Journal of New Theory

ISSN: 2149-1402

48 (2024) 24-39

Journal of New Theory

https://dergipark.org.tr/en/pub/jnt

Open Access

Bi-f-Harmonic Legendre Curves on (α, β)-Trans-Sasakian Generalized Sasakian Space Forms

Şerife Nur Bozdağ[1], Selcen Yüksel Perktaş[2], Feyza Esra Erdoğan[3]

Article Info

Received: 01 Jul 2024

Accepted: 14 Sep 2024

Published: 30 Sep 2024

doi:10.53570/jnt.1508392

Research Article

Abstract — In this study, we consider bi-f-harmonic Legendre curves on (α, β)-trans-Sasakian generalized Sasakian space form. We provide the necessary and sufficient conditions for a Legendre curve to be bi-f-harmonic on (α, β)-trans-Sasakian generalized Sasakian space form without any restrictions by a main theorem. Afterward, we investigate these conditions under nine different cases. As a result of these investigations, we obtain the original theorems and corollaries as well as the nonexistence theorems. We perform these investigations according to the ρ_2 and ρ_3 functions from the curvature tensor of the (α, β)-trans-Sasakian generalized Sasakian space form, the curvature and torsion of the bi-f-harmonic Legendre curve, and finally, the positions of the basis vectors relative to each other.

Keywords *Bi-f-harmonic curves, Legendre curves, trans-Sasakian space forms, generalized Sasakian space forms*

Mathematics Subject Classification (2020) 53C25, 53C43

1. Introduction

Let $\mathbb{M}$ and $\mathbb{N}$ be Riemannian manifolds. Then, a map $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$ is called harmonic if it is a critical point of energy functional given by

$$E(\dot{\vartheta}) = \frac{1}{2} \int_{\mathbb{M}} \left| d\dot{\vartheta} \right|^2 v_g$$

Moreover, harmonic maps are defined as solutions of the corresponding Euler-Lagrange equation which is a non-linear elliptic partial differential equation characterized by the vanishing of the tension field $\dot{\tau}(\dot{\vartheta}) = \mathrm{trace} \nabla d\dot{\vartheta}$.

The bienergy functional of a map $\dot{\vartheta}$ is introduced by Eells and Sampson [1] as follows:

$$V_2(\dot{\vartheta}) = \frac{1}{2} \int_{\mathbb{M}} \left| \dot{\tau}(\dot{\vartheta}) \right|^2 v_g$$

Here, if $\dot{\vartheta}$ is a critical point of the bienergy functional, then it is called a biharmonic map. The Euler-Lagrange equation of $V_2(\dot{\vartheta})$ which is characterized by the vanishing of the bitension field is obtained by Jiang [2] as

$$\dot{\tau}_2(\dot{\vartheta}) = -\triangle \dot{\tau}(\dot{\vartheta}) - \mathrm{trace} \mathcal{R}^{\mathbb{N}}(d\dot{\vartheta}, \dot{\tau}(\dot{\vartheta}))d\dot{\vartheta}$$

Here, $\mathcal{R}^{\mathbb{N}}(\mathcal{X}, \mathcal{Y}) = [\nabla_{\mathcal{X}}, \nabla_{\mathcal{Y}}] - \nabla_{[\mathcal{X},\mathcal{Y}]}$ is the curvature operator of $\mathbb{N}$ and $\Delta = -\mathrm{trace}(\nabla^{\dot{\vartheta}} \nabla^{\dot{\vartheta}} - \nabla^{\dot{\vartheta}}_{\nabla})$ is the rough Laplacian on the sections of $\dot{\vartheta}^{-1}T\mathbb{N}$. If $\dot{\tau}_2(\dot{\vartheta}) = 0$, then $\dot{\vartheta}$ is called as a biharmonic map.

[1]serife.nur.yalcin@ege.edu.tr (Corresponding Author) [2]sperktas@adiyaman.edu.tr; [3]feyza.esra.erdogan@ege.edu.tr
[1,3]Department of Mathematics, Faculty of Science, Ege University, İzmir, Türkiye
[2]Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University, Adıyaman, Türkiye

f-harmonic maps are defined as critical points of f-energy functional

$$V_f(\dot{\vartheta}) = \frac{1}{2}\int_{\mathbb{M}} f\left|d\dot{\vartheta}\right|^2 v_g$$

for the maps $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$ where $f \in C^\infty(\mathbb{M}, \mathbb{R})$ [3]. The Euler-Lagrange equation is given by $\dot{\tau}_f(\dot{\vartheta}) = f\dot{\tau}(\dot{\vartheta}) + d\dot{\vartheta}(\text{grad}f)$ where $\dot{\tau}(\dot{\vartheta}) \equiv \text{trace}\nabla d\dot{\vartheta}$ is the tension field of $\dot{\vartheta}$.

The critical points of the f-bienergy functional

$$V_{2,f}\left(\dot{\vartheta}\right) = \frac{1}{2}\int_{\mathbb{M}} f\left|\dot{\tau}(\dot{\vartheta})\right|^2 v_g$$

for maps $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$ is called as f-biharmonic maps. The Euler-Lagrange equation provides the f-biharmonic map equation as

$$\dot{\tau}_{2,\,f}\left(\dot{\vartheta}\right) \equiv f\,\dot{\tau}_2(\dot{\vartheta}) - (\Delta f)\dot{\tau}(\dot{\vartheta}) - 2\nabla^{\dot{\vartheta}}_{gradf}\,\dot{\tau}(\dot{\vartheta})$$

which is called f-bitension field of map $\dot{\vartheta}$ [4].

