Communications in Advanced Mathematical Sciences
Vol. 7, No. 3, 147-156, 2024
Research Article
e-ISSN: 2651-4001
https://doi.org/10.33434/cams.1503610

On Some Cauchy Type Mean-Value Theorems with Applications

Uğur Selamet Kırmacı[1]

Abstract
Some Cauchy-type mean-value theorems for Chebychev's inequality, Steffensen's inequality, midpoint rule, and Simpson's rule are presented. Furthermore, we give some applications for the obtained results using the exponential and logarithmic functions, their Taylor polynomials, and some trigonometric functions. Further, we obtain some exponential, logarithmic, and trigonometric equations and give two inequalities for midpoint and Simpson's rules.

Keywords: Cauchy mean-value theorem, Chebychev's inequality, Midpoint rule, Steffensen's inequality, Simpson's rule
2010 AMS:26D15, 26D10, 26D07, 26A33

[1] *Atatürk University, Kazim Karabekir Education Faculty, Erzurum, Türkiye, 25240, kirmaci@atauni.edu.tr, ORCID: 0000-0002-8177-6649*
Received: 23 June 2024, **Accepted:** 06 September 2024, **Available online:** 08 September 2024
How to cite this article: U. S. Kırmacı *On some Cauchy type mean-value theorems with applications*, Commun. Adv. Math. Sci., 7(3) (2024), 147-156.

1. Introduction

Mean-value theorems play a fundamental role in the modern mathematics. In recent years, a number of authors have written about extensions of the mean-value theorems, which are considered in [1]-[4]. Cauchy mean-value theorem is of huge importance in mathematical analysis. A meaningful advancement in the theory of Cauchy type means is given in [5, 6]. Also, see [7, 8] for more information about the means.

In [5], A. McD. Mercer gave a formula, from which one can obtain a family of two-sided inequalities involving the elementary means values $\left(\sum_1^n w_k x_k^r\right)^{1/r}$ and showed that one member of this family provides a new refinement of the arithmetic mean-geometric mean inequality. In [6], J.E. Pečarić et al. gave a general method for deducing Cauchy type formulas and obtained two Cauchy type formulas which are connected with Jensen's inequality and classical trapezoid rule. In [9], M. Anwar et al. discussed the log-convexity for the differences of the Popoviciu inequalities and introduced some mean-value theorems and related results. Also, they gave the Cauchy means of the Popoviciu type and showed that these means are monotonic. In [10], S. Abramovich et al. gave new results associated with Hermite-Hadamard inequalities for superquadratic functions, derived a set of Cauchy type means from these Hermite-Hadamard-type inequalities and proved its log-convexity and monotonicity. In [11], N. Mehreen and M. Anwar established Jensen's inequality for s-convex functions in the first sense. By using Jensen's inequalities, they obtained some Cauchy type means for p-convex and s-convex functions in the first sense. Also, by using Hermite–Hadamard inequalities for the respective generalized convex functions, they found new generalized Cauchy type means. In [12], L. Horvath et al. defined some weighted mixed symmetric means and Cauchy type means. They investigated the exponential convexity of some functions, studied some mean-value theorems, and proved the monotonicity of the introduced means. In [13], M. Anwar et al. proved the positive semidefiniteness of matrices generated by differences deduced from majorization type results which implied exponential convexity and log-convexity of these differences and also obtained Lyapunov's and Dresher's inequalities for these differences. They introduced new Cauchy means and showed that these means were monotonic. In [14], J.E. Pečarić et al. presented several further generalizations and applications of some mean-value theorems of the Cauchy type, which are connected with Jensen's inequality. In [15], S.S. Dragomir established some two points Taylor type representations with integral remainders and applied them for the logarithmic and exponential functions.

In [16], Chebychev's inequality is given by the following theorem:

Theorem 1.1. *Let f and g be real and integrable functions on [a,b] and let them both be either increasing or decreasing. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx.$$

If one function is increasing and the other decreasing, the reverse inequality holds.

In [16], Steffensen's inequality is given by the following theorem:

Theorem 1.2. *Assume that two integrable functions f and g are defined on the interval (a,b), that f never increases and that $0 \leq g(t) \leq 1$ in (a,b). Then*

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt,$$

where $\lambda = \int_a^b g(t)dt$.

In this paper we give some Cauchy type mean-value theorems for Chebychev's inequality, Steffensen's inequality, midpoint rule and Simpson's rule and apply them to exponential and logarithmic functions, their Taylor polynomials and to some trigonometric functions. Further, we write some applications for midpoint and Simpson's rules.

