}\left(\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-s-1}\frac{f(\theta)\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}\right)^{q}\right)^{\frac{1}{q}}\\ &\leq\frac{\|f\|_{\tau_{\gamma}}}{(re(\beta)-s)\Gamma(\beta-s)}\left(\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{(re(\beta)-s)}.\end{aligned}$$

For $\tau=\phi$, we say that

$$(4.9)\qquad {}^{\gamma}_{\phi}J^{\beta-s,\alpha}f(\phi)=0.$$

The proof is done. □

Lemma 4.4. *Let* $\alpha>0$, $Re(\beta)\geq 0$ *and* $n=[Re(\beta)]+1$. *Furthermore, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *If we get* ${}^{\gamma}_{\phi}T^{n,\alpha}\in C^{n}_{\alpha,\phi}[\phi,\delta]$ $\left({}^{\gamma}T^{n,\alpha}_{\delta}\in C^{n}_{\alpha,\delta}[\phi,\delta]\right)$, *then we obtain*

$$(4.10)\qquad \begin{aligned}&{}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\phi)=0,\\ &{}^{\gamma,C}D^{\beta,\alpha}_{\delta}f(\delta)=0.\end{aligned}$$

Proof. It is clearly seen that

$$(4.11)\qquad \left|{}^{\gamma,C}_{\phi}D^{\beta,\alpha}f(\tau)\right|\leq\frac{\left\|{}^{\gamma}_{\phi}T^{n,\alpha}\right\|_{X_{\gamma}}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{(n-re(\beta))}$$

and

$$\left| {}^{\gamma,C}D_{\delta}^{\beta,\alpha} f(\tau) \right| \leq \frac{\left\| {}^{\gamma}T_{\delta}^{n,\alpha} \right\|_{X_{\gamma}}}{(n-re(\beta))\Gamma(n-\beta)} \left(\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha} \right)^{(n-re(\beta))}. \tag{4.12}$$

The proof is done. □

Theorem 4.5. *Let* $Re(\beta) \geq 0$, $n = [Re(\beta)]+1$ *and* $f \in C_{\alpha,\phi}^{n}[\phi,\delta]$ ($f \in C_{\alpha,\delta}^{n}[\phi,\delta]$). *Moreover, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0) = 0$. *We can say that*

(1) If we take $Re(\beta) \notin \mathbb{N}$ *or* $\beta \in \mathbb{N}$, *then we acquire*

$$\begin{aligned} {}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha} f(\tau)\right) &= f(\tau), \\ {}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha} f(\tau)\right) &= f(\tau). \end{aligned} \tag{4.13}$$

(2) If we take $Re(\beta) \neq 0$ *or* $Re(\beta) \in \mathbb{N}$, *then we get*

$$\begin{aligned} {}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha} f(\tau)\right) &= f(\tau) - \frac{{}_{\phi}^{\gamma}J^{\beta-n+1,\alpha} f(\phi)}{\Gamma(n-\beta)} \left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta}, \\ {}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma}J_{\delta}^{\beta,\alpha} f(\tau)\right) &= f(\tau) - \frac{{}^{\gamma}J_{\delta}^{\beta-n+1,\alpha} f(\phi)}{\Gamma(n-\beta)} \left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{n-\beta}. \end{aligned} \tag{4.14}$$

Proof. By using *Definition* 6, we have,

$$\begin{aligned} &{}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha} f(\tau)\right) \\ &= f(\tau) - \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \int_{\phi}^{\tau} \left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1} \\ &\times \left(\sum_{m=0}^{n-1} \frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha} f(\phi)}{m!} \left(\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right)^{m}\right) \frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}} \\ &= f(\tau) - \sum_{m=0}^{n-1} \frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha} f(\phi)}{m!} \cdot \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \\ &\times \int_{\phi}^{\tau} \left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}-(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{n-\beta-1} \left[\frac{(\gamma(\theta)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m} \frac{\gamma'(\theta)d\theta}{(\gamma(\theta)-\gamma(\phi))^{1-\alpha}}. \end{aligned} \tag{4.15}$$

In here, by using the following the change of variable,

$$(\gamma(\theta)-\gamma(\phi))^{\alpha} = z(\gamma(\tau)-\gamma(\phi))^{\alpha} \tag{4.16}$$

we can hold

$${}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{\beta,\alpha} f(\tau)\right) = f(\tau) - \sum_{m=0}^{n-1} \frac{{}_{\phi}^{\gamma}J^{m-\beta,\alpha} f(\phi)}{\Gamma(m-\beta+1)} \cdot \left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m-\beta}. \tag{4.17}$$

