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Spectral Characteristics of the Sturm-Liouville Problem with Spectral Parameter-Dependent Boundary Conditions

Aynur Çöl[1]

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Abstract — We consider the Sturm-Liouville problem on the half line ($0 \leq x < \infty$), where the boundary conditions contain polynomials of the spectral parameter. We define the scattering function and present the spectrum of the boundary value problem. The continuity of the scattering function is discussed. In a special case, the Levinson-type formula is introduced, demonstrating that the increment of the scattering function's logarithm is related to the number of eigenvalues.

Keywords *Levinson-type formula, scattering function, spectral parameter dependent boundary condition*

Mathematics Subject Classification (2020) 34L25, 81U40

1. Introduction

Consider the boundary value problem

$$\ell v := -v'' + \varphi(x)v = \lambda^2 v,\ 0 \leq x < \infty \tag{1.1}$$

$$(\beta_3 v(0) - \alpha_3 v'(0))i\lambda^3 + (\beta_2 v(0) - \alpha_2 v'(0))\lambda^2 - (\beta_1 v(0) - \alpha_1 v'(0))i\lambda - \beta_0 v(0) + \alpha_0 v'(0) = 0 \tag{1.2}$$

known as a Sturm Liouville problem, where λ is a spectral parameter, the potential function $\varphi(x)$ is real valued such that

$$\int_0^\infty (1+x)\left|\varphi(x)\right| dx < \infty \tag{1.3}$$

and for $\alpha_i,\ \beta_i \in \mathbb{R},\ i = \overline{0,3},\ \alpha_3 \neq 0$, and $\beta_3 \neq 0$,

$$(-1)^k \delta_{ik} \leq 0,\ k \in \{1,2\};\ \delta_{ik} = 0,\ k = 3 \text{ where } \delta_{ik} = \alpha_{i+k}\beta_i - \alpha_i\beta_{i+k} \tag{1.4}$$

An important part of scattering theory is the study of boundary value problems involving the spectral parameter. Sturm-Liouville problems with spectral parameter-dependent boundary conditions arise in studies of heat conduction problems and vibrating string problems. Cohen introduced a method to solve an initial-boundary value problem arising in the diffusion and heat flow theory [1]. Various examples of spectral problems that occur in mechanical engineering and contain an eigenparameter in the boundary conditions were presented in [2]. Moreover, problems with boundary conditions concerning spectral parameters were investigated in finite intervals [3–10] and on the half line in [11–14].

[1]aynurcol@sinop.edu.tr (Corresponding Author)
[1]Department of Mathematics and Science Education, Faculty of Education, Sinop University, Sinop, Türkiye

Levinson's theorem provides a relation between the number of bound states of a quantum mechanical system and the phase shift of that system [15–17]. It is a fundamental tool in quantum mechanics and scattering theory, as it is responsible for solving the inverse scattering problem [18]. In work [19], the Levinson formula was obtained for Sturm Liouville operator, which is not only a necessary condition but also sufficient for the given collection $\{S(\lambda);\ \lambda_j;\ m_j\ (j=\overline{1,n})\}$ to be scattering data of the reconstructed equation. The Levinson-type formulas for boundary conditions containing a spectral parameter were studied in [20–22].

The paper aims to analyze the spectral characteristics of the Sturm-Liouville problem with a nonlinear spectral parameter in the boundary condition. In progress, we provide the scattering function and the spectrum of the boundary value problem (1.1) and (1.2), and present the relation between the number of eigenvalues and the argument's variation of the scattering function. This relation is referred to as the Levinson-type formula.

The remaining paper is structured as follows: Section 2 presents the scattering function and the spectrum for (1.1) and (1.2). Section 3 investigates the scattering function's continuity. Finally, section 4 derives the Levinson-type formula.