For several recent results concerning Hermite-Hadamard's inequality and convex functions, we refer the reader to [17]-[30].

2. Main Results

Firstly, we start by the following theorem, which is connected with Chebychev's inequality.

Theorem 2.1. *Let $f,g:[a,b]\subset R \to R$ be integrable functions and let them both be either increasing or decreasing. Let $h,w:[a,b]\subset R\to R$ be integrable functions and let them both be either increasing or decreasing and $f,h\in C^2([a,b])$. Then we have, for some $\zeta \in [a,b]$*

$$\frac{\int_a^b f(x)dx\int_a^b g(x)dx - \int_a^b f(x)g(x)dx}{\int_a^b h(x)dx\int_a^b w(x)dx - \int_a^b h(x)\,w(x)\,dx} = \frac{(Qf)''(\zeta)}{(Qh)''(\zeta)} \tag{2.1}$$

provided that the denominator on the left-hand side of (2.1) is non-zero.

Proof. We consider the function

$$(Qf)(t) = \int_a^b f(tx+(1-t)A)dx\int_a^b g(x)\,dx - A,$$

where $A=\int_a^b f(x)g(x)dx$. Taking the first and second derivatives of this function, we obtain

$$(Qf)'(t) = \int_a^b (x-A)f'(tx+(1-t)A)dx\int_a^b g(x)dx$$

and

$$(Qf)''(t) = \int_a^b (x-A)^2 f''(tx+(1-t)A)dx\int_a^b g(x)dx.$$

For the function

$$(Qh)(t) = \int_a^b h(tx+(1-t)B)dx\int_a^b w(x)\,dx - B,$$

we obtain

$$(Qh)''(t) = \int_a^b (x-B)^2 h''(tx+(1-t)B)dx \int_a^b w(x)dx,$$

where $B = \int_a^b h(x)w(x)dx$. We now write the function $W(t)$ given by

$$W(t) = (Qh)(1)(Qf)(t) - (Qf)(1)(Qh)(t).$$

Hence, we have

$$W(0) = W'(0) = W(1) = 0.$$

Since the function $W(t)$ satisfies the conditions of the mean-value theorem, we can write two successive applications of the mean-value theorem. Hence, we have

$$W''(\eta) = 0 \text{ for some } \eta \in (0,1).$$

This implies that

$$(Qh)(1)\left[\int_a^b (x-A)^2 f''(\eta x+(1-\eta)A)dx \int_a^b g(x)\,dx\right] - (Qf)(1)\left[\int_a^b (x-B)^2 h''(\eta x+(1-\eta)B)dx \int_a^b w(x)\,dx\right] = 0.$$

For any fixed η, the expressions in the two square brackets are continuous functions of x and hence they vanish for some value of x in (a,b). Corresponding to this value of $x \in (a,b)$, we get a number $\zeta \in [a,b]$ such that

$$(Qh)(1)(Qf)''(\zeta) - (Qf)(1)(Qh)''(\zeta) = 0.$$

This gives equality (2.1). □

The following result is connected with midpoint rule.

Theorem 2.2. *Let $f,g:[a,b]\to R$ be two functions, each of which possesses a continuous derivative of order $n \geq 2$. If*

$$f^{(k)}(a) = g^{(k)}(a) = 0, (k = 2, \ldots, n-2),$$

then we have

$$\frac{(b-a)f\left(\frac{a+b}{2}\right) - \int_a^b f(x)dx}{(b-a)g\left(\frac{a+b}{2}\right) - \int_a^b g(x)dx} = \frac{f^{(n)}\left(\frac{a+\varsigma}{2}\right)(\varsigma-a) + 2nf^{(n-1)}\left(\frac{a+\varsigma}{2}\right) - 2^n f^{(n-1)}(\varsigma)}{g^{(n)}\left(\frac{a+\varsigma}{2}\right)(\varsigma-a) + 2ng^{(n-1)}\left(\frac{a+\varsigma}{2}\right) - 2^n g^{(n-1)}(\varsigma)}, \tag{2.2}$$

for some $\varsigma \in (a,b)$.