Bi-f-harmonic maps are defined as critical points of the bi-f-energy functional

$$V_{f,2}\left(\dot{\vartheta}\right) = \frac{1}{2}\int_{\mathbb{M}} \left|\dot{\tau}_f\left(\dot{\vartheta}\right)\right|^2 v_g$$

for maps $\dot{\vartheta} : (\mathbb{M}, g) \to (\mathbb{N}, h)$. The Euler-Lagrange equation provides the bi-f-harmonic map equation [5]:

$$\dot{\tau}_{f,2}\left(\dot{\vartheta}\right) \equiv f\,J^{\dot{\vartheta}}(\dot{\tau}_f\left(\dot{\vartheta}\right)) - \nabla^{\dot{\vartheta}}_{\text{grad}f}\,\dot{\tau}_f\left(\dot{\vartheta}\right) \tag{1.1}$$

where $J^{\dot{\vartheta}}$ is the Jacobi operator of the map defined by

$$J^{\dot{\vartheta}}(\mathcal{X}) = -Tr\,(\nabla^{\dot{\vartheta}}\nabla^{\dot{\vartheta}}\mathcal{X} - \nabla^{\dot{\vartheta}}_{\nabla^{\mathbb{M}}}\mathcal{X} - \mathcal{R}^{\mathbb{N}}(d\dot{\vartheta}, \mathcal{X})d\dot{\vartheta})$$

It is obvious that if f is a constant function, then f-biharmonic and bi-f-harmonic maps become biharmonic maps. Bi-f-harmonic and f-biharmonic maps which are not biharmonic are called proper bi-f-harmonic and proper f-biharmonic maps, respectively. For more details about bi-f-harmonic maps, see [4–6].

The notion of generalized Sasakian space forms was introduced by Alegre et al. [7]. Sarkar et.al. [8] studied Legendre curves in 3-dimensional trans-Sasakian manifolds. Then, Fetcu [9] handled biharmonic Legendre curves in Sasakian space forms. Moreover, Güvenç and Özgür [10, 11] investigated some classes of biharmonic Legendre curves in generalized Sasakian space forms and f-biharmonic Legendre curves in Sasakian space forms. In addition, for recent studies, see [12–14].

In this paper, we study bi-f-harmonic Legendre curves in (α, β)-trans-Sasakian generalized Sasakian space forms and provide some characterizations for bi-f-harmonicity of such curves under some special assumptions.

2. Generalized Sasakian Space Forms

In this section, we provide some basic definitions about almost contact metric manifolds and generalized Sasakian space forms in [7, 15].

$\mathbb{M}^{(2n+1)}$ is defined as an almost contact manifold with the almost contact structure $(\dot{\vartheta}, \varsigma, \dot{\eta})$ if a tensor field $\dot{\vartheta}$ of type $(1, 1)$, a vector field ς, and a $1-$form $\dot{\eta}$ satisfy the followings

$$\dot{\vartheta}^2 = -I + \dot{\eta} \otimes \varsigma \tag{2.1}$$

and

$$\dot{\eta}(\varsigma) = 1$$

Here, I denotes the identity transformation. As an consequence of the conditions (2.1), $\dot{\vartheta}\varsigma = 0$ and $\dot{\eta}\circ\dot{\vartheta} = 0$.

Let $\mathbb{M}^{(2n+1)}$ be an almost contact manifold with an almost contact structure $(\dot{\vartheta},\varsigma,\dot{\eta})$. If it admits a Riemannian metric g such that

$$g(\dot{\vartheta}\mathcal{X},\dot{\vartheta}\mathcal{Y}) = g(\mathcal{X},\mathcal{Y}) - \dot{\eta}(\mathcal{X})\dot{\eta}(\mathcal{Y}), \quad \mathcal{X},\,\mathcal{Y}\in\Gamma(T\mathbb{M}) \tag{2.2}$$

then it becomes an almost contact metric manifold with an almost contact metric structure $(\dot{\vartheta},\varsigma,\dot{\eta},g)$. From (2.2),

$$g(\mathcal{X},\dot{\vartheta}\mathcal{Y}) = -g(\dot{\vartheta}\mathcal{X},\mathcal{Y})$$

and

$$g(\mathcal{X},\varsigma) = \dot{\eta}(\mathcal{X})$$

for any $\mathcal{X},\,\mathcal{Y}\in T\mathbb{M}$. The fundamental 2-form of $\mathbb{M}$ is defined by

$$\Phi(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\dot{\vartheta}\mathcal{Y})$$

An almost contact metric structure becomes a contact metric structure if

$$g(\mathcal{X},\dot{\vartheta}\mathcal{Y}) = d\dot{\eta}(\mathcal{X},\mathcal{Y})$$

for all vector fields $\mathcal{X},\mathcal{Y}\in\Gamma(T\mathbb{M})$, where

$$d\dot{\eta}(\mathcal{X},\mathcal{Y}) = \frac{1}{2}\{\mathcal{X}\dot{\eta}(\mathcal{Y}) - \mathcal{Y}\dot{\eta}(\mathcal{X}) - \dot{\eta}([\mathcal{X},\mathcal{Y}])\}$$