In here, we establish ${}_{\phi}^{\gamma}J^{\beta-s,\alpha} f(\phi) = 0$ and ${}^{\gamma}J_{\delta}^{\beta-s,\alpha} f(\delta) = 0$ for $Re(\beta) \notin \mathbb{N}$ by using *Lemma* 4. The case $\beta \in \mathbb{N}$ is inconsequential. Additionaly, if $Re(\beta) \in \mathbb{N}$, then we assert that ${}_{\phi}^{\gamma}J^{\beta-s,\alpha} f(\phi) = 0$ and ${}^{\gamma}J_{\delta}^{\beta-s,\alpha} f(\delta) = 0$ for $s = 0, 1, ..., n-2$ by using *Lemma* 4. □

Theorem 4.6. *Let* $\beta \in \mathbb{C}$ *and* $f \in C_{\alpha,\phi}^{n}[\phi,\delta]$ $\left(f \in C_{\alpha,\delta}^{n}[\phi,\delta]\right)$. *Furthermore, we will consider* γ *as a monotonically increasing and positive function defined on the* *interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *We have*

$$
\begin{aligned}
&{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right)=f(\tau)-\sum_{m=0}^{n-1}\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)}{\Gamma(m+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m},\\
&{}^{\gamma}J_{\delta}^{\beta,\alpha}\left({}^{\gamma,C}D_{\delta}^{\beta,\alpha}f(\tau)\right)=f(\tau)-\sum_{m=0}^{n-1}\frac{{}^{\gamma}T_{\delta}^{m,\alpha}f(\delta)}{\Gamma(m+1)}\left[\frac{(\gamma(\delta)-\gamma(\tau))^{\alpha}}{\alpha}\right]^{m}.
\end{aligned}
\tag{4.18}
$$

Proof. In here, we can write the following as,

$$
\begin{aligned}
{}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\beta,\alpha}f(\tau)\right)&={}_{\phi}^{\gamma}J^{\beta,\alpha}\left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f(\tau)\right)\right)\\
&={}_{\phi}^{\gamma}J^{n,\alpha}\left({}_{\phi}^{\gamma}T^{n,\alpha}f(\tau)\right)\\
&=f(\tau)-\frac{{}_{\phi}^{\gamma}D^{\beta-j,\alpha}f(\phi)}{\Gamma(\beta-j+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{\beta-j}\\
&=f(\tau)-\frac{{}_{\phi}^{\gamma}T^{m,\alpha}f(\phi)}{\Gamma(m+1)}\left[\frac{(\gamma(\tau)-\gamma(\phi))^{\alpha}}{\alpha}\right]^{m}.
\end{aligned}
\tag{4.19}
$$

The proof is done. □

Theorem 4.7. *Let* $f\in C_{\alpha,\phi}^{p+r}[\phi,\delta]\left(f\in C_{\alpha,\delta}^{p+r}[\phi,\delta]\right)$, $Re(\beta)\geq 0$, $Re(\mu)\geq 0$, $r-1<[Re(\beta)]\leq r$ *and* $p-1<[Re(\beta)]\leq p$. *Moreover, we will consider* γ *as a monotonically increasing and positive function defined on the interval* $[0,\infty)$, *with its derivative* γ' *being continuous and* $\gamma(0)=0$. *Then, we write*

$$
\begin{aligned}
&{}_{\phi}^{\gamma,C}D^{\beta,\alpha}\left({}_{\phi}^{\gamma,C}D^{\mu,\alpha}(f(\tau))\right)={}_{\phi}^{\gamma,C}D^{\beta+\mu,\alpha}f(\tau),\\
&{}^{\gamma,C}D_{\delta}^{\beta,\alpha}\left({}^{\gamma,C}D_{\delta}^{\mu,\alpha}(f(\tau))\right)={}^{\gamma,C}D_{\delta}^{\beta+\mu,\alpha}f(\tau).
\end{aligned}
\tag{4.20}
$$

Proof. The proof can be successfully completed by using $Theorem$ 1, $Theorem$ 4, $Theorem$ 6 and $Lemma$ 5. □

5. FRACTIONAL INTEGRALS CLASS

1. Taking $\gamma(\tau)=\tau$ in $Definition$ 2,

$$
{}_{\phi}^{\beta}J^{\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left[\frac{(\tau-\phi)^{\alpha}-(\theta-\phi)^{\alpha}}{\alpha}\right]^{\beta-1}\frac{f(\theta)d\theta}{(\theta-\phi)^{1-\alpha}}.
$$

We acquire the left fractional conformable integrals in [1].

2. Taking $\gamma(\tau)=\tau$ and $\alpha=1$ in $Definition$ 2,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.
$$

We obtain the left Riemann-Lioville fractional integrals.

3. Taking $\gamma(\tau)=\tau$, $\alpha=1$ and $\phi=-\infty$ in $Definition$ 2,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{-\infty}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.
$$

We get the left Liouville fractional integrals.