2. The Scattering Function $S(\lambda)$ and the Discrete Spectrum

Let (1.3) hold. Then, as known in [19], there exists a unique solution $e(\lambda,x)$ of (1.1) which holds the asymptotic behavior $\lim\limits_{x\to+\infty} e^{-i\lambda x}e(\lambda,x)=1$, for $\Im\geq 0$, and can be expressed as

$$e(\lambda,x)=e^{i\lambda x}+\int_x^{\infty}K(x,t)\,e^{i\lambda t}dt \tag{2.1}$$

called Jost solution. The function $e(\lambda,x)$ is analytic with respect to λ in the upper-half plane ($\Im>0$) and continuous on the real line. Moreover, the kernel function $K(x,t)$ is related as follows:

$$K(x,x)=\frac{1}{2}\int_x^{\infty}\varphi(\zeta)\,d\zeta$$

Let $\psi(\lambda,x)$ represent the solution of (1.1) with the conditions:

$$\psi(\lambda,0)=\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3 \quad \text{and} \quad \psi'(\lambda,0)=\beta_0+i\beta_1\lambda-\beta_2\lambda^2-i\beta_3\lambda^3$$

It is obvious that the solution $\psi(\lambda,x)$ holds (1.2).

Let $W[y;z]:=y'z-yz'$ denote the Wronskian. For any solutions $e(\lambda,x)$ and $e(-\lambda,x)$ of (1.1), the Wronskian $W[e(\lambda,x);e(-\lambda,x)]$ is independent of x and is equal to $2i\lambda$. Therefore, for all $\lambda\in\mathbb{R}\backslash\{0\}$, $e(\lambda,x)$ and $e(-\lambda,x)$ constitute a fundamental set of solutions of (1.1), and any solution $\psi(\lambda,x)$ of (1.1) can be expressed as

$$\psi(\lambda,x)=e(\lambda,x)\,\gamma_1(\lambda)+e(-\lambda,x)\,\gamma_2(\lambda) \tag{2.2}$$

By evaluating the following Wronskians of $e(\lambda,x)$ and $\psi(\lambda,x)$,

$$W[e(\lambda,x),\psi(\lambda,x)]=\gamma_2(\lambda)2i\lambda=\psi(\lambda,0)e'(\lambda,0)-\psi'(\lambda,0)e(\lambda,0)$$

and

$$W[e(-\lambda,x),\psi(\lambda,x)]=-\gamma_1(\lambda)2i\lambda=\psi(\lambda,0)e'(-\lambda,0)-\psi'(\lambda,0)e(-\lambda,0)$$

we find $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ and substitute in (2.2). Let $\Theta(\lambda)$ and $\Theta_1(\lambda)$ be functions such that

$$\Theta(\lambda)=\psi(\lambda,0)e'(\lambda,0)-\psi'(\lambda,0)e(\lambda,0) \tag{2.3}$$

and

$$\Theta_1(\lambda) = \psi(\lambda, 0)e'(-\lambda, 0) - \psi'(\lambda, 0)e(-\lambda, 0) \tag{2.4}$$

Therefore, we obtain the solution of (1.1) with (1.2) such that

$$\psi(\lambda, x) = (2i\lambda)^{-1}\left[-\Theta_1(\lambda)e(\lambda, x) + \Theta(\lambda)e(-\lambda, x)\right] \tag{2.5}$$

Define the function

$$S(\lambda) = \Theta_1(\lambda)[\Theta(\lambda)]^{-1} \tag{2.6}$$

called the scattering function of (1.1) and (1.2).

We state some properties of $S(\lambda)$. Show that $\Theta(\lambda) \neq 0$, for all $\lambda \in \mathbb{R}\backslash\{0\}$. Assuming the contrary, then there exists a $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$, such that

$$(\alpha_0 + i\alpha_1\lambda_0 - \alpha_2\lambda_0^2 - i\alpha_3\lambda_0^3)e'(\lambda_0, 0) = (\beta_0 + i\beta_1\lambda_0 - \beta_2\lambda_0^2 - i\beta_3\lambda_0^3)e(\lambda_0, 0)$$

Besides,

$$\begin{aligned}
2i\lambda_0 &= W[e(\lambda_0, 0), \overline{e(\lambda_0, 0)}] \\
&= e'(\lambda_0, 0)\overline{e(\lambda_0, 0)} - e(\lambda_0, 0)\overline{e'(\lambda_0, 0)} \\
&= |e(\lambda_0, 0)|^2\, 2i\Im\left(\frac{\beta_0 + i\beta_1\lambda_0 - \beta_2\lambda_0^2 - i\beta_3\lambda_0^3}{\alpha_0 + i\alpha_1\lambda_0 - \alpha_2\lambda_0^2 - i\alpha_3\lambda_0^3}\right)
\end{aligned}$$

From the result,

$$\frac{|e(\lambda_0, 0)|^2\left[\alpha_1\beta_0 - \alpha_0\beta_1 + (\alpha_2\beta_1 - \alpha_1\beta_2)|\lambda_0|^2 + (\alpha_3\beta_2 - \alpha_2\beta_3)|\lambda_0|^4\right]}{|\alpha_0 + i\alpha_1\lambda_0 - \alpha_2\lambda_0^2 - i\alpha_3\lambda_0^3|^2} = -1$$

This is a contradiction since the left hand is positive, which proves the claim.

