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Controllability Analysis of Fractional-Order Delay Differential Equations via Contraction Principle

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Abstract

This paper investigates the existence of solutions and the controllability for three distinct types of fractional-order delay differential equations, aiming to establish sufficient conditions for both existence and uniqueness while demonstrating controllability. Beginning with a fractional-order delayed system containing a nonzero control function, we apply the Banach fixed-point theorem to show that this system has a unique solution and satisfies the controllability property. Extending our analysis, we introduce an integral function with a delay term on the right-hand-side of the system, forming a more complex integro-fractional delay system. With a Lipschitz condition imposed on this newly introduced function, we establish the existence and uniqueness of solution, as well as the controllability of this system. In the final system, an integro-fractional hybrid model, an additional delayed function is embedded within the Caputo derivative operator, introducing distinct analytical challenges. Despite these complexities, we use the Banach fixed-point theorem and certain assumptions to demonstrate that the systems are controllable. Our approach is distinctive in incorporating delay functions on both sides of the related systems, which we support with theoretical results and illustrative examples. The paper outlines the fundamentals of fractional calculus, specifies the necessary assumptions, and uses fixed-point criteria to establish controllability with the existence of a solution, providing a clear framework for analyzing fractional-order control systems with delay functions.

1. Introduction

Control theory, a branch of applied mathematics, deals with the key theoretical and practical aspects of designing and analysing control systems. These systems can be viewed as dynamical systems in which the rules of behaviour are determined by parameters called control functions. As in structures such as dynamical systems on manifolds, Lie groups and semigroups [1, 2], the question of controllability is also essential for fractional systems [3–6]. Recently, fractional-order differential equations have been increasingly recognised for their applications in physics, engineering and finance, and for their ability to model real-world problems. Unlike classical integer-order differential equations, they offer a more general approach [7–10]. However, similar to classical differential equations, there is no universal method to solve them explicitly, making existence and uniqueness theorems an important topic of discussion [11–14].
The controllability of fractional differential equations has been the subject of extensive study by numerous authors. In [15], M. Benchohra examined the sufficient conditions of controllability property by using semi-group theory. N.I. Mahmudov et al. addressed the controllability of semilinear integro-differential systems in Hilbert spaces, while V. Vijayakumar et al. provided results for these type of systems using resolvent operators [16, 17]. Moreover, M.M. Raja et al. obtain some results with the help of sectorial operators and Bohnenblust–Karlin's fixed point theorem [18]. Subsequently, many authors also explored fractional systems with delays, incorporating delay terms to make the models more applicable to real-life problems. Recently, K.S. Nisar et al. employed degree theory to analyze the controllability of delayed impulsive fractional integro-differential equations through numerical computations [19], while another study by Nisar et al. utilized integrated resolvent operator theory with Lipschitz conditions to demonstrate the existence and uniqueness of solutions for this type equations with nonlocal conditions and finite delay functions [20]. To emphasize the importance of the theory, it is worthy of note that there are studies on various spaces and different types of fractional type differential equations. In [21], K. Kavitha et al. proposed existence results for

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Sobolev-type Hilfer fractional integro-differential systems with infinite delay, and in [22], K. Kavitha et al. investigated the controllability of Hilfer fractional neutral Volterra-Fredholm delay integro-differential systems by using Dhage's fixed point theorem. In obtaining results related to controllability and existence/uniqueness of solutions, the use of tools such as fixed point theories like Banach, Schauder and Krasnoselskii, and Gronwall type inequalities, fractional calculus analysis and even topological degree theory is frequently seen, which shows how important and broad the field the theory studied is. For further details, we refer the reader to the papers [3–6] with [14–25] and the references therein.