4. Taking $\gamma(\tau)=\tau$, $\phi=0$ and $\alpha=1$ in $Definition$ 2,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{0}^{\tau}(\tau-\theta)^{\beta-1}f(\theta)\,d\theta.
$$

We have the left Riemann fractional integrals.

5. Taking $\gamma(\tau)=ln\tau$ and $\alpha=1$ in $Definition$ 2,

$$
{}_{\phi}^{\gamma}J^{\beta,\alpha}f(\tau)=\frac{1}{\Gamma(\beta)}\int_{\phi}^{\tau}\left(\ln\frac{\tau}{\theta}\right)^{\beta-1}\frac{f(\theta)}{\theta}d\theta.
$$

We achieve the left Hadamard fractional integrals [11].

6. Taking $\gamma(\tau) = \tau^m$, $g(\tau) = \tau^{m\eta} f(\tau)$ and $\alpha = 1$ in $Definition$ 2,

$$\tau^{-m(\beta+\eta)} \cdot {}_{\phi}^{\gamma}J^{\beta,\alpha} g(\tau) = \frac{m\tau^{-m(\beta+\eta)}}{\Gamma(\beta)} \int_{\phi}^{\tau} \theta^{m\eta+m-1} (\tau^m - \theta^m)^{\beta-1} f(\theta)\, d\theta.$$

We acquire the left Erdélyi-Kober fractional integrals.

7. Taking $\gamma(\tau) = \tau^m$, $g(\tau) = \tau^{m\eta} f(\tau)$, $\phi = 0$ and $\alpha = 1$ in $Definition$ 2,

$$\tau^{-m(\beta+\eta)} \cdot {}_{0}^{\gamma}J^{\beta,\alpha} g(\tau) = \frac{m\tau^{-m(\beta+\eta)}}{\Gamma(\beta)} \int_{o}^{\tau} \theta^{m\eta+m-1} (\tau^m - \theta^m)^{\beta-1} f(\theta)\, d\theta.$$

We obtain the left Erdélyi fractional integrals.

8. Taking $\gamma(\tau) = \tau$, $g(\tau) = \tau^{\eta} f(\tau)$, $\phi = 0$ and $\alpha = 1$ in $Definition$ 2,

$$\tau^{-(\beta+\eta)} \cdot {}_{0}^{\gamma}J^{\beta,\alpha} g(\tau) = \frac{\tau^{-(\beta+\eta)}}{\Gamma(\beta)} \int_{o}^{\tau} \theta^{\eta} (\tau - \theta)^{\beta-1} f(\theta)\, d\theta.$$

We get the left Kober fractional integrals.

9. Taking $\gamma(\tau) = \tau^m$, $g(\tau) = \tau^{m\eta} f(\tau)$ and $\alpha = 1$ in $Definition$ 2,

$$\frac{\tau^K}{m^\beta} \cdot {}_{\phi}^{\gamma}J^{\beta,\alpha} g(\tau) = \frac{\tau^K m^{1-\beta}}{\Gamma(\beta)} \int_{\phi}^{\tau} \theta^{m\eta+m-1} (\tau^m - \theta^m)^{\beta-1} f(\theta)\, d\theta.$$

We have the left generalized fractional integrals that unify another six fractional integrals.

10. Taking $\gamma(\tau) = \tau^m$ and $\alpha = 1$ in $Definition$ 2,

$$\frac{1}{m^\beta} \cdot {}_{\phi}^{\gamma}J^{\beta,\alpha} f(\tau) = \frac{m^{1-\beta}}{\Gamma(\beta)} \int_{\phi}^{\tau} \theta^{m-1} (\tau^m - \theta^m)^{\beta-1} f(\theta)\, d\theta.$$

We achieve the left Katugampola fractional integrals.

6. FRACTIONAL DERIVATIVES CLASS

1. Taking $\phi = 0$, $\alpha = 1$ and $\gamma(\tau) = \tau$ in $Definition$ 3, we acquire Riemann-liouville fractional derivative

$$\begin{aligned} {}_{0}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{0}^{\gamma}T^{n} \left({}_{0}^{\gamma}J^{n-\beta,\alpha}\right) f(\tau) \\ &= \frac{{}_{0}^{\gamma}T^{n}}{\Gamma(n-\beta)} \int_{0}^{\tau} [\tau - \theta]^{n-\beta-1} f(\theta)\, d\theta. \end{aligned}$$