Therefore, firstly, $S(\lambda)$ is defined on $(-\infty, 0)$ and $(0, \infty)$, and secondly, it is continuous in these intervals, which can be observed from the definition of $\Theta(\lambda)$. In section 3, the continuity of $S(\lambda)$ at $\lambda = 0$ is investigated. Next, $\Theta(\lambda)$ is analytic function of λ since $e(\lambda, 0)$ and $e'(\lambda, 0)$ are analytic in the upper half plane.

From the definition of $S(\lambda)$, we derive that the function $-1 - S(\lambda)$ belongs to the space $L_2(-\infty, \infty)$. Using (2.1) and substituting related expressions into the $\Theta(\lambda)$,

$$\begin{aligned}
\Theta(\lambda) &= (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)\left\{i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt\right\} \\
&\quad -(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)\left\{1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right\} \\
&= (i\lambda)^4\left[\alpha_3 + O\left(\frac{1}{\lambda}\right)\right]
\end{aligned} \tag{2.7}$$

as $|\lambda| \to \infty$. Similarly,

$$\begin{aligned}
\Theta_1(\lambda) &= (\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)\left\{-i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{-i\lambda t}dt\right\} \\
&\quad -(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)\left\{1 + \int_0^\infty K(0,t)e^{-i\lambda t}dt\right\} \\
&= (i\lambda)^4\left[-\alpha_3 + O\left(\frac{1}{\lambda}\right)\right]
\end{aligned} \tag{2.8}$$

Then, the following result is obtained:

$$-1 - S(\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty \tag{2.9}$$

Therefore, $-1 - S(\lambda) \in L_2(-\infty, \infty)$.

Lemma 2.1. For all $\lambda \in \mathbb{R}\backslash\{0\}$,

$$S(\lambda) = \overline{S(-\lambda)}, \quad |S(\lambda)| < 1$$

Proof. Since $q(x)$ is real, it follows that $\overline{e(\lambda,0)} = e(-\lambda,0)$. For $\lambda \in \mathbb{R}\backslash\{0\}$, $\overline{\psi(\lambda,0)} = \psi(-\lambda,0)$ and $\overline{\psi'(\lambda,0)} = \psi'(-\lambda,0)$, it follows from (2.3) and (2.4) that $\overline{\Theta(\lambda)} = \Theta(-\lambda)$ and $\overline{\Theta_1(\lambda)} = \Theta_1(-\lambda)$, which shows $\overline{S(\lambda)} = S(-\lambda)$, for all $\lambda \in \mathbb{R}\backslash\{0\}$.

To show $|S(\lambda)| < 1$, the following equality is obtained:

$$\begin{aligned} |S(\lambda)|^2 &= S(\lambda)\cdot\overline{S(\lambda)} \\ &= \frac{|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right)}{|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2+|\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right)} \end{aligned}$$

for all $\lambda \in \mathbb{R}\backslash\{0\}$. Using (1.4),

$$\begin{aligned} &\left[\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)} - \psi'(\lambda,0)\cdot\overline{\psi(\lambda,0)}\right]\cdot\left[e'(\lambda,0)\,\overline{e(\lambda,0)} - \overline{e'(\lambda,0)}e(\lambda,0)\right] \\ &= 2i\Im\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\right)\cdot W[e(\lambda,0),\overline{e(\lambda,0)}] \\ &= -4\lambda^2\left[\alpha_1\beta_0 - \alpha_0\beta_1 + (\alpha_2\beta_1 - \alpha_1\beta_2)\lambda^2 + (\alpha_3\beta_2 - \alpha_2\beta_3)\lambda^4\right] \\ &< 0 \end{aligned}$$