A contact metric manifold with a Killing Reeb vector field ς is called a K-contact manifold. An almost contact metric manifold is called normal if

$$\mathcal{N}_{\dot{\vartheta}}(\mathcal{X},\mathcal{Y}) + 2d\dot{\eta}(\mathcal{X},\mathcal{Y})\varsigma = 0$$

where $\mathcal{N}$ is the Nijenhuis torsion tensor of $\dot{\vartheta}$ given by

$$\mathcal{N}_{\dot{\vartheta}}(\mathcal{X},\mathcal{Y}) = \dot{\vartheta}^2[\mathcal{X},\mathcal{Y}] + \left[\dot{\vartheta}\mathcal{X},\dot{\vartheta}\mathcal{Y}\right] - \dot{\vartheta}\left[\dot{\vartheta}\mathcal{X},\mathcal{Y}\right] - \dot{\vartheta}\left[\mathcal{X},\dot{\vartheta}\mathcal{Y}\right]$$

for all $\mathcal{X},\,\mathcal{Y}\in\Gamma(T\mathbb{M})$. A contact normal metric manifold is said to be a Sasakian manifold. Besides, an almost contact metric manifold is called a Sasakian manifold if and only if

$$(\nabla_{\mathcal{X}}\dot{\vartheta})\mathcal{Y} = g(\mathcal{X},\mathcal{Y})\varsigma - \dot{\eta}(\mathcal{Y})\mathcal{X}$$

for any $\mathcal{X},\,\mathcal{Y}\in\Gamma(T\mathbb{M})$.

An almost contact metric manifold is called a Kenmotsu manifold if and only if $d\dot{\eta} = 0$ and $d\Phi = 2\dot{\eta}\wedge\Phi$, or equivalently

$$(\nabla_{\mathcal{X}}\dot{\vartheta})\mathcal{Y} = -\dot{\eta}(\mathcal{Y})\dot{\vartheta}\mathcal{X} - g(\mathcal{X},\dot{\vartheta}\mathcal{Y})\varsigma$$

Hence,

$$\nabla_{\mathcal{X}}\varsigma = \mathcal{X} - \dot{\eta}(\mathcal{X})\varsigma$$

Finally, an almost contact metric manifold is called a cosymplectic manifold if and only if $d\dot{\eta} = 0$ and $d\Phi = 0$, or equivalently $\nabla\dot{\vartheta} = 0$ and thus $\nabla\varsigma = 0$.

As a generalization of Kenmotsu and Sasakian manifolds, (α,β)-trans-Sasakian manifolds were introduced by Oubiña [16]. If there exist two functions α and β on an almost contact metric manifold $\mathbb{M}$ satisfying

$$\left(\nabla_{\mathcal{X}}\dot{\vartheta}\right)\mathcal{Y} = \alpha\left(g(\mathcal{X},\mathcal{Y})\varsigma - \dot{\eta}(\mathcal{Y})\mathcal{X}\right) + \beta\left(g\left(\dot{\vartheta}\mathcal{X},\mathcal{Y}\right)\varsigma - \dot{\eta}(\mathcal{Y})\dot{\vartheta}\mathcal{X}\right)$$

for any $\mathcal{X},\,\mathcal{Y}\in\Gamma(T\mathbb{M})$, then $\mathbb{M}$ is called a trans-Sasakian manifold.

Here,

i. if $\beta = 0$, then $\mathbb{M}$ is called a α-Sasakian manifold,

ii. if $\beta = 0$ and $\alpha = 1$, then $\mathbb{M}$ is called a Sasakian manifold,

iii. if $\alpha = 0$, then $\mathbb{M}$ is called a β-Kenmotsu manifold,

iv. if $\beta = 1$ and $\alpha = 0$, then $\mathbb{M}$ is called a Kenmotsu manifolds, and

v. if $\alpha = \beta = 0$, then $\mathbb{M}$ is a cosymplectic manifold.

For a trans-Sasakian manifold,

$$\nabla_{\mathcal{X}} \varsigma = -\alpha \dot{\vartheta} \mathcal{X} + \beta \left(\mathcal{X} - \dot{\eta}(\mathcal{X}) \varsigma \right)$$

and

$$d\dot{\eta} = \alpha \Phi$$

De and Tripathi [17] showed that on an (α,β)-trans-Sasakian manifold the following relation is hold:

$$\varsigma(\alpha) + 2\alpha\beta = 0$$

It was shown in [18] that an (α,β)-trans-Sasakian manifold with dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.

A $\dot{\vartheta}$-section of an almost contact metric manifold $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ at a point $p \in \mathbb{M}$ is a section $\Pi \subseteq T_p\mathbb{M}$ spanned by a unit vector field $\mathcal{X}_p$ orthogonal to ς_p and $\dot{\vartheta}\mathcal{X}_p$. The $\dot{\vartheta}$-sectional curvature $\mathcal{K}(\mathcal{X} \wedge \dot{\vartheta}\mathcal{X})$ is defined by

$$\mathcal{K}(\mathcal{X} \wedge \dot{\vartheta}\mathcal{X}) = \mathcal{R}(\mathcal{X}, \dot{\vartheta}\mathcal{X}, \dot{\vartheta}\mathcal{X}, \mathcal{X})$$

If $\dot{\vartheta}$-sectional curvature of $\mathbb{M}$ is constant, then it is called a space form.