Proof. Let us define the function

$$(Qf)(t) = f\left(\frac{a+t}{2}\right)(t-a) - \int_a^t f(s)ds.$$

Taking the first three derivatives of this function gives us

$$(Qf)'(t) = \frac{1}{2}f'\left(\frac{a+t}{2}\right)(t-a) + f\left(\frac{a+t}{2}\right) - f(t),$$

$$(Qf)''(t) = \frac{1}{2^2}f''\left(\frac{a+t}{2}\right)(t-a) + f'\left(\frac{a+t}{2}\right) - f'(t),$$

and

$$(Qf)'''(t) = \frac{1}{2^3}f'''\left(\frac{a+t}{2}\right)(t-a) + \frac{1}{2^2}f''\left(\frac{a+t}{2}\right) + \frac{1}{2}f''\left(\frac{a+t}{2}\right) - f''(t).$$

Hence we write the k th derivative as follows

$$(Qf)^{(k)}(t) = \frac{1}{2^k} f^{(k)}\left(\frac{a+t}{2}\right)(t-a) + \frac{k}{2^{k-1}} f^{(k-1)}\left(\frac{a+t}{2}\right) - f^{(k-1)}(t), \qquad 2 \le k \le n.$$

We note that

$$(Qf)(a) = (Qf)'(a) = (Qf)''(a) = \cdots = (Qf)^{(n)}(a) = 0.$$

We now write the function $W(t)$ given by

$$W(t) = (Qg)(b)(Qf)(t) - (Qf)(b)(Qg)(t),$$

where

$$(Qg)(t) = g\left(\frac{a+t}{2}\right)(t-a) - \int_a^t g(s)ds.$$

Hence, we obtain

$$W(a) = W'(a) = W''(a) = \cdots = W^{(n-1)}(a) = W(b) = 0.$$

Since the functions f *and* g possess continuous derivatives of order $n \ge 2$, the function $W(t)$ satisfies the conditions of the mean-value theorem. For this reason, we can write n successive applications of the mean-value theorem. Hence, we have

$$W^{(n)}(\varsigma) = 0 \text{ for some } \varsigma \in (a,b).$$

That is,

$$W^{(n)}(\varsigma) = (Qg)(b)(Qf)^{(n)}(\varsigma) - (Qf)(b)(Qg)^{(n)}(\varsigma) = 0,$$

which is (2.2). □

Now we give the following result, which is connected with Simpson's rule.

Theorem 2.3. *Let $f, g : [a,b] \to R$ be two functions, each of which possesses a continuous derivative of order $n \ge 2$. If*

$$f^{(k)}(a) = g^{(k)}(a) = 0, (k = 2, \ \ldots \ , n-2),$$

then we have

$$\frac{\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(x)dx}{\frac{b-a}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \int_a^b g(x)dx} = \frac{(Qf)^{(n)}(\varsigma)}{(Qg)^{(n)}(\varsigma)}, \tag{2.3}$$

for some $\varsigma \in (a,b)$, where

$$(Qf)^{(n)}(\varsigma) = f^{(n-1)}\left(\frac{a+\varsigma}{2}\right) + 2^{n-3} f^{(n-1)}(\varsigma) + (\varsigma - a)\left[\frac{1}{2n} f^{(n)}\left(\frac{a+\varsigma}{2}\right) + \frac{2^{n-3}}{n} f^{(n-1)}(\varsigma)\right] - 3\frac{2^{n-2}}{n} f^{(n-1)}(\varsigma).$$

Proof. Let us consider the function

$$(Qf)(t) = (t-a)\left[\frac{2}{3} f\left(\frac{a+t}{2}\right) + \frac{f(a) + f(t)}{6}\right] - \int_a^t f(x)dx.$$

Taking the first three derivatives of this function, we obtain

$$(Qf)'(t) = \left[\frac{2}{3} f\left(\frac{a+t}{2}\right) + \frac{f(a) + f(t)}{6}\right] + (t-a)\left[\frac{1}{3} f'\left(\frac{a+t}{2}\right) + \frac{1}{6} f'(t)\right] - f(t),$$

$$(Qf)''(t) = 2\left[\frac{1}{3} f'\left(\frac{a+t}{2}\right) + \frac{1}{6} f'(t)\right] + (t-a)\left[\frac{1}{2.3} f''\left(\frac{a+t}{2}\right) + \frac{1}{6} f''(t)\right] - f'(t),$$

and

$$(Qf)'''(t) = 3\left[\frac{1}{2.3}f''\left(\frac{a+t}{2}\right)+\frac{1}{6}f''(t)\right]+(t-a)\left[\frac{1}{3.2^2}f'''\left(\frac{a+t}{2}\right)+\frac{1}{6}f'''(t)\right]-f''(t).$$

Hence we write the k th derivative as

$$(Qf)^{(k)}(t) = k\left[\frac{1}{3.2^{k-2}}f^{(k-1)}\left(\frac{a+t}{2}\right)+\frac{1}{6}f^{(k-1)}(t)\right]+(t-a)\left[\frac{1}{3.2^{k-1}}f^{(k)}\left(\frac{a+t}{2}\right)+\frac{1}{6}f^{(k)}(t)\right]-f^{(k-1)}(t).$$