which yields

$$\begin{aligned} &-\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)} - \overline{\psi(\lambda,0)}\cdot\psi'(\lambda,0)\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)} \\ &< -\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)} - \overline{\psi(\lambda,0)}\cdot\psi'(\lambda,0)\cdot\overline{e'(\lambda,0)}\cdot e(\lambda,0) \end{aligned}$$

i.e.,

$$-2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right) < -2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right)$$

and then

$$\begin{aligned} &|\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2 + |\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2 - 2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e(\lambda,0)\cdot\overline{e'(\lambda,0)}\right) \\ &< |\psi(\lambda,0)|^2\cdot|e'(\lambda,0)|^2 + |\psi'(\lambda,0)|^2\cdot|e(\lambda,0)|^2 - 2\Re\left(\psi(\lambda,0)\cdot\overline{\psi'(\lambda,0)}\cdot e'(\lambda,0)\cdot\overline{e(\lambda,0)}\right) \end{aligned}$$

which shows $|S(\lambda)|^2 < 1$, that is, $|S(\lambda)| < 1$, for all $\lambda \in \mathbb{R}\backslash\{0\}$. Thus, the lemma is proved. □

We proceed to research the spectrum of the boundary value problem (1.1) and (1.2). Therefore, we investigate the scattering function in more detail. It is a meromorphic function in the upper half plane $\Im\lambda > 0$, with poles at the zeros of the function $\Theta(\lambda)$.

Lemma 2.2. The function $\Theta(\lambda)$ has only finitely many zeros in the upper half plane $\Im > 0$. The zeros of $\Theta(\lambda)$ are simple and pure imaginary.

Proof. If we assume that $\rho(x) \equiv 1$ in [11], the proof of the lemma can be obtained similarly. □

Let $i\lambda_j$ such that $\lambda_j > 0$, for all $j = \overline{1,n}$, be the zeros of the function $\Theta(\lambda)$, called the singular values of (1.1) and (1.2). Thus, the numbers m_j, for all $j = \overline{1,n}$, are defined by

$$m_j^{-2} \equiv \int_0^\infty |e(i\lambda_j, x)|^2 dx + \frac{|e(i\lambda_j, 0)|^2 \left[\frac{1}{2}\sum_{k=0}^{2}(\alpha_{1+k}\beta_k - \alpha_k\beta_{1+k})\lambda_j^{2k-1} + \sum_{k=0}^{1}(\alpha_k\beta_{2+k} - \alpha_{2+k}\beta_k)\lambda_j^{2k}\right]}{|\alpha_0 - \alpha_1\lambda_j + \alpha_2\lambda_j^2 - \alpha_3\lambda_j^3|^2}$$

and called the normalized numbers for (1.1) and (1.2). As a result, we can give the following definition.

Definition 2.3. The collection of quantities $\{S(\lambda);\ i\lambda_j;\ m_j\ (j=\overline{1,n})\}$ is called the scattering data of the boundary value problem (1.1) and (1.2).

Based on the scattering data, form an integral equation for the kernel $K(x,y)$.

Theorem 2.4. For every fixed $x \geq 0$, the kernel $K(x,y)$ of the solution (2.1) satisfies the integral equation, called the main equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,t)F(t+y)dt = 0, \quad y > x \tag{2.10}$$

where

$$F(x) = \sum_{j=1}^{n} m_j^2 e^{-\lambda_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}(-1 - S(\lambda))e^{i\lambda x}d\lambda$$

Proof. The proof is obtained similarly for the case $\rho(x) \equiv 1$ in [11]. □

3. The Scattering Function's Continuity

This section presents the scattering function's continuity.

Theorem 3.1. For all $\lambda \in \mathbb{R}$, the function $S(\lambda)$ is continuous.