The primary aim of this paper is to establish the existence and uniqueness of solutions for some fractional-order delay differential equations and to demonstrate that these systems possess the controllability property using a clear and more understandable way. For this we first consider the following fractional-order delayed system with the continuous function f from $[0,T]\times\mathbb{R}^2$ to $\mathbb{R}$:

$$ {}^{c}\mathscr{D}^{\kappa}\rho(\varsigma) = f(\varsigma,\rho(\varsigma),\rho(g(\varsigma))) - Bu(\varsigma), \quad \varsigma\in[0,T] \tag{1.1}$$

$$ \rho(\varsigma) = \varpi(\varsigma), \quad \varsigma\in[-\tau,0]. \tag{1.2}$$

where our initial condition $\varpi\in C([-\tau,0],\mathbb{R})$, delayed term $g\in C([0,T],[-\tau,T])$ satisfying $g(\varsigma)\leq\varsigma$, and control function $u(\cdot)$ is given in $L^2([0,T],U)$ with admissible control functions space U, B is a bounded linear operator and ${}^{c}\mathscr{D}^{\kappa}$ is the fractional derivative of order $0<\kappa<1$ w.r.t. Caputo. The case where the control function in the system (1.1)-(1.2) is zero is discussed in [14], where a Lipschitz condition is imposed on the right-hand side with respect to the second and third variables to ensure the existence of a solution. In this study, we extend this by treating the nonzero control function, introducing some hypotheses based only on the control function and some hypotheses related to it. We then show, just using the Banach fixed-point theorem, that the system (1.1)-(1.2) satisfies both the existence and uniqueness of solution as well as the controllability property. We then add an integral function with delay function to the right hand side of the system (1.1)-(1.2), and we get the following integro-fractional-order delayed system:

$$ {}^{c}\mathscr{D}^{\kappa}\rho(\varsigma) = f(\varsigma,\rho(\varsigma),\rho(g(\varsigma))) - Bu(\varsigma) + F\Big(\varsigma,\rho(\varsigma),\int_0^{\varsigma}\zeta(\varsigma,s,\rho(g(s)))\Big)ds, \quad \varsigma\in[0,T] \tag{1.3}$$

$$ \rho(\varsigma) = \varpi(\varsigma), \quad \varsigma\in[-\tau,0]. \tag{1.4}$$

Since the system (1.3)-(1.4) is more complex than the initial system, we demonstrate the existence and uniqueness of the solution, as well as the controllability property, using the Banach fixed-point theorem and the Lipschitz condition related to the newly added function on the right-hand side, along with the hypotheses established in the first system. Next, we consider the system of and integro-fractional-order hybrid delayed system, defined as follows, which is obtained when another function involving delay function is added to the Caputo derivative operator in the system (1.3)-(1.4):

$$ {}^{c}\mathscr{D}^{\kappa}\Big(\rho(\varsigma)+\xi(\varsigma,v(g(\varsigma)))\Big) = f(\varsigma,\rho(\varsigma),\rho(g(\varsigma))) - Bu(\varsigma) + F\Big(\varsigma,\rho(\varsigma),\int_0^{\varsigma}\zeta(\varsigma,s,\rho(g(s)))\Big)ds, \quad \varsigma\in[0,T] \tag{1.5}$$

$$ \rho(\varsigma) = \varpi(\varsigma), \quad \varsigma\in[-\tau,0]. \tag{1.6}$$

Here, in addition to the conditions of the previous system, we also impose the Lipschitz condition on the function within the Caputo derivative operator. This allows us to establish the existence and uniqueness of solutions for this system and to verify the controllability property by Banach fixed-point theorem. Unlike most studies focusing on these type of systems, we introduce delay functions in the functions on both the right and left sides for related systems. This addition of delay functions brings a new perspective to the problem, making our approach distinctive and challenging. The paper is organized as follows: Section 2 provides an introduction to the Caputo fractional-order derivative and discusses the concept of controllability. Here we outline the fundamental properties of the Caputo fractional-order derivative, the relevant inequalities and the main assumptions. In the Section 3, we explore the controllability of the systems and establish the existence and uniqueness of solutions for our problems. By converting the problems into well-defined fixed point statements, we prove our results relying mainly on Lipschitz conditions and the contraction mapping theorem. Finally, in Section 4 we illustrate our results with examples.

2. Preliminaries

This section introduces the notations, definitions and basic concepts employed in the all of the paper.