Moreover, an almost contact metric manifold is called a generalized Sasakian space form [7] if there exist functions ρ_1, ρ_2, and ρ_3 on $\mathbb{M}$ such that

$$\begin{aligned} \mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = {} & \rho_1 \left\{ g(\mathcal{Y},\mathcal{Z})\mathcal{X} - g(\mathcal{X},\mathcal{Z})\mathcal{Y} \right\} + \rho_2 \left\{ g(\mathcal{X}, \dot{\vartheta}\mathcal{Z})\dot{\vartheta}\mathcal{Y} - g(\mathcal{Y}, \dot{\vartheta}\mathcal{Z})\dot{\vartheta}\mathcal{X} + 2g(\mathcal{X}, \dot{\vartheta}\mathcal{Y})\dot{\vartheta}\mathcal{Z} \right\} \\ & + \rho_3 \left\{ \dot{\eta}(\mathcal{X})\dot{\eta}(\mathcal{Z})\mathcal{Y} - \dot{\eta}(\mathcal{Y})\dot{\eta}(\mathcal{Z})\mathcal{X} + g(\mathcal{X},\mathcal{Z})\dot{\eta}(\mathcal{Y})\varsigma - g(\mathcal{Y},\mathcal{Z})\dot{\eta}(\mathcal{X})\varsigma \right\} \end{aligned} \tag{2.3}$$

for any vector fields on $\mathbb{M}$, where $\mathcal{R}$ denotes the curvature tensor of $\mathbb{M}$.

For a generalized Sasakian-space-form;

i. if $\rho_1 = \frac{c+3}{4}$ and $\rho_2 = \rho_3 = \frac{c-1}{4}$, then it becomes a Sasakian-space-form,

ii. if $\rho_1 = \frac{c-3}{4}$ and $\rho_2 = \rho_3 = \frac{c+1}{4}$, then it becomes a Kenmotsu-space-form, and

iii. if $\rho_1 = \rho_2 = \rho_3 = \frac{c}{4}$, then it becomes a cosymplectic-space-form

where c is the constant $\dot{\vartheta}$-sectional curvature. The contact distribution of an almost contact metric manifold $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is defined by

$$\{\mathcal{X} \in \Gamma(T\mathbb{M}) : \dot{\eta}(\mathcal{X}) = 0\}$$

and an integral curve of the contact distribution is called a Legendre curve [15].

3. Bi-f-Harmonic Curves

Recall the bi-f-harmonic map equation for curves in Riemannian and start with the important proposition for Euler-Lagrange equation of bi-f-harmonic maps [5].

Proposition 3.1. Let $\dot{\vartheta}:(\mathbb{M},g)\to(\mathbb{N},h)$ be a smooth map between Riemannian manifolds. Then, $\dot{\vartheta}$ is a bi-f-harmonic map if and only if its bi-f-tension field

$$\dot{\tau}_{f,2}(\dot{\vartheta})=\Delta_f^2\dot{\tau}_f\left(\dot{\vartheta}\right)-f\,\mathrm{trace}_g\mathcal{R}^{\mathbb{N}}\left(\dot{\tau}_f\left(\dot{\vartheta}\right),d\dot{\vartheta}\right)d\dot{\vartheta} \tag{3.1}$$

vanishes, where

$$\Delta_f^2\dot{\tau}_f\left(\dot{\vartheta}\right)=-\mathrm{trace}_g\left(\nabla^{\dot{\vartheta}}f\left(\nabla^{\dot{\vartheta}}\dot{\tau}_f\left(\dot{\vartheta}\right)\right)-f\,\nabla^{\dot{\vartheta}}_{\nabla^{\mathbb{M}}}\dot{\tau}_f\left(\dot{\vartheta}\right)\right) \tag{3.2}$$

and $\dot{\tau}_f\left(\dot{\vartheta}\right)$ is the f-tension field given by (1.1).

By considering a curve, from (3.1) and (3.2), from [6], the following proposition is hold:

Proposition 3.2. Let $\sigma:I\to(\mathbb{N},h)$ be a curve parameterized by arclenght on a Riemannian manifold $(\mathbb{N},h)$ and $\sigma'=T$. Then, σ is a bi-f-harmonic curve if and only if

$$(ff'')'T+\left(2\left(f'\right)^2+3f''f\right)\nabla_T^{\mathbb{N}}T+4f'f\,\nabla_T^2T+f^2\nabla_T^3T+f^2\mathcal{R}^{\mathbb{N}}\left(\nabla_T^{\mathbb{N}}T,T\right)T=0$$

where $f:I\to\mathbb{R}^+$, I is an interval, $\nabla_T^2T=\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}T$, and $\nabla_T^3T=\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}T$.