Proof. From section 2, $\Theta(\lambda) \neq 0$, for all λ in the intervals $(-\infty, 0)$ and $(0,\infty)$, and $S(\lambda)$ is defined on $(-\infty, 0)$ and $(0,\infty)$ and continuous in these intervals. From the form (2.7) of $\Theta(\lambda)$, if $\Theta(0) \neq 0$, then $S(\lambda)$ is continuous at zero and $S(0) = 1$. It remains to investigate the case:

$$\Theta(0) = \alpha_0\left\{-K(0,0) + \int_0^\infty K_x(0,t)dt\right\} - \beta_0\left\{1 + \int_0^\infty K(0,t)dt\right\} = 0$$

Moreover, if we substitute $x = 0$ into (2.10), then

$$K(0,y) + F(y) + \int_0^\infty K(0,t)F(t+y)dt = 0 \tag{3.1}$$

Integrating (3.1) according to y from z to ∞, letting $t + y = \xi$, and applying the integration by parts,

$$\left\{1 + \int_0^\infty K(0,y)dy\right\}\int_z^\infty F(y)dy + \int_z^\infty K(0,y)dy - \int_0^\infty F(t+z)\left\{\int_t^\infty K(0,\xi)d\xi\right\}dt = 0 \tag{3.2}$$

We now apply the same procedure to the derivation of the main equation concerning x for obtaining

$$\left\{-K(0,0) + \int_0^\infty K_x(0,y)dy\right\}\int_z^\infty F(y)dy - F(z) + \int_z^\infty K_x(0,y)dy - \int_0^\infty F(t+z)\left\{\int_t^\infty K_x(0,\xi)d\xi\right\}dt = 0 \tag{3.3}$$

Multiplying (3.3) by $(\alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3)$ and (3.2) by $(\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)$ and subtracting the latter from the former,

$$0=\left[(\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3)\left\{-K(0,0)+\int\limits_0^\infty K_x(0,y)dy\right\}\right.$$

$$\left.-\ (\beta_0+i\beta_1\lambda-\beta_2\lambda^2-i\beta_3\lambda^3)\left\{1+\int\limits_0^\infty K(0,y)dy\right\}\right]\int\limits_z^\infty F(y)dy$$

$$+(\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3)\int\limits_z^\infty K_x(0,y)dy$$

$$-(\beta_0+i\beta_1\lambda-\beta_2\lambda^2-i\beta_3\lambda^3)\int\limits_z^\infty K(0,y)dy-(\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3)F(z)$$

$$-\int\limits_0^\infty\left\{\int\limits_t^\infty\left[(\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3)K_x(0,\xi)-(\beta_0+i\beta_1\lambda-\beta_2\lambda^2-i\beta_3\lambda^3)K(0,\xi)\right]d\xi\right\}F(t+z)dt$$

Letting $\lambda\to 0$,

$$\alpha_0F(z)=\left[\alpha_0\left\{-K(0,0)+\int\limits_0^\infty K_x(0,y)dy\right\}-\beta_0\left\{1+\int\limits_0^\infty K(0,y)dy\right\}\right]\int\limits_z^\infty F(y)dy$$

$$+\int\limits_z^\infty\left[\alpha_0K_x(0,y)dy-\beta_0K(0,y)\right]dy$$

$$-\int\limits_0^\infty\left\{\int\limits_t^\infty\left[\alpha_0K_x(0,\xi)-\beta_0K(0,\xi)\right]d\xi\right\}F(t+z)dt$$

Define the functions $G(z)$ and $H(z)$ as follows:

$$G(z):=\int\limits_z^\infty\left[\alpha_0K_x(0,y)dy-\beta_0K(0,y)\right]dy$$

and

$$H(z):=\alpha_0F(z)$$

Hence, the integral equation is as follows:

$$G(z)-\int\limits_0^\infty F(t+z)G(t)dt=H(z)$$

$G(z)$ is a bounded solution of the equation

$$G(z)-\int\limits_0^\infty F(t+z)G(t)dt=0,\quad 0\le z<\infty$$

and every bounded solution of this equation is summable on the half line $[0,\infty)$. It means that $G(z)\in L_1(0,\infty)$ (see p. 211 [19]). Thus, for

$$\widehat{K_1}(\lambda)=\alpha_0+i\alpha_1\lambda-\alpha_2\lambda^2-i\alpha_3\lambda^3-\beta_1-i\beta_2\lambda+\beta_3\lambda^2$$

$$-(\alpha_1+i\alpha_2\lambda-\alpha_3\lambda^2)K(0,0)+(\alpha_1+i\alpha_2\lambda-\alpha_3\lambda^2)\int\limits_0^\infty K_x(0,t)e^{i\lambda t}dt$$