We note that

$$(Qf)(a) = (Qf)'(a) = (Qf)''(a) = \cdots = (Qf)^{(n)}(a) = 0.$$

We now define the function $W(t)$ given by

$$W(t) = (Qg)(b)(Qf)(t)-(Qf)(b)(Qg)(t),$$

where

$$(Qg)(t) = (t-a)\left[\frac{2}{3}g\left(\frac{a+t}{2}\right)+\frac{g(a)+g(t)}{6}\right]-\int_a^t g(x)dx.$$

Hence, we obtain

$$W(a) = W'(a) = W''(a) = \cdots = W^{(n-1)}(a) = W(b) = 0.$$

Since the functions f and g possess continuous derivatives of order $n \geq 2$, the function $W(t)$ satisfies the conditions of the mean-value theorem. For this reason, we can write n successive applications of the mean-value theorem. Hence, we have

$$W^{(n)}(\varsigma) = 0 \text{ for some } \varsigma \in (a,b)$$

.

That is,

$$W^{(n)}(\varsigma) = (Qg)(b)(Qf)^{(n)}(\varsigma)-(Qf)(b)(Qg)^{(n)}(\varsigma) = 0.$$

Thus we get (2.3). □

Other results are given in the following theorems, which are connected with Steffensen's inequality.

Theorem 2.4. *Let $f,g,h,w : [a,b] \subset R \to R$ be integrable functions and let $f,h \in C^2([a,b])$. Let f,h be never increasing and $0 \leq g(t) \leq 1$ and $0 \leq w(t) \leq 1$, for each $t \in (a,b)$. Then we have, for some $\zeta \in [a,b]$,*

$$\frac{\int_{b-\lambda_1}^b f(x)dx - \int_a^b f(x)g(x)dx}{\int_{b-\lambda_2}^b h(x)dx - \int_a^b h(x)w(x)dx} = \frac{(\varphi fg)''(\zeta)}{(\varphi hw)''(\zeta)}, \tag{2.4}$$

provided that the denominator on the left-hand side of (2.4) is non-zero.

Proof. Let us define the function

$$(\varphi fg)(t) = \int_{b-\lambda_1}^b f(ts+(1-t)A)ds - A,$$

where $A = \int_a^b f(s)g(s)ds$ and $\lambda_1 = \int_a^b g(s)ds$. Taking the first and second derivatives of this function, we obtain

$$(\varphi fg)'(t) = \int_{b-\lambda_1}^b (s-A)f'(ts+(1-t)A)ds,$$

and

$$(\varphi fg)''(t) = \int_{b-\lambda_1}^b (s-A)^2 f''(ts+(1-t)A)ds.$$

We now consider the function $W(t)$ given by

$$W(t) = (\varphi fg)(1)(\varphi hw)(t) - (\varphi hw)(1)(\varphi fg)(t),$$

where $(\varphi hw)(t) = \int_{b-\lambda_2}^{b} h(ts+(1-t)B)ds - B$, $B = \int_a^b h(s)w(s)ds$ and $\lambda_2 = \int_a^b w(s)ds$. Hence, we write $W(0) = W'(0) = W(1) = 0$. Since the function $W(t)$ satisfies the conditions of the mean-value theorem, we can write two successive applications of the mean-value theorem. Hence, we have $W''(\varsigma) = 0$ for some $\varsigma \in (0,1)$.
Since

$$W''(t) = (\varphi fg)(1)(\varphi hw)''(t) - (\varphi hw)(1)(\varphi fg)''(t),$$

we get

$$(\varphi fg)(1)\left[\int_{b-\lambda_2}^{b} (s-B)^2 h''(\varsigma s+(1-\varsigma)B)\,ds\right] - (\varphi hw)(1)\left[\int_{b-\lambda_1}^{b} (s-A)^2 f''(\varsigma s+(1-\varsigma)A)\,ds\right] = 0.$$

For any fixed ς, the expressions in the two square brackets are continuous functions of s and hence they vanish for some value of s in (a,b). Corresponding to this value of $s \in (a,b)$, we get a number $\zeta \in [a,b]$ such that

$$(\varphi fg)(1)(\varphi hw)''(\zeta) - (\varphi hw)(1)(\varphi fg)''(\zeta) = 0.$$

This gives equality (2.4). □

Theorem 2.5. *If the conditions of Theorem 2.4 hold, then we have for some $\zeta \in [a,b]$*

$$\frac{\int_a^{a+\lambda_1} f(x)dx - \int_a^b f(x)g(x)dx}{\int_a^{a+\lambda_2} h(x)dx - \int_a^b h(x)w(x)dx} = \frac{(\varphi fg)''(\zeta)}{(\varphi hw)''(\zeta)}, \tag{2.5}$$

provided that the denominator on the left-hand side of (2.5) is non-zero.