Assume that $\sigma:I\to(\mathbb{N},h)$ is a arclenght parameterized curve in an $n-$dimensional Riemannian manifold $(\mathbb{N},h)$. If there exist ortonormal vector fields $V_1,V_2,\cdots,V_r$ along σ such that

$$\begin{aligned}\nabla_TV_1&=k_1V_2\\ \nabla_TV_2&=-k_1V_1+k_2V_3\\ &\vdots\\ \nabla_TV_r&=-k_{r-1}V_{r-1}\end{aligned} \tag{3.3}$$

then σ is called a Frenet curve of osculating order r, for $1\le r\le n$. Here, $V_1=\sigma'=T$ is the unit tangent vector field of σ, V_2 is the unit normal vector field of σ with the same direction as ∇_TV_1, and the vectors $V_3,V_4,\cdots,V_r$ are the unit vectors obtained from the Frenet equations for σ, where $k_1=\|\nabla_TV_1\|$ and $k_2,k_3,\cdots,k_{r-1}$ are real-valued positive functions.

From (3.3),

$$\nabla_T^2T=\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}T=-k_1^2V_1+k_1'V_2+k_1k_2V_3$$

$$\nabla_T^3T=\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}\nabla_T^{\mathbb{N}}T=-3k_1k_1'V_1+\left(k_1''-k_1^3-k_1k_2^2\right)V_2+\left(2k_1'k_2+k_1k_2'\right)V_3+k_1k_2k_3V_4$$

and

$$\mathcal{R}^{\mathbb{N}}\left(\nabla_T^{\mathbb{N}}T,T\right)T=k_1\mathcal{R}^{\mathbb{N}}\left(V_2,T\right)T$$

Then,

$$\begin{aligned}\dot{\tau}_{f,2}(\sigma)=&\left((ff'')'-3k_1k_1'f^2-4k_1^2ff'\right)T+\left(\left(-k_1^3-k_1k_2^2+k_1''\right)f^2+4k_1'ff'+3k_1f''f+2k_1(f')^2\right)V_2\\ &+\left(4k_1k_2ff'+f^2\left(2k_2k_1'+k_1k_2'\right)\right)V_3+\left(k_1k_2k_3f^2\right)V_4+k_1f^2\mathcal{R}^{\mathbb{N}}(V_2,T)T\end{aligned}$$

Theorem 3.3. Let $\sigma:I\to(\mathbb{N},h)$ be a arclenght parameterized curve on a Riemannian manifold $(\mathbb{N},h)$. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned}0=&\left((ff'')'-3k_1k_1'f^2-4k_1^2ff'\right)T+\left(\left(-k_1^3-k_1k_2^2+k_1''\right)f^2+4k_1'ff'+3k_1f''f+2k_1(f')^2\right)V_2\\ &+\left(4k_1k_2ff'+f^2\left(2k_2k_1'+k_1k_2'\right)\right)V_3+\left(k_1k_2k_3f^2\right)V_4+k_1f^2\mathcal{R}^{N}(V_2,T)T\end{aligned} \tag{3.4}$$

4. Bi-f-harmonic Curves in (α, β)-Trans-Sasakian Generalized Sasakian Space Forms

In this section, we first obtain bi-f-harmonic equation of a curve $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ on an (α, β)-trans-Sasakian generalized Sasakian space form. Note that, throughout this paper, we use (α, β)-TSGSSF instead of (α, β)-trans-Sasakian generalized Sasakian space form and cons instead of constant in equations for the sake of simplicity. By using (2.3),

$$\begin{aligned}\mathcal{R}^{\mathbb{M}}(V_2, T)T =& \rho_1 \{g(T,T)V_2 - g(V_2,T)T\} + \rho_2 \left\{g(V_2, \dot{\vartheta}T)\dot{\vartheta}T - g(T, \dot{\vartheta}T)\dot{\vartheta}V_2 - 2g(T, \dot{\vartheta}V_2)\dot{\vartheta}T\right\} \\ &+\rho_3 \{\dot{\eta}(V_2)\dot{\eta}(T)T - \dot{\eta}(T)\dot{\eta}(T)V_2 + g(V_2,T)\dot{\eta}(T)\varsigma - g(T,T)\dot{\eta}(V_2)\varsigma\}\end{aligned}$$

which implies

$$\mathcal{R}^{\mathbb{M}}(V_2, T)T = \rho_3\dot{\eta}(T)\dot{\eta}(V_2)T + \left(\rho_1 - \rho_3 \left(\dot{\eta}(T)\right)^2\right) V_2 - 3\rho_2 g(T, \dot{\vartheta}V_2)\dot{\vartheta}T - \rho_3\dot{\eta}(V_2)\varsigma$$

From (3.4), we get bi-f-tension field of σ.

Theorem 4.1. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned}0 =& \left((ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' + k_1 f^2\rho_3\dot{\eta}(T)\dot{\eta}(V_2)\right) T \\ &+ \left(\left(-k_1^3 - k_1k_2^2 + k_1''\right) f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2 \left(\rho_1 - \rho_3 \left(\dot{\eta}(T)\right)^2\right)\right) V_2 \\ &+ \left(4k_1k_2 ff' + f^2 \left(2k_2k_1' + k_1k_2'\right)\right) V_3 + \left(k_1k_2k_3 f^2\right) V_4 + 3\rho_2 k_1 f^2 g(\dot{\vartheta}T, V_2)\dot{\vartheta}T - \rho_3 k_1 f^2\dot{\eta}(V_2)\varsigma\end{aligned}$$

For the remaining parts of this study, we consider that $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a Legendre curve in an (α, β)-TSGSSF. If σ is a Legendre curve, then

$$\dot{\eta}(V_2) = -\frac{\beta}{k_1} \tag{4.1}$$

Since σ is a Legendre curve, from (4.1), it is obvious that $V_2 \perp \varsigma$ if and only if $\beta = 0$ [19].