$$-(\beta_1+i\beta_2\lambda-\beta_3\lambda^2)\int\limits_0^\infty K(0,t)e^{i\lambda t}dt+\int\limits_0^\infty G(t)e^{i\lambda t}dt$$

$$
\begin{aligned}
\Theta(\lambda) = {} & \alpha_0 \left(i\lambda - K(0,0) + \int_0^\infty K_x(0,t) e^{i\lambda t} dt \right) - \beta_0 \left(1 + \int_0^\infty K(0,t) e^{i\lambda t} dt \right) \\
& + (i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3) \left(i\lambda - K(0,0) + \int_0^\infty K_x(0,t) e^{i\lambda t} dt \right) \\
& - (i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3) \left(1 + \int_0^\infty K(0,t) e^{i\lambda t} dt \right) \\
= {} & \alpha_0 \left\{ i\lambda - K(0,0) + \int_0^\infty K_x(0,t) dt + i\lambda \int_0^\infty \left(\int_t^\infty K_x(0,y) dy \right) e^{i\lambda t} dt \right\} \\
& - \beta_0 \left\{ 1 + \int_0^\infty K(0,t) dt + i\lambda \int_0^\infty \left(\int_t^\infty K(0,y) dy \right) e^{i\lambda t} dt \right\} \\
& + i\lambda \left[i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 - \beta_1 - i\beta_2\lambda + \beta_3\lambda^2 - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) K(0,0) \right] \\
& + i\lambda \left[(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t) e^{i\lambda t} dt - (\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t) e^{i\lambda t} dt \right] \\
= {} & \alpha_0 \left\{ -K(0,0) + \int_0^\infty K_x(0,t) dt \right\} - \beta_0 \left\{ 1 + \int_0^\infty K(0,t) dt \right\} \\
& + i\lambda \int_0^\infty \left\{ \int_t^\infty \left[\alpha_0 K_x(0,y) - \beta_0 K(0,y) \right] dy \right\} e^{i\lambda t} dt \\
& + i\lambda \left[i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 - \beta_1 - i\beta_2\lambda + \beta_3\lambda^2 - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) K(0,0) \right] \\
& + i\lambda \left[(\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t) e^{i\lambda t} dt - (\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t) e^{i\lambda t} dt \right] \\
= {} & i\lambda \widehat{K_1}(\lambda)
\end{aligned}
\tag{3.4}
$$

In a similar manner, from (2.8),

$$\Theta_1(\lambda) = -i\lambda \widehat{K_2}(\lambda) \tag{3.5}$$

where

$$
\begin{aligned}
\widehat{K_2}(\lambda) = {} & \alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3 + \beta_1 + i\beta_2\lambda - \beta_3\lambda^2 \\
& + (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) K(0,0) - (\alpha_1 + i\alpha_2\lambda - \alpha_3\lambda^2) \int_0^\infty K_x(0,t) e^{-i\lambda t} dt \\
& + (\beta_1 + i\beta_2\lambda - \beta_3\lambda^2) \int_0^\infty K(0,t) e^{-i\lambda t} dt + \int_0^\infty G(t) e^{-i\lambda t} dt
\end{aligned}
$$

According to (2.6), (3.4), and (3.5),

$$S(\lambda) = -\frac{\widehat{K_2}(\lambda)}{\widehat{K_1}(\lambda)}$$

Taking into account the identity (2.5),

$$2\psi(\lambda, x) = \widehat{K_1}(\lambda) \left\{ e(-\lambda, x) - S(\lambda) e(\lambda, x) \right\}$$

from which it follows that $\widehat{K_1}(0) \neq 0$, otherwise it would be $\psi(0,x) = 0$ and it contradicts $\psi(0,0) \neq 0$. This shows that $S(\lambda)$ is continuous at $\lambda = 0$ and completes the proof. □

Consequently, from these results and section 2, $S(\lambda)$ is defined over $(-\infty, \infty)$ and continuous in this interval. Moreover, in the case $\alpha_1 = \beta_1 = 0$,

$$S(0) = \begin{cases} 1, & \Theta(\lambda) \neq 0 \\ -1, & \Theta(\lambda) = 0 \end{cases} \tag{3.6}$$

4. The Levinson-Type Formula

This section describes the Levinson-type formula for the considered boundary value problem.