Definition 2.1 ([26,27]). *Let $\kappa>0$ be a number and $\Gamma(\cdot)$ be the Gamma function. Assume that ς is any real number in the interval $[0,T]$.*

(1) The Riemann–Liouville integral for the function ρ is defined as follows w.r.t. the order κ

$$ I^{\kappa}\rho(\varsigma) = \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\rho(s)ds. \tag{2.1}$$

(2) The Caputo derivative for the function ρ is defined as follows w.r.t. the order κ

$$ D^{\kappa}\rho(\varsigma) = \frac{1}{\Gamma(n-\kappa)}\int_0^{\varsigma}(\varsigma-s)^{n-\kappa-1}\rho^{(n)}(s)ds,$$

where $n=[\kappa]+1$ and $[\kappa]$ denotes the integer part of κ.

Before presenting our main assumptions, we give the definition of the concept of controllability given in [28], adapted to the differential equation of interest.

Definition 2.2. *The fractional control system (1.1)-(1.2) is called to be controllable on the given interval if for every points $\rho_0,\rho_1\in\mathscr{C}$ there exists a control function $u\in L^2([0,T],U)$ such that the solution $\rho(\cdot)$ of (1.1)-(1.2) satisfies $\rho(0)=\rho_0$ and $\rho(\varsigma)=\rho_1$. The same definition applies to systems (1.3)-(1.4) and (1.5)-(1.6).*

Note that throughout this paper all operations on continuous function spaces are conducted using the standard uniform convergence norm. Otherwise, it will be specified.

Let us now state the hypotheses we will use in the proofs of our results:

(C1) Assume that there exists $L>0$. Let the following inequality hold for all $\rho_i, \overline{\rho}_i \in \mathbb{R}$ $(i=1,2)$ and $\varsigma \in [0,T]$

$$|f(\varsigma,\rho_1,\overline{\rho}_1)-f(\varsigma,\rho_2,\overline{\rho}_2)| \leq L(|\rho_1-\rho_2|+|\overline{\rho}_1-\overline{\rho}_2|).$$

(C2) Suppose that $\mathscr{X}$ be a linear operator from $L^2([0,T],U)$ to $\mathbb{R}$ defined as follows

$$\mathscr{X}u := \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}Bu(s)ds.$$

Then, we see that it induces an inverse operator $\overline{\mathscr{X}}^{-1}$ which is bounded and defined on the coset space $L^2([0,T],U)/ker\mathscr{X}$, and there is a constant $K>0$ satisfying $\left|B\overline{\mathscr{X}}^{-1}\right| \leq K$.

Remark 2.3 ([29]). *We give the sketch for the construction of $\overline{\mathscr{X}}^{-1}$ as follows. Let us think a Banach space M and let J be a closed interval of $\mathbb{R}$. Now take into account of the coset sapce $Y = L^2[J,U]/\ker(\mathscr{X})$. Since $\ker(\mathscr{X})$ is closed, Y is a Banach space w.r.t. the following norm*

$$\|[u]\|_Y = \inf_{u\in[u]}\|u\|_{L^2[J,U]} = \inf_{\mathscr{X}\hat{u}=0}\|u+\hat{u}\|_{L^2[J,U]}$$

where $[u]$ are classes of equivalence for u. Define $\overline{\mathscr{X}}: Y \to M$ by $\overline{\mathscr{X}}[u] = \mathscr{X}u$ for all $u\in[u]$. Then we have that $\overline{\mathscr{X}}$ is one-to-one and

$$\|\overline{\mathscr{X}}[u]\|_M \leq \|\mathscr{X}\| \,.\, \|[u]\|_Y.$$

Moreover, $V := R(\mathscr{X})$ (i.e. range of $\mathscr{X}$) is a Banach space with the following norm

$$\|v\|_V = \left\|\overline{\mathscr{X}}^{-1}v\right\|_Y.$$

To see this, note that this norm is equivalent to the graph norm on the domain of $\overline{\mathscr{X}}^{-1}$, i.e., we have that $D\left(\overline{\mathscr{X}}^{-1}\right) = R(\overline{\mathscr{X}})$. On the other hand $\overline{\mathscr{X}}$ is bounded, and since $D(\overline{\mathscr{X}}) = Y$ is closed, $\overline{\mathscr{X}}^{-1}$ must be closed. Then we get that $R(\mathscr{X}) = V$ is a Banach space with respect to the above norm. Also, we get the following relation

$$\|\mathscr{X}u\|_V = \left\|\overline{\mathscr{X}}^{-1}\mathscr{X}u\right\|_Y = \left\|\overline{\mathscr{X}}^{-1}\overline{\mathscr{X}}[u]\right\| = \|[u]\| = \inf_{u\in[u]}\|u\| \leq \|u\|.$$