Corollary 4.2. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a Legendre curve parameterized by its arclenght on an (α, β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

$$\begin{aligned}0 =& \left((ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f'\right) T \\ &\left((-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f'' f + 2(f')^2 k_1 + k_1 f^2\rho_1\right) V_2 \\ &+ f \left(2k_2k_1' f + k_1k_2' f + 4k_1k_2 f'\right) V_3 + \left(k_1k_2k_3 f^2\right) V_4 + 3\rho_2 k_1 f^2 g(\dot{\vartheta}T, V_2)\dot{\vartheta}T + \beta\rho_3 f^2\varsigma\end{aligned} \tag{4.2}$$

Let $m = \min\{r, 4\}$. From (4.2), σ is a bi-f-harmonic Legendre curve if and only if

i. $\rho_2 = 0$ or $\dot{\vartheta}T \perp V_2$ or $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$

ii. $\rho_3 = 0$ or $\varsigma \in \text{span}\{V_2, V_3 \cdots, V_m\}$

iii. $g(\dot{\tau}_{f,2}(\sigma), V_i) = 0$, for all $i \in \{1, 2, \cdots, m\}$

Theorem 4.3. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a Legendre curve parameterized by its arclenght on an (α, β)-TSGSSF. Then, σ is a bi-f-harmonic curve if and only if

i. $\rho_2 = 0$ or $\dot{\vartheta}T \perp V_2$ or $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$

ii. $\rho_3 = 0$ or $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$

iii. The following equations are satisfied:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2ff' = 0 \\ \left\{ \begin{array}{c} (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1'f'f + 3k_1f''f + 2(f')^2k_1 \\ +3f^2k_1\rho_2 g(\dot{\vartheta}T, V_2)^2 + f^2\rho_3\beta\dot{\eta}(V_2) \end{array} \right. = 0 \\ 4k_1k_2ff' + f^2\ (2k_2k_1' + k_1k_2') + 3\rho_2k_1f^2g(\dot{\vartheta}T, V_2)g(\dot{\vartheta}T, V_3) + \beta\rho_3f^2\dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + 3\rho_2k_1g(\dot{\vartheta}T, V_2)g(\dot{\vartheta}T, V_4) + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases} \tag{4.3}$$

CASE I. Let $\rho_2 = \rho_3 = 0$. Then, the manifold $\mathbb{M}$ is a Riemannian space form of constant sectional curvature ρ_2. In this case, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2ff' = 0 \\ (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1'ff' + 3k_1f''f + 2k_1(f')^2 = 0 \\ 4k_1k_2f' + 2k_2k_1'f + k_1k_2'f = 0 \\ k_1k_2k_3 = 0 \end{cases} \tag{4.4}$$

Theorem 4.4. There is no any proper bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$.

From (4.4), if σ is a geodesic curve, then it is a bi-f-harmonic curve if and only if $ff'' = \text{cons}$.

Theorem 4.5. A geodesic curve in an (α, β)-TSGSSF is bi-f-harmonic if and only if $ff'' = \text{cons}$.

This theorem proves that there are bi-f harmonic curves that are not harmonic. Afterward, we investigate bi-f-harmonicity of $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ considering some special subcases:

CASE I. 1. If $k_1 = \text{cons} \neq 0$ and $k_2 = 0$, then, from (4.4),

$$\begin{cases} (ff'')' - 4k_1^2ff' = 0, \\ (\rho_1 - k_1^2)f^2 + 2(f')^2 + 3f''f = 0 \end{cases} \tag{4.5}$$

From the second equation of (4.5), $ff'' = \frac{(k_1^2-\rho_1)f^2-2(f')^2}{3}$ which implies

$$10k_1^2ff' + \rho_1'f^2 + 2\rho_1ff' + 4f'f'' = 0 \tag{4.6}$$

via the first equation of (4.5).

Theorem 4.6. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if f, k_1, and ρ_1 satisfy following differential equation

$$10k_1^2ff' + \rho_1'f^2 + 2\rho_1ff' + 4f'f'' = 0$$

Further, if (4.6) is solved by assuming ρ_1 constant, the the following result is obtained.

Theorem 4.7. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an α-Sasakian generalized Sasakian space form dimension ≥ 5 with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = 0$. Then, σ is a proper bi-f-harmonic Legendre curve if and only if f is a function defined by

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 + \rho_1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 + \rho_1}{2}}s\right)$$

where $s \in I$ and ρ_1 is a constant.

CASE I. 2. If $k_1 = \text{cons} \neq 0$ and $k_2 = \text{cons} \neq 0$, then (4.4) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' = 0 \\ f^2(-k_1^2 - k_2^2 + \rho_1) + 3f''f + 2(f')^2 = 0 \\ f' = 0 \\ k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} f = \text{cons} \\ k_1^2 + k_2^2 = \rho_1 \\ k_3 = 0 \end{cases}$$

Theorem 4.8. There is no any proper bi-f-harmonic Legendre curve on an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 = \text{cons} \neq 0$, and $k_2 = \text{cons} \neq 0$.