Theorem 4.1. The following formula holds:

$$n - t(\Theta) = \frac{\mu(+\infty) - \mu(+0)}{\pi} \tag{4.1}$$

where n is the number of the zeros of the function $\Theta(\lambda)$ in the upper half plane,

$$\mu(\lambda) = \arg\Theta(\lambda), \text{ and } t(\Theta) = \begin{cases} 2, & \Theta(0) \neq 0 \\ \frac{3}{2}, & \Theta(0) = 0 \end{cases} \tag{4.2}$$

Proof. To achieve formula (4.1), the function $\Theta(\lambda)$ is analyzed using the argument principle. We now assume that

$$\Gamma_{R,\epsilon} = C_R^+ \cup C_R^- \cup [-R, -\epsilon] \cup [\epsilon, R]$$

for sufficiently large $R > 0$ and sufficiently small ϵ, where C_R^+ is a circle oriented counterclockwise and centered at the origin with radius R, and C_ϵ^- is a circle oriented clockwise and centered at the origin with radius ϵ.

Define the function $\arg\Theta(\lambda) = \mu(\lambda)$. Then, the function $\Theta(\lambda)$ is analytic in the upper half plane and continuous along the real axis. Hence, the increment of $\mu(\lambda)$ equals the number of zeros of $\Theta(\lambda)$ multiplied by 2π as λ runs over the real axis from $-\infty$ to ∞, bypassing the point $\lambda = 0$ along semicircle of sufficiently small radius ϵ in the upper half-plane.

As $R \to \infty$,

$$\{\mu(-\epsilon) - \mu(-\infty)\} + \{\mu(+\epsilon) - \mu(-\epsilon)\} + \{\mu(+\infty) - \mu(+\epsilon)\} + 4\pi = 2\pi n$$

because

$$\Theta(\lambda) = (i\lambda)^4 \left[\alpha_3 + O(\frac{1}{\lambda})\right], \quad |\lambda| \to \infty$$

for $\Im \geq 0$. If $\Theta(0) \neq 0$, then

$$\lim_{\epsilon \to 0} \{\mu(+\epsilon) - \mu(-\epsilon)\} = 0$$

However, if $\Theta(0) = 0$, then $\Theta(\lambda) = i\lambda\widehat{K_1}(\lambda), \widehat{K_1}(0) \neq 0$ by (3.4). Hence,

$$\lim_{\epsilon \to 0} \{\mu(+\epsilon) - \mu(-\epsilon)\} = -\pi$$

When $\epsilon \to 0$,

$$2\{\mu(\infty) - \mu(0)\} + \begin{cases} 0, & \Theta(0) \neq 0 \\ -\pi, & \Theta(0) = 0 \end{cases} + 4\pi = 2\pi n$$

since $\lim_{\epsilon \to 0} \{\mu(-\epsilon) - \mu(-\infty)\} = \lim_{\epsilon \to 0} \{\mu(\infty) - \mu(\epsilon)\}$. Thus,

$$n - t(\Theta) = \frac{\mu(+\infty) - \mu(0)}{\pi}$$

where $t(\Theta)$ is defined by the formula (4.2), which proves the theorem. □

Proposition 4.2. For $\alpha_1 = \beta_1 = 0$, the increase in the logarithm of the scattering function is associated with the number of eigenvalues of the problem (1.1) and (1.2) by the following equality

$$n - 2 = \frac{\ln S(+0) - \ln S(\infty)}{2\pi i} - \frac{1 - S(0)}{4} \tag{4.3}$$

Proof. According to (2.9) and (3.6), $|S(0)| = |S(\infty)| = 1$, and hence $\ln S(+0) = -2i\mu(0)$ and $\ln S(\infty) = -2i\mu(\infty)$. Considering these results in (4.1), (4.3) holds. □

Definition 4.3. (4.3) is called *the Levinson-type formula* for (1.1) and (1.2).

5. Conclusion

Levinson's theorem is a valuable tool for understanding quantum scattering phenomena. In this work, we have provided the scattering function and the spectrum for (1.1) and (1.2). The scattering function's continuity has been studied. The formula connecting the number of eigenvalues of (1.1) and (1.2) to the argument's variation of the function $\Theta(\lambda)$ over the interval $(-\infty, \infty)$ has been introduced. In a special case, we have derived the Levinson-type formula.