Since the space $L^2[J,U]$ is reflexive and the set $\ker(\overline{\mathscr{X}})$ is closed (weakly sense), the infimum value mentioned above has been attained. Hence, for any $v\in V$, there exists a control $u\in L^2[J,U]$ such that $u = \overline{\mathscr{X}}^{-1}v$.

3. Controllability Results

In this section, we present the controllability results for the systems (1.1)-(1.2), (1.3)-(1.4) and (1.5)-(1.6). For the sake of simplicity, the space $C([-\tau,T],\mathbb{R})$ will be referred to as $\mathscr{C}$ for short in this and the following sections.

Theorem 3.1. *If the assumptions (C1)-(C2) are satisfied, then the control system (1.1)-(1.2) is controllable provided that*

$$\Lambda_1 := L\Big(\frac{T^{\kappa}}{\Gamma(\kappa+1)} + K\frac{T^{2\kappa}}{\Gamma(\kappa+1)^2}\Big) < \frac{1}{2}.$$

Proof. First we reconsider the our problem (1.1)-(1.2) as a fixed point problem. Then we make a detailed analysis of the following operator

$$\mathscr{F}:\mathscr{C}\to\mathscr{C}$$

defined by

$$\mathscr{F}\rho(\varsigma) = \begin{cases}\varpi(\varsigma), & \varsigma\in[-\tau,0]\\ \varpi(0)+\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}f(s,\rho(s),\rho(g(s)))ds - \int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}Bu(s)ds, & \varsigma\in[0,T].\end{cases}$$

Now, using a suitable control function u, our goal is to identify a unique fixed point of $\mathscr{F}$. If any $\rho(\varsigma)$ and $\overline{\rho}(\varsigma)$ satisfy $\mathscr{F}\rho(\varsigma) = \mathscr{F}\overline{\rho}(\varsigma)$ for $\varsigma\in[-\tau,0]$, then this equality is extended to $\varsigma\in[0,T]$. In addition, let us choose the control function u as follows:

$$u(\varsigma) = \overline{\mathscr{X}}^{-1}\Big(\rho_0-\rho_1+\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}f(s,\rho(s),\rho(g(s)))ds\Big) \quad \text{for all} \quad \rho_0,\rho_1\in\mathbb{R}.$$

Now, using this control, let us determine the fixed point of the operator $\mathscr{F}$.

$$
\begin{aligned}
&\left|\mathscr{F}\rho(\varsigma)-\mathscr{F}\overline{\rho}(\varsigma)\right| \\
&\quad\leq \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\left|f(s,\rho(s),\rho(g(s)))-f(s,\overline{\rho}(s),\overline{\rho}(g(s)))\right|ds \\
&\quad+ \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\big(Bu(s)-B\overline{u}(s)\big)ds \\
&\quad\leq \frac{L}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\big(|\rho(s)-\overline{\rho}(s)|+|\rho(g(s))-\overline{\rho}(g(s))|\big)ds \\
&\quad+ \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\left[B\overline{\mathscr{X}}^{-1}\left(\left(\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}(f(s,\rho(s),\rho(g(s))-f(s,\overline{\rho}(s),\overline{\rho}(g(s)))ds))\right)\right)\right] \\
&\quad\leq 2L\frac{T^{\kappa}}{\Gamma(\kappa+1)}|\rho-\overline{\rho}|+\frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}|B\overline{\mathscr{X}}^{-1}|\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}\left|(f(s,\rho(s),\rho(g(s))-f(s,\overline{\rho}(s),\overline{\rho}(g(s)))\right|ds \\
&\quad\leq 2L\frac{T^{\kappa}}{\Gamma(\kappa+1)}|\rho-\overline{\rho}|+2KL\frac{T^{2\kappa}}{\Gamma(\kappa+1)^2}|\rho-\overline{\rho}| \\
&\quad\leq \left[2L\Big(\frac{T^{\kappa}}{\Gamma(\kappa+1)}+K\frac{T^{2\kappa}}{\Gamma(\kappa+1)^2}\Big)\right]|\rho-\overline{\rho}|.
\end{aligned}
$$