Proof. The proof is analogous to that of Theorem 2.4, taking $(\varphi fg)(t) = \int_a^{a+\lambda_1} f(ts+(1-t)B)ds - B$, where $B = \int_a^b f(s)g(s)ds$ and the values λ_1, λ_2 are as in Theorem 2.4.

□

3. Applications

Now, using Theorems 2.1-2.4, we give some applications for the exponential, logarithmic functions, their Taylor polynomials and for some trigonometric functions. Finally, we write two inequalities for midpoint and Simpson's rules.

Corollary 3.1. *Let $f(x) = -x$, $g(x) = e^{-x}$, $h(x) = x$, $w(x) = lnx$. Here f,g are decreasing functions and h,w are increasing functions. For $[a,b] = [1,x]$ and $1 < x \le 2$, we have*

$$\int_a^b f(x)dx \int_a^b g(x)dx - \int_a^b f(x)g(x)dx = \int_1^x (-x)dx \int_1^x e^{-x}dx - \int_1^x (-xe^{-x})dx = \frac{x^2}{2}e^{-x} - xe^{-x} - \frac{3}{2}e^{-x} - \frac{x^2}{2}e^{-1} + \frac{5}{2}e^{-1},$$

$$\int_a^b h(x)dx \int_a^b w(x)dx - \int_a^b h(x)w(x)dx = \int_1^x xdx \int_1^x lnxdx - \int_1^x xlnx\,dx = \frac{x^3}{2}lnx - \frac{x}{2}lnx + \frac{x^2-x^3}{2} + \frac{x}{2} - \frac{1}{2},$$

and

$$(\varphi f)''(\zeta) = c\int_1^x e^{-x}dx = c\left(-e^{-x}+e^{-1}\right), \ (\varphi h)''(\zeta) = c\,(xlnx - x + 1),$$

where c is a constant. From Theorem 2.1, we have

$$\frac{\frac{x^2}{2}e^{-x} - xe^{-x} - \frac{3}{2}e^{-x} - \frac{x^2}{2}e^{-1} + \frac{5}{2}e^{-1}}{\frac{x^3}{2}lnx - \frac{x}{2}lnx + \frac{x^2-x^3+x-1}{2}} = \frac{-e^{-x}+e^{-1}}{xlnx - x + 1}.$$

For $x = 3/2$, we have the following exponential equation

$$5e^{-3/2} - 3e^{-1} = 0.$$

From here, we obtain

$$e^{-1/2} = 3/5.$$

We note that the approximate value of $e^{-1/2}$ obtained with a calculator is 0,6065...

Corollary 3.2. *i. For $n=2$, Theorem 2.2 can be applied to Taylor polynomials. Let $f(x)=e^x$. The Taylor expansion of this function is $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n}$. Let $g(x)=1+x+\frac{x^2}{2}$. For $n=0,1,2,\quad f^{(n)}(0)=g^{(n)}(0)=1$. From the equation (2.2) and considering the interval $[a,b]=[0,x]$, we obtain*

$$xe^{\frac{x}{2}}-e^x+1=\frac{\varsigma e^{\frac{\varsigma}{2}}+4e^{\frac{\varsigma}{2}}-4e^{\varsigma}}{-\varsigma}\left(x+\frac{x^2}{2}+\frac{x^3}{8}\right)-\int_0^x(1+x+\frac{x^2}{2})dx.$$

For $\varsigma=\frac{1}{2}$ and $\frac{1}{2}<x\leq 1$, we get

$$xe^{\frac{x}{2}}-e^x+1=\frac{\frac{9}{2}e^{\frac{1}{4}}-4e^{\frac{1}{2}}}{-1/2}\left(x+\frac{x^2}{2}+\frac{x^3}{8}\right)-\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right).$$

For $x=1$, we obtain the exponential equation

$$e-\frac{4}{3}e^{\frac{1}{2}}+\frac{3}{8}e^{\frac{1}{4}}=1.$$

ii. For $\frac{1}{3}<x\leq 1$ and taking $\varsigma=\frac{1}{3}$ in the equation (2.2), we get

$$xe^{\frac{x}{2}}-e^x+1=\frac{\frac{13}{3}e^{\frac{1}{6}}-4e^{\frac{1}{3}}}{-1/3}\left(x+\frac{x^2}{2}+\frac{x^3}{8}\right)-\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right).$$

For $x=1$, we obtain the exponential equation

$$e-e^{\frac{1}{2}}+\frac{13}{24}e^{\frac{1}{6}}-\frac{1}{2}e^{1/3}=1.$$

For different x and ς values, different exponential equations can be obtained.