CASE I. 3. If $k_1 \neq \text{cons}$ and $k_2 = \text{cons} \neq 0$, then (4.4) reduces to

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f''f + 2k_1(f')^2 + k_1\rho_1 f^2 = 0 \\ 2k_1 f' + k_1' f = 0 \\ k_3 = 0 \end{cases}$$

Theorem 4.9. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 \neq \text{cons}$, and $k_2 = \text{cons} \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$f = \pm c k_1^{-\frac{1}{2}}$$

for some real constant c, $k_3 = 0$, and the curvature k_1 solves the following second order non-linear differential equations system

$$\begin{cases} 9(k_1')^3 + 4k_1'k_1^4 - 10k_1''k_1'k_1 + 2k_1'''k_1^2 = 0 \\ -3(k_1')^2 + 4k_1^4 + 4k_1^2k_2^2 + 2k_1''k_1 - 4k_1^2\rho_1 = 0 \end{cases}$$

CASE I. 4. If $k_1 \neq \text{cons}$ and $k_2 \neq \text{cons}$, then by using the third equation in (4.4),

$$f = \pm c k_1^{-\frac{1}{2}} k_2^{-\frac{1}{4}}$$

for some real constant c. Besides, from the last equation in (4.4), $k_3 = 0$.

Theorem 4.10. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-TSGSSF with $\rho_2 = \rho_3 = 0$, $k_1 \neq \text{cons}$, and $k_2 \neq \text{cons}$. Then, σ is a bi-f-harmonic Legendre curve if and only if $f = \pm c k_1^{-\frac{1}{2}} k_2^{-\frac{1}{4}}$, c is a constant, $k_3 = 0$, and k_1 and k_2 satisfy the following second order non-linear differential equation system

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1)f^2 + 4k_1' f f' + 3k_1 f''f + 2k_1(f')^2 = 0 \end{cases}$$

Before calculating Case II, we recall the following results [20]:

Proposition 4.11. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be an α-Sasakian generalized Sasakian space form. Therefore,

α is independent of the direction of ς and the following equation is valid

$$\rho_1 - \rho_3 = \alpha^2$$

Moreover, if $\mathbb{M}$ is connected, then α is a constant.

Theorem 4.12. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a connected α-Sasakian generalized Sasakian space form with dimension $m \geq 5$. Then, ρ_1, ρ_2, and ρ_3 are constant functions related as follows:

i. If $\alpha = 0$, then $\rho_1 = \rho_2 = \rho_3$ and $\mathbb{M}$ is a cosymplectic manifold of constant $\dot{\vartheta}$-sectional curvature

ii. If $\alpha \neq 0$, then $\rho_1 - \alpha^2 = \rho_2 = \rho_3$

CASE II. Let $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, from (4.1), it is obvious that the manifold $\mathbb{M}$ is an α-Sasakian generalized Sasakian space form. By using Proposition 4.11, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2 (\rho_3 + \alpha^2) = 0 \\ 4k_1k_2 f' + (2k_2 f k_1' + k_1k_2')f = 0 \\ k_1k_2k_3 = 0. \end{cases} \tag{4.7}$$

Theorem 4.13. There is no any bi-f-harmonic Legendre curve of osculating order $r > 3$ satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in an α-Sasakian generalized Sasakian space form.

Theorem 4.14. There is no any bi-f-harmonic Legendre curve satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in a connected α-Sasakian generalized Sasakian space form with dimension ≥ 5.

CASE II.1. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $V_2 \perp \varsigma$, and $\alpha \neq 0$.

In this case, we consider bi-f-harmonic Legendre curves satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$ in a connected 3-dimensional α-Sasakian generalized Sasakian space forms. In a 3-dimensional α-Sasakian manifold, a Legendre curve is a Frenet curve of osculating order 3 and its torsion is always α [21]. Then, (4.7) reduces to

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 + k_1 f^2 (\rho_3 + \alpha^2) = 0 \\ 2k_1 f' + f k_1' = 0 \end{cases} \tag{4.8}$$

Theorem 4.15. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a 3-dimensional connected α-Sasakian generalized Sasakian space form satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a bi-f-harmonic Legendre curve if and only if $f = \pm c k_1^{-\frac{1}{2}}$, where c is a constant and k_1 solves the following second order non-linear differential equation system

$$\begin{cases} 9(k_1')^3 + 4k_1'k_1^4 - 10k_1''k_1'k_1 + 2k_1'''k_1^2 = 0 \\ -3(k_1')^2 + 4k_1^4 + 4k_1^2k_2^2 + 2k_1''k_1 - 4k_1^2(\rho_3 + \alpha^2) = 0 \end{cases}$$

If $k_1 = \text{cons} \neq 0$, then f is constant from the third equation in (4.8).

Corollary 4.16. There is no any proper bi-f-harmonic Legendre helix in a 3-dimensional connected α-Sasakian generalized Sasakian space form satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, and $V_2 \perp \varsigma$.

CASE II.2. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $V_2 \perp \varsigma$, and $\alpha = 0$.