Since $\Lambda_1<\frac{1}{2}$, then there exists a fixed point $\rho(\cdot)$ of the operator $\mathscr{F}$ w.r.t. the control function u by the Banach Contraction Principle. Thus this fixed point is the solution of the systems (1.1)-(1.2). Also,these control systems is controllable since (i) $(\mathscr{F}\rho)(\varsigma)=\varpi(\varsigma)=\rho(\varsigma)$ on $[-\tau,0]$ and (ii) $(\mathscr{F}\rho)(\varsigma)=\rho(\varsigma)$ with $(\mathscr{F}\rho)(T)=\rho(T)=\rho_1$ on $[0,\tau]$. Thus, it is concluded that the our systems is controllable on the whole interval $[-\tau,T]$. □

Theorem 3.2. *Let the assumptions (C1)-(C2) be satisfied. Suppose that*

(C3) The function $\zeta:\Delta\times\mathbb{R}\to\mathbb{R}$ is continuous and there is a constant $H>0$ such that

$$|\zeta(\varsigma,s,\rho)-\zeta(\varsigma,s,\overline{\rho})|\leq H\,|\rho-\overline{\rho}|.$$

(C4) The function $F:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there is a constant $P>0$ such that

$$|F(s,\rho_1,\overline{\rho}_1)-F(s,\rho_2,\overline{\rho}_2)|\leq P\left(|\rho_1-\rho_2|+|\overline{\rho}_1-\overline{\rho}_2|\right).$$

for all $\rho_i,\overline{\rho}_i\in\mathbb{R}$ $(i=1,2)$ and $\varsigma,s\in[0,T]$. Also we denote here that $\Delta:=\{(\varsigma,s):0\leq s\leq\varsigma\leq T\}$ and $\mathscr{H}\rho(\varsigma)=\int_0^{\varsigma}\zeta(\varsigma,s,\rho(s))ds$ for brevity.

Then the control system (1.3)-(1.4) is controllable provided that

$$\Lambda_2:=\Lambda_1+P(1+H)\frac{T^{\kappa}}{\Gamma(\kappa+1)}<1.$$

Proof. By analogy, we need to make the problem (1.3)-(1.4) into a fixed point problem. In the next step we shall analyse the operator

$$\Psi:\mathscr{C}\to\mathscr{C}$$

defined by

$$\Psi\rho(\varsigma)=\begin{cases}\varpi(\varsigma), & \varsigma\in[-\tau,0]\\ \varpi(0)+\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}f(s,\rho(s),\rho(g(s))ds-\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}Bu(s)ds+\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)ds, & \varsigma\in[0,T]\end{cases}$$

Similar to what was done above, we find the fixed point of the operator Ψ via a suitable control function u. If any $\rho(\varsigma),\overline{\rho}(\varsigma)$ satisfying $\Psi\rho(\varsigma)=\Psi\overline{\rho}(\varsigma)$ for $\varsigma\in[-\tau,0]$, then we take it in the interval $[0,T]$. Now let us determine the control function as follow:

$$u(\varsigma)=\overline{\mathscr{X}}^{-1}\Big(\rho_0-\rho_1+\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}f(s,\rho(s),\rho(g(s))ds\Big)-\int_0^{\varsigma}\frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)}F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)ds \quad \text{for all} \quad \rho_0,\rho_1\in\mathbb{R}.$$