Corollary 3.3. *Let $f(x)=lnx$. The Taylor expansion of this function is $lnx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}$ and convergent interval of this series is $(0,2]$. Let $g(x)=x-1-\frac{(x-1)^2}{2}$. Here $f(1)=g(1)=0,\ f'(1)=g'(1)=1,\ f''(1)=g''(1)=-1$. From the equation (2.2) and considering the interval $[a,b]=[1,x]\ \ for\ \ 1<x\leq 2$, we have*

$$(x-1)\,ln\frac{x+1}{2}-\int_1^x lnx\,dx=\frac{\frac{-1}{\left(\frac{1+\varsigma}{2}\right)^2}(\varsigma-1)+4\left(\frac{2}{1+\varsigma}\right)-\frac{4}{\varsigma}}{\varsigma-1}\left(\frac{3}{8}-\int_1^x\left(x-1-\frac{(x-1)^2}{2}\right)dx\right).$$

Taking $\varsigma=\frac{3}{2}$ in the equation above, we obtain

$$(x-1)\,ln\frac{x+1}{2}-lnx^x+x-1=\frac{32}{75}\left(\frac{3}{8}-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}\right).$$

Getting $x=2$ in the equation above, we have the following logarithmic equation $ln\frac{3}{8}=-\frac{221}{225}=-0,9822\ldots$
We note that the approximate value of $ln\frac{3}{8}$ obtained with a calculator is -0,9808...

Corollary 3.4. *Let $f(x)=-\frac{1}{2}x,\ g(x)=x^2,\ h(x)=-x,\ w(x)=x^3$. Here f and h are never increasing and $0\leq g(x)\leq 1$ and $0\leq w(x)\leq 1,\ for\ 0\leq x\leq 1$. Since*

$$\lambda_1=\int_0^x g(t)dt=\frac{x^3}{3}\ and\ \lambda_2=\int_0^x w(t)dt=\frac{x^4}{4}\ ,$$

we have

$$\int_{b-\lambda_1}^b f(x)dx-\int_a^b f(x)\,g(x)\,dx=\int_{x-\frac{x^3}{3}}^x-\frac{1}{2}xdx-\int_0^x-\frac{1}{2}x^3\,dx=-\frac{x^2}{4}+\frac{\left(3x-x^3\right)^2}{36}+\frac{x^4}{8}$$

and

$$\int_{b-\lambda_2}^b h(x)dx-\int_a^b h(x)\,w(x)\,dx=\int_{x-\frac{x^4}{4}}^x-xdx-\int_0^x-x^4dx=\frac{\left(x-\frac{x^4}{4}\right)^2}{2}-\frac{x^2}{2}+\frac{x^5}{5}.$$

Since $f''(x) = h''(x) = 0$ *and using* $\frac{(\varphi fg)''(\zeta)}{(\varphi hw)''(\zeta)} = c$ *in the equation* (2.4)*, we obtain*

$$-\frac{x^2}{4} + \frac{\left(3x - x^3\right)^2}{36} + \frac{x^4}{8} = c\left(\frac{\left(4x - x^4\right)^2}{32} - \frac{x^2}{2} + \frac{x^5}{5}\right).$$

For $c = 1$ *and* $0 \le x \le 1$*, we have the following equation*

$$\frac{\left(3x - x^3\right)^2}{36} - \frac{\left(4x - x^4\right)^2}{32} + \frac{x^2}{4} + \frac{x^4}{8} - \frac{x^5}{5} = 0.$$

Setting $x = e^{-1}$ *in the equation above, where* $0 < e^{-1} < 1$*, we get the exponential equation*

$$\frac{\left(3e^{-1} - e^{-3}\right)^2}{36} - \frac{\left(4e^{-1} - e^{-4}\right)^2}{32} + \frac{e^{-2}}{4} + \frac{e^{-4}}{8} - \frac{e^{-5}}{5} = 0.$$