2. Taking $\phi = 0$ and $\alpha = 1$ in $Definition$ 3, we obtain the $\gamma-$Riemann-liouville fractional derivative

$$\begin{aligned} {}_{0}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{0}^{\gamma}T^{n} \left({}_{0}^{\gamma}J^{n-\beta,\alpha}\right) f(\tau) \\ &= \frac{{}_{0}^{\gamma}T^{n}}{\Gamma(n-\beta)} \int_{0}^{\tau} [\gamma(\tau) - \gamma(\theta)]^{n-\beta-1} \gamma'(\theta) f(\theta)\, d\theta. \end{aligned}$$

3. Taking $\gamma(\tau) = \tau$ and $\alpha = 1$ in $Definition$ 3, we get the Caputo fractional derivative

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{\phi}^{\gamma}J^{n-\beta,\alpha} \left({}_{\phi}^{\gamma}T^{n}\right) f(\tau) \\ &= \frac{1}{\Gamma(n-\beta)} \int_{\phi}^{\tau} [\tau - \theta]^{n-\beta-1} \left({}_{\phi}^{\gamma}T^{n}\right) f(\theta)\, d\theta. \end{aligned}$$

4. Taking $\alpha = 1$,in $Definition$ 3 we have the left $\gamma-$Caputo fractional derivatives

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{\phi}^{\gamma}J^{n-\beta,\alpha} \left({}_{\phi}^{\gamma}T^{n}\right) f(\tau) \\ &= \frac{1}{\Gamma(n-\beta)} \int_{\phi}^{\tau} [\gamma(\tau) - \gamma(\theta)]^{n-\beta-1} \gamma'(\theta) {}_{\phi}^{\gamma}T^{n} f(\theta)\, d\theta. \end{aligned}$$

5. Taking $\gamma(\tau) = \tau$ in $Definition$ 3, we achieve the left fractional conformable derivatives in [1] ,

$$\begin{aligned} {}_{\phi}^{\gamma}D^{\beta,\alpha} f(\tau) &= {}_{\phi}^{\gamma}T^{n,\alpha} \left({}_{\phi}^{\gamma}J^{n-\beta,\alpha}\right) f(\tau) \\ &= \frac{{}_{\phi}^{\gamma}T^{n,\alpha}}{\Gamma(n-\beta)} \int_{\phi}^{\tau} \left[\frac{(\tau-\phi)^{\alpha} - (\theta-\phi)^{\alpha}}{\alpha}\right]^{n-\beta-1} \frac{f(\theta)d\theta}{(\theta-\phi)^{1-\alpha}}. \end{aligned}$$

6. Taking $\gamma(\tau)=\tau^{\rho}$ $\beta=0$ and $\alpha=1$,in $Definition$ 3, we acquire the Katugampola fractional derivative

$$\rho^{\beta}\cdot{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\rho^{\beta}\left(\frac{1}{\rho\tau^{\rho-1}}\frac{d}{d\tau}\right)^{n}\cdot{}_{\phi}^{\gamma}J^{n-\beta,\alpha}f(\tau).$$

7. Taking $\gamma(\tau)=\tau$ and $\alpha=1$ in $Definition$ 3, we obtain the Riemann-Liouville fractional derivative

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\left(\frac{d}{d\tau}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}[\tau-\theta]^{n-\beta-1}f(\theta)\,d\theta.$$

8. Taking $\gamma(\tau)=\tau^{\rho}$ and $\alpha=1$ in $definition$ 3, we get the Caputo$-$Katugampola fractional derivative

$$\rho^{\beta}\cdot{}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\rho^{\beta}\cdot{}_{\phi}^{\gamma}J^{n-\beta,\alpha}\left(\frac{1}{\rho\tau^{\rho-1}}\frac{d}{d\tau}\right)^{n}f(\tau).$$

9. Taking $\gamma(\tau)=\ln\tau$ and $\alpha=1$ in $Definition$ 3, we have the Caputo$-$Hadamard fractional derivative in $[12-14]$

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\ln\frac{\tau}{\theta}\right]^{n-\beta-1}\left(\theta\frac{d}{d\theta}\right)^{n}f(\theta)\,\frac{d\theta}{\theta}.$$

10. Taking $\gamma(\tau)=\ln\tau$ and $\alpha=1$ in $Definition$ 3, we achiev the hadamard fractional derivative

$${}_{\phi}^{\gamma}D^{\beta,\alpha}f(\tau) \quad =\left(\tau\frac{d}{d\tau}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{\phi}^{\tau}\left[\ln\frac{\tau}{\theta}\right]^{n-\beta-1}f(\theta)\,\frac{d\theta}{\theta}.$$

7. CONCLUSION

In this study, we introduced the left and right generalized conformable fractional integrals and derivatives. We explored significant implications and fundamental properties of these operators. Additionally, we derived the generalized conformable fractional derivatives within the Caputo framework. Ultimately, we presented classical consequences in the context of generalized conformable derivatives and integrals.

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