Next we show that the operator Ψ has a fixed point with the following steps:

$$
\begin{aligned}
\left|\Psi\rho(\varsigma)-\Psi\overline{\rho}(\varsigma)\right| &\leq \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\left|f(s,\rho(s),\rho(g(s)))-f(s,\overline{\rho}(s),\overline{\rho}(g(s)))\right|ds \\
&+ \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\big(Bu(s)-B\overline{u}(s)\big)ds \\
&+ \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\left[F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)-F\Big(s,\overline{\rho}(s),\mathscr{H}\overline{\rho}(g(s))\Big)\right]ds \\
&\leq \Lambda_1+\frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\left|F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)-F\Big(s,\overline{\rho}(s),\mathscr{H}\overline{\rho}(g(s))\Big)\right|ds \\
&\leq \Lambda_1+P|\rho-\overline{\rho}|\frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}ds+PH|\rho-\overline{\rho}|\frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}ds \\
&\leq \underbrace{\left[\Lambda_1+P(1+H)\frac{T^{\kappa}}{\Gamma(\kappa+1)}\right]}_{\Lambda_2}|\rho-\overline{\rho}|.
\end{aligned}
$$

Since $\Lambda_2 < 1$, then there exists a fixed point $\rho(\cdot)$ of the operator Ψ w.r.t. the control function u by the Banach Contraction Principle. Therefore, this fixed point serves as the solution to the systems (1.3)-(1.4). Also,these control systems is controllable since (i) $(\Psi\rho)(\varsigma) = \varpi(\varsigma) = \rho(\varsigma)$ on $[-\tau, 0]$ and (ii) $(\Psi\rho)(\varsigma) = \rho(\varsigma)$ with $(\Psi\rho)(T) = \rho(T) = \rho_1$ on $[0, \tau]$. Hence, we conclude that the systems is controllable on the interval $[-\tau, T]$. Thus the proof is complete. □

Theorem 3.3. *Let the assumptions (C1)-(C4) be satisfied. Suppose that the function $\xi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a constant $\delta > 0$ such that*

$$|\xi(\varsigma,\rho(\varsigma)) - \xi(\varsigma,\overline{\rho}(\varsigma))| \leq \delta\,|\rho - \overline{\rho}|.$$

Then the control system (1.5)-(1.6) is controllable provided that

$$\Lambda_3 := (\delta + M)\left(\frac{KT^{\kappa}}{\Gamma(\kappa+1)} + 1\right) < 1$$

where the constant $M = \frac{T^{\kappa}}{\Gamma(\kappa+1)}(2L + P(1+HT))$.

Proof. Analogously,let us turn the problem (1.5)-(1.6) into a fixed point problem.

$$\Theta : \mathscr{C} \to \mathscr{C}$$

defined by

$$\Theta\rho(\varsigma) = \begin{cases} \varpi(\varsigma), \quad \varsigma \in [-\tau, 0] \\ \varpi(0) + \xi(0,\rho_0) - \xi(\varsigma,\rho(g(\varsigma))) + \int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} f(s,\rho(s),\rho(g(s))ds \\ -\int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} Bu(s)ds + \int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)ds, \ \varsigma \in [0,T] \end{cases}$$

Analogously, let us begin with the steps above. If any $\rho(\varsigma), \overline{\rho}(\varsigma)$ satisfying $\Theta\rho(\varsigma) = \Theta\overline{\rho}(\varsigma)$ for $\varsigma \in [-\tau, 0]$, then we take $\varsigma \in [0, T]$. Now let us determine the control function as follow:

$$\begin{aligned} u(\varsigma) \quad &= \quad \overline{\mathscr{X}}^{-1}\Big(\rho_0 - \rho_1 + \xi(0,\rho_0) - \xi(\varsigma,\rho(g(\varsigma))) + \int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} f(s,\rho(s),\rho(g(s))ds\Big) \\ &- \quad \int_0^{\varsigma} \frac{(\varsigma-s)^{\kappa-1}}{\Gamma(\kappa)} F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big)ds \quad \text{for all} \quad \rho_0,\rho_1 \in \mathbb{R}. \end{aligned}$$