The study described in the text focuses on conducting spectral analysis of a second-order differential operator with nonlinear dependence on spectral parameters in boundary conditions. In future research, this methodology can be extended to various boundary value problems, and the boundary value problem (1.1) and (1.2) can be generalized for boundary conditions involving higher order polynomials of the spectral parameter.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] D. S. Cohen, *An integral transform associated with boundary conditions containing an eigenvalue parameter*, SIAM Journal on Applied Mathematics 14 (5) (1966) 1164–1175.

[2] L. Collatz, Eigenwertaufgaben mit technischen anwendungen, Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1949.

[3] P. A. Binding, P. J. Browne, B. A. Watson, *Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions*, Journal of the London Mathematical Society 62 (1) (2000) 161–182.

[4] P. A. Binding, P. J. Browne, B. A. Watson, *Sturm Liouville problems with boundary conditions rationally dependent on the eigenparameter, II*, Journal of Computational and Applied Mathematics 148 (1) (2002) 147–168.

[5] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics 77 (3-4) (1977) 293–308.

[6] Ch. G. Ibadzadeh, L. I. Mammadova, I. M. Nabiev, *Inverse problem of spectral analysis for diffusion operator with nonseparated boundary conditions and spectral parameter in boundary condition*, Azerbaijan Journal of Mathematics 9 (1) (2019) 171–189.

[7] I. M. Nabiev, *Reconstruction of the differential operator with spectral parameter in the boundary condition*, Mediterranean Journal of Mathematics 19 (3) (2022) 1–14.

[8] L. I. Mammadova, I. M. Nabiev, *Spectral properties of the Sturm–Liouville operator with a spectral parameter quadratically included in the boundary condition*, Vestnik Udmurtskogo Universiteta Matematika Mekhanika Komp'yuternye Nauki 30 (2) (2020) 237—248.

[9] A. A. Nabiev, *On a boundary value problem for a polynomial pencil of the Sturm-Liouville equation with spectral parameter in boundary conditions*, Applied Mathematics 7 (18) (2016) 2418–2423.

[10] V. N. Pivovarchik, *Direct and inverse problems for a damped string*, Journal of Operator Theory 42 (1999) 189–220.

[11] A. Çöl, *Inverse spectral problem for Sturm-Liouville operator with discontinuous coefficient and cubic polynomials of spectral parameter in boundary condition*, Advances in Difference Equations 2015 (2015) 1–12.

[12] Kh. R. Mamedov, *Uniqueness of the solution to the inverse problem of scattering theory for the Sturm–Liouville operator with a spectral parameter in the boundary condition*, Mathematical Notes 74 (2003) 136—140.

[13] Kh. R. Mamedov, F. A. Cetinkaya, *Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient*, Hacettepe Journal of Mathematics and Statistics 44 (4) (2015) 867—874.

[14] Kh. R. Mamedov, H. Menken, *On the inverse problem of scattering theory for a differential operator of the second order*, North-Holland Mathematics Studies 197 (2004) 185–194.

[15] D. Bollé, *Sum rules in scattering theory and applications to statistical mechanics*, Mathematics + Physics, Lectures on Recent Results 2 (1986) 84—153.

[16] Z. Q. Ma, *The Levinson theorem*, Journal of Physics A: Mathematical and General 39 (48) (2006) R625.

[17] R. G. Newton, Scattering theory of waves and particles, Springer-Verlag, New York, 1982.

[18] N. Levinson, *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase*, Danske Videnskab Selskab Matematisk-Fysiske Meddelelser 25 (9) (1949) 29.

[19] V. A. Marchenko, Sturm–Liouville operators and applications, Birkhäuser Verlag, Basel, 1986.

[20] S. Goktas, Kh. R. Mamedov, *The Levinson-type formula for a class of Sturm-Liouville equation*, Facta Universitatis, Series: Mathematics and Informatics 35 (4) (2020) 1219—1229.

[21] Kh. R. Mamedov, N. P. Kosar, *Continuity of the scattering function and Levinson type formula of a boundary-value problem*, International Journal of Contemporary Mathematical Sciences 5 (4) (2010) 159–170.

[22] Ö. Mızrak, Kh. R. Mamedov, A. M. Akhtyamov, *Characteristic properties of scattering data of a boundary value problem*, Filomat 31 (12) (2017) 3945–3951.