Corollary 3.5. *Let* $f(x) = cosx$ *and* $g(x) = sinx$*, where* $|x| < \infty$*. For* $n = 2$*, we can apply Theorem* 2.3*. For* $[a,b] = [0,x]$ *and* $0 < x \le \frac{\pi}{2}$*, we have*

$$\frac{x\left(\frac{2}{3}cos\frac{x}{2} + \frac{1+cosx}{6}\right) - sinx}{x\left(\frac{2}{3}sin\frac{x}{2} + \frac{sinx}{6}\right) + cosx - 1} = \frac{-2\left[\frac{1}{3}sin\frac{\varsigma}{2} + \frac{1}{6}sin\varsigma\right] - \frac{\varsigma}{6}\left[cos\frac{\varsigma}{2} + cos\varsigma\right] + sin\varsigma}{2\left[\frac{1}{3}cos\frac{\varsigma}{2} + \frac{1}{6}cos\varsigma\right] - \frac{\varsigma}{6}\left[sin\frac{\varsigma}{2} + sin\varsigma\right] - cos\varsigma}.$$

When we arrange the equation above for $x = \frac{\pi}{2}$*, we get, for* $0 < \varsigma < \pi/2$

$$\left(\frac{\varsigma}{6} - \frac{2}{3}\right)\left(sin\frac{\varsigma}{2} - cos\varsigma\right) + \left(\frac{\varsigma}{6} + \frac{2}{3}\right)\left(sin\varsigma - cos\frac{\varsigma}{2}\right) = 0,$$

or

$$\frac{sin\varsigma - cos\frac{\varsigma}{2}}{cos\varsigma - sin\frac{\varsigma}{2}} = \frac{\varsigma - 4}{\varsigma + 4}.$$

For $\varsigma = \frac{\pi}{4}$, $\varsigma = \frac{\pi}{8}$ *and* $\varsigma = \frac{\pi}{16}$ *respectively, we get the following trigonometric equations*

$$\frac{sin\frac{\pi}{4} - cos\frac{\pi}{8}}{cos\frac{\pi}{4} - sin\frac{\pi}{8}} = \frac{\pi - 16}{\pi + 16},$$

$$\frac{sin\frac{\pi}{8} - cos\frac{\pi}{16}}{cos\frac{\pi}{8} - sin\frac{\pi}{16}} = \frac{\pi - 32}{\pi + 32},$$

and

$$\frac{sin\frac{\pi}{16} - cos\frac{\pi}{32}}{cos\frac{\pi}{16} - sin\frac{\pi}{32}} = \frac{\pi - 64}{\pi + 64},$$

respectively. For different trigonometric functions, x and ς *values, different trigonometric equations can be obtained. Furthermore, Theorem* 2.3 *can be applied for hyperbolic functions to obtain hyperbolic equations.*

Corollary 3.6. *Let* $g(x) = (x-a)^n$ *and let* $f^{(n-1)}(x) = 0,\ for\ x \in [a,b]$ *in Theorem* 2.2 *and Theorem* 2.3 *Then the following inequalities are satisfied from the equations* (2.2) *and* (2.3)*, respectively:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \le \frac{(b-a)^n}{2^n.n!}M_n,$$

and

$$\frac{1}{6}[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] - \frac{1}{b-a}\int_a^b f(x)dx \le \frac{(b-a)^n}{6.n!(n-5)2^{n-2}}M_n,$$

where $M_n = max_{x\in[a,b]} f^{(n)}(x)$ *and in the second inequality, we have used the facts that* $2^n \ge n+1$ *and* $\frac{n-1}{n+1} < 1$.

4. Conclusion

We proved some Cauchy type mean-value theorems for Chebychev's inequality, Steffensen's inequality, midpoint rule and Simpson's rule and gave some applications for the obtained results using the exponential and logarithmic functions, their Taylor polynomials and for some trigonometric functions. Further, we wrote some exponential, logarithmic and trionometric equations and gave two inequalities for midpoint and Simpson's rules.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] U. Abel, M. Ivan, T. Riedel, *The mean-value theorem of flett and divided differences*, J. Math. Anal. Appl., **295** (2004), 1-9.

[2] G. Farid, M. Marwan, A. Ur Rehman, *New mean-value theorems and generalization of Hadamard inequality via coordinated m-convex functions,* J. Inequal. Appl., **283** (2015), 1-11.

[3] J. Matkowski, *A mean-value theorem and its applications,* J. Math. Anal. Appl., **373** (2011), 227–234.

[4] J. A. Reyna*, A generalized mean-value theorem*, Mh. Math., **106** (1988), 95-97.

[5] A. McD. Mercer, *Some new inequalities involving elementary mean values,* J. Math. Anal. Appl., **229** (1999), 677-681.