Next we show that the operator Θ has a fixed point:

$$\begin{aligned} \big|\Theta\rho(\varsigma) - \Theta\overline{\rho}(\varsigma)\big| \quad &\leq \quad \big|\xi(\varsigma,\rho(g(\varsigma))) - \xi(\varsigma,\overline{\rho}(g(\varsigma)))\big| \\ &\leq \quad \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\big|f(s,\rho(s),\rho(g(s))) - f(s,\overline{\rho}(s),\overline{\rho}(g(s)))\big|ds \\ &+ \quad \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\big(Bu(s) - B\overline{u}(s)\big)ds \\ &+ \quad \frac{1}{\Gamma(\kappa)}\int_0^{\varsigma}(\varsigma-s)^{\kappa-1}\Big[F\Big(s,\rho(s),\mathscr{H}\rho(g(s))\Big) - F\Big(s,\overline{\rho}(s),\mathscr{H}\overline{\rho}(g(s))\Big)\Big]ds \\ &\leq \quad \delta\big|\rho - \overline{\rho}\big| + 2L\frac{T^{\kappa}}{\Gamma(\kappa+1)}\big|\rho - \overline{\rho}\big| \\ &+ \quad \frac{KT^{\kappa}}{\Gamma(\kappa+1)}\Big[\delta + \frac{T^{\kappa}}{\Gamma(\kappa+1)}(2L + P(1+HT))\Big]\big|\rho - \overline{\rho}\big| \\ &+ \quad \frac{T^{\kappa}}{\Gamma(\kappa+1)}P(1+HT)\big|\rho - \overline{\rho}\big| = (\delta + M)\left(\frac{KT^{\kappa}}{\Gamma(\kappa+1)} + 1\right) \end{aligned}$$

Since $\Lambda_3 < 1$, then there exists a fixed point $\rho(\cdot)$ of the operator Θ w.r.t. the control function u by the Banach Contraction Principle. Consequently, this fixed point gives the solution to the systems (1.5)-(1.6). Also, these control systems is controllable since (i) $(\Theta\rho)(\varsigma) = \varpi(\varsigma) = \rho(\varsigma)$ on $[-\tau, 0]$ and (ii) $(\Theta\rho)(\varsigma) = \rho(\varsigma)$ with $(\Theta\rho)(T) = \rho(T) = \rho_1$ on $[0, \tau]$. Hence, we conclude that the systems is controllable on the entire interval $[-\tau, T]$. Thus the proof is complete.

□

4. Examples

Example 4.1. *Let us examine the fractional-order differential equation below:*

$$\begin{cases} {}^{c}D^{\frac{1}{2}}\rho(\varsigma) = \frac{|\rho(\varsigma)|}{8+8|\rho(\varsigma)|} + \frac{\cos\rho(\varsigma^2)}{8} - u(\varsigma) & \varsigma \in [0,1] \\ \quad \rho(\varsigma) = \varsigma & \varsigma \in [-1,0]. \end{cases} \tag{4.1}$$

Let $f(\varsigma,\rho,\overline{\rho}) = \frac{|\rho|}{8+8|\rho|} + \frac{\cos\overline{\rho}}{8}$ and $g(\varsigma) = \varsigma^2$. It is obtained that

$$|f(\varsigma,\rho_1,\overline{\rho}_1) - f(\varsigma,\rho_2,\overline{\rho}_2,)| \leq \frac{1}{8}\Big(|\rho_1 - \rho_2| + |\overline{\rho}_1 - \overline{\rho}_2|\Big)$$

for all $\rho_i, \overline{\rho}_i \in \mathbb{R}$ *where* $i = 1,2$ *with* $\varsigma \in [0,1]$ *and* u *is a control function. Now, assume that the operator* $\mathscr{X} : L^2([0,T],U) \to \mathbb{R}$ *defined by*

$$\mathscr{X}u := \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\varsigma}(\varsigma - s)^{-\frac{1}{2}}Bu(s)ds.$$

induces a bounded inverse operator $\overline{\mathscr{X}}^{-1}$ *on the coset* $L^2([0,T],U)/ker\mathscr{X}$. *Choose* $K < 2$ *in such a way* $\Lambda_1 < \frac{1}{2}$. *Indeed,*

$$\Lambda_1 = \frac{1}{8}\left(\frac{1}{\Gamma(\frac{1}{2}+1)} + K\frac{1}{\Gamma(\frac{1}{2}+1)^2}\right) < \frac{1}{2} \iff \Lambda_1 = \left(\frac{1}{\Gamma(\frac{1}{2}+1)} + K\frac{1}{\Gamma(\frac{1}{2}+1)^2}\right) < 4 \iff K < 2,2552$$

where $\Gamma(\frac{1}{2}+1) \approx 0,8862$. *Since the all required conditions of Theorem 3.1 are satisfied, then we obtain that the problem (4.1) is controllable.*