[6] J. E. Pečarić, I. Perić, H. M. Srivastava, *A family of the Cauchy type mean-value Theorems,* J. Math. Anal. Appl., **306** (2005), 730-739.

[7] C. E. M. Pearce, *Stolarsky means and Hadamard's inequality*, J. Math. Anal. Appl., **220** (1998), 99-109.

[8] F. Qi, *Generalized abstracted mean values*, J. Inequal. Pure Appl. Math., **1**(1) (2000), Article 4, 9 pages.

[9] M. Anwar, N. Latif, J.E. Pečarić, *Cauchy means of the Popoviciu type*, J. Inequal. Appl., (2009), Article ID 628051, 16 pages.

[10] S. Abramovich, G. Farid, J. E. Pečarić, *More about Hermite-Hadamard Inequalities, Cauchy's means, and superquadracity,* J. Inequal. Appl., (2010), Article ID 102467, 14 pages.

[11] N. Mehreen, M. Anwar, *Cauchy type means for some generalized convex functions,* J. Inequal. Appl., **114** (2021), 1-13.

[12] L. Horváth, K. Ali Khan, J. E. Pečarić, *Refinements of results about weighted mixed symmetric means and related Cauchy means,* J. Inequal. Appl., (2011), Article ID 350973, 19 pages.

[13] M. Anwar, N. Latif, J.E. Pečarić, *Positive semidefinite matrices, exponential convexity for majorization, and related Cauchy means*, J. Inequal. Appl., (2010), Article ID 728251, 19 pages.

[14] J.E. Pečarić, M. R. Lipanović, H. M. Srivastava, *Some mean-value theorems of the Cauchy type,* Fract. Calc. Appl. Anal., **9**(2) (2006), 143-158.

[15] D. Andrica, T. M. Rassias, *Differential and Integral Inequalities*, Springer Optimization and Its Applications, 151, 2019.

[16] D. S. Mitrinovic, *Analytic Inequalities*, Springer-VerlagNew-York, Heidelberg, Berlin, 1970.

[17] M. Alomari, M. Darus, U. S.Kirmaci, *Refinements of Hadamard-type inequalities for quasi-convex functions with applictions to trapezoidal formula and to special means,* Comput. Math. Appl., **59** (1) (2010), 225-232.

[18] O. Hutnik, *On Hadamard type inequalities for generalized weighted quasi-arithmetic means,* J. Inequal. Pure Appl. Math.,**7**(3) (2006), Article 96, 10 pages.

[19] U. S. Kirmaci, *On new integral inequalities with applications*, American Review of Mathematics and Statistics, **2**(2) (2014),107-124.

[20] U. S. Kirmaci, M. K. Bakula, M. E.Özdemir, J. E. Pečarić, *On some inequalities for p-norms,* J. Inequal. Pure Appl. Math.,**9**(1) (2008), Article 27, 8 pages.

[21] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,* Appl. Math. Comput., **147** (2004), 137-146.

[22] U. S. Kirmaci, R. Dikici, *On some Hermite-Hadamard type inequalities for twice differentiable mappings and applications*, Tamkang J. Math., **44**(1) (2013), 41-51.

[23] U.S. Kirmaci, M. K. Bakula, M.E. Özdemir, J.E. Pečarić, *Hadamard-type inequalities for s-convex functions,* Appl. Math. Comput., **193** (2007), 26–35.

[24] U.S. Kirmaci, *Improvement and further generalization of inequalities for differentiable mappings and applications*, Comput. Math. Appl., **55** (2008), 485–493.

[25] U. S. Kirmaci, M. E. Özdemir, *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., **153**(2004), 361–368.

[26] U. S. Kirmaci, M. E. Özdemir, *Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers*, Appl. Math. Lett., **17**(2004), 641–645.

[27] U. S. Kirmaci, *Refinements of Hermite-Hadamard type inequalities for s-convex functions with applications to special means*, *Univers. J. Math. Appl.,* **4**(3) (2021), 114-124.

[28] U. S. Kirmaci, *On generalizations of Holder's and Minkowski's inequalities*, Math. Sci. Appl. E-Notes, **11** (2023), 213–225.

[29] S. S. Dragomir, *Ostrowski and trapezoid type inequalities for the generalized k-g-fractional integrals of functions with bounded variation*, Commun. Adv. Math. Sci., **II**(4) (2019), 309-330.

[30] M. Tunc, *On new inequalities for h-convex functions via Riemann-Liouville fractional integration,* Filomat, **27**(4) (2013), 559–565.