Example 4.2. *Now, let us think the following integro-fractional order differential equation:*

$$\begin{cases} {}^cD^{\frac{1}{2}}\rho(\varsigma) = \frac{\sin\rho(\varsigma)}{10} + \frac{\rho(\varsigma-1)}{10} - u(\varsigma) + \frac{e^{-\varsigma}|\rho(\varsigma)|}{(2+e^{-\varsigma})(1+|\rho(\varsigma)|)} & \\ \qquad + \frac{1}{3}\int_0^{\varsigma} e^{\frac{-1}{3}\rho(g(s))}ds & \varsigma \in [0,1] \\ \rho(\varsigma) = e^{\varsigma} & \varsigma \in [-1,0]. \end{cases} \tag{4.2}$$

Let $f(\varsigma,\rho,\overline{\rho}) = \frac{1}{10}\left(\sin\rho + \overline{\rho}\right)$, $g(\varsigma) = \varsigma - 1$ *and* $F\left(\varsigma,\rho,\mathscr{H}\rho\right) = \frac{e^{-\varsigma}|\rho(\varsigma)|}{(2+e^{-\varsigma})(1+|\rho(\varsigma)|)} + \mathscr{H}\rho$ *where* $\mathscr{H}\rho(\varsigma) = \frac{1}{3}\int_0^{\varsigma} e^{\frac{-1}{3}\rho(s)}ds$. *It is hold that*

$$\begin{aligned} |f(\varsigma,\rho_1,\overline{\rho}) - f(\varsigma,\rho_2,\overline{\rho})| &\leq \frac{1}{10}\Big(|\rho_1 - \rho_2| + |\overline{\rho}_1 - \overline{\rho}_2|\Big) \\ |\mathscr{H}\rho - \mathscr{H}\overline{\rho}| &\leq \frac{1}{3}|\rho - \overline{\rho}| \\ |F(\varsigma,\rho,\mathscr{H}\rho) - F(\varsigma,\overline{\rho},\mathscr{H}\overline{\rho})| &\leq \frac{1}{3}(|\rho - \overline{\rho}| + |\mathscr{H}\rho - \mathscr{H}\overline{\rho}|). \end{aligned}$$

for all $\rho_i, \overline{\rho} \in \mathbb{R}$ *where* $i = 1,2$ *with* $\varsigma \in [0,1]$ *and* u *is a control function. Therefore,* $H = \frac{1}{3}$, $P = \frac{1}{3}$. *Using similar calculations above we have that* $\Lambda_1 = 0,240161$ *and find that* $\Lambda_2 = 0,741663 < 1$. *Since the remaining conditions of Theorem 3.2 are satisfied, then we obtain that the problem (4.2) is controllable.*

Conclusion

In this paper we have established the existence and uniqueness of solutions for several fractional-order delay differential equations, while demonstrating their controllability properties. We began by analysing a fractional-order delay control system with a nonzero control function, and successfully applied the Banach fixed point theorem to show that the solution exists and is unique. Meanwhile, in Theorem 3.1, we have shown that the system is controllable using the function detailed in Remark 2.3. We then introduced an additional function involving integral part on the right-handside with delay function, which led to a more complex integro-fractional-order delayed system. Here we imposed a Lipschitz condition on the new function to verify the existence and uniqueness of solutions and the controllability property in Theorem 3.2. In addition, we extended our work to an integro-fractional-order hybrid delayed system by incorporating another delayed function into the Caputo derivative operator. This presented different challenges, particularly in formulating appropriate conditions for our analyses in Theorem 3.3. Despite these difficulties, our new approach, which included the introduction of delay functions on both the right and left sides of the equations, provides a uncomplicated perspective on control systems in fractional calculus and enhances the understanding of their dynamics. The use of Banach fixed point theory as the basis for the proofs of these theorems emphasizes the simplicity and clarity of our method and makes it applicable to different types of control systems.

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