Theorem 4.17. Let $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a cosymplectic generalized Sasakian space form satisfying $\rho_2 = 0$,

$\rho_3 \neq 0$, and $V_2 \perp \varsigma$. Then, $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a bi-f-harmonic Legendre curve if and only if $\rho_1 = \rho_3$ and the following differential equation system is satisfied

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 + k_1 f^2(\rho_3 + \alpha^2) = 0 \\ 4k_1k_2 f' + (2k_2 f k_1' + k_1 k_2') f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

CASE III. Let $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, from (4.2), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1 f^2 k_1' - 4k_1^2 f f' = 0 \\ \begin{cases} (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 \\ +k_1 f^2 \rho_1 + f^2 \beta\rho_3\dot{\eta}(V_2) = 0 \end{cases} \\ 2k_2 f k_1' + k_1 f k_2' + 4k_1k_2 f' + \beta\rho_3 f \dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases}$$

Let $m = \min\{r, 4\} = 4$, which implies $r \geq 4$. Then,

$$\varsigma = \cos\theta_1 V_2 + \sin\theta_1\cos\theta_2 V_3 + \sin\theta_1\sin\theta_2 V_4$$

which implies

$$\dot{\eta}(V_2) = \cos\theta_1, \quad \dot{\eta}(V_3) = \sin\theta_1\cos\theta_2, \quad \text{and} \quad \dot{\eta}(V_4) = \sin\theta_1\sin\theta_2$$

Here, $\theta_1 : I \to \mathbb{R}$ denotes the angle function between ς and V_2 and $\theta_2 : I \to \mathbb{R}$ is the angle function between V_3 and the orthogonal projection of ς onto $\text{span}\{V_3, V_4\}$.

Theorem 4.18. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$ and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1 f^2 k_1' - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'')f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 + k_1 f^2 \rho_1 + f^2 \beta\rho_3\cos\theta_1 = 0 \\ 2k_2 f k_1' + k_1 f k_2' + 4k_1k_2 f' + \beta\sin\theta_1\cos\theta_2\rho_3 f = 0 \\ k_1k_2k_3 + \beta\sin\theta_1\sin\theta_2\rho_3 = 0 \end{cases} \tag{4.9}$$

provided $r \geq 4$.

As a particular case, if $\beta = 0$, that is, $(\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is an α-Sasakian generalized Sasakian space form, then the following results is obtained:

Corollary 4.19. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$.

If ρ_1, ρ_3, and the first three curvatures of σ are constants, then the following result is valid:

Theorem 4.20. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$ and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if f is one of the followings:

$$f(s)=c_1\cos\left(\sqrt{\frac{-5k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2}{2}}s\right)+c_2\sin\left(\sqrt{\frac{-5k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2}{2}}s\right) \tag{4.10}$$

$$f(s)=c_3s+c_4 \tag{4.11}$$

and

$$f(s)=c_5e^{-\sqrt{\frac{5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2}{2}}s}+c_6e^{\sqrt{\frac{5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2}{2}}s} \tag{4.12}$$

provided that

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2>0$$

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2=0$$

and

$$5k_1^2-k_2^2+\rho_1+\rho_3(cos\theta_1)^2<0$$

respectively, and

$$f(s)=e^{\frac{k_3}{4}\int\cot\theta_2\,ds} \tag{4.13}$$

where $c_1,c_2,\cdots,c_6$, θ_1 and θ_2 are constants.

Proof. By using (4.9),

$$\begin{cases}(ff'')'-4k_1^2ff'=0\\ 3f''f+2(f')^2+f^2\left(-k_1^2-k_2^2+\rho_1-\rho_3\,(cos\theta_1)^2\right)=0\\ 4k_1k_2f'+\beta\sin\theta_1\cos\theta_2\rho_3f=0\\ k_1k_2k_3+\beta\sin\theta_1\sin\theta_2\rho_3=0\end{cases} \tag{4.14}$$

From the second equation of (4.14),

$$f''f=\frac{-2(f')^2+\left(k_1^2+k_2^2-\rho_1+\rho_3(cos\theta_1)^2\right)f^2}{3} \tag{4.15}$$

If (4.15) is used in the first equation of (4.14),

$$2f''+\left(5k_1^2-k_2^2+\rho_1-\rho_3(cos\theta_1)^2\right)f=0 \tag{4.16}$$

By solving the differential equation (4.16), the first assertion of the theorem is obtained. Besides,

$$\beta\rho_3\sin\theta_1\left(\cos\theta_2k_3f-4\sin\theta_2f'\right)=0$$

via the last two equations of (4.14) which implies (4.13). □

Let $r=3$. This implies that $\varsigma\in\text{span}\{V_2,V_3\}$ and by choosing $\theta_2=0$, $\varsigma=\cos\theta_1V_2+\sin\theta_1V_3$ where $\theta_1:I\to\mathbb{R}$ denotes the angle function between ς and V_2.

Theorem 4.21. Let $\sigma:I\to(\mathbb{M},\dot{\vartheta},\varsigma,\dot{\eta},g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2=0$, $\rho_3\neq0$, $\dot{\eta}(V_2)\neq0$, and $\varsigma\in\text{span}\{V_2,V_3\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases}(ff'')'-3k_1k_1'f^2-4k_1^2ff'=0\\ (3f''f+2(f')^2)k_1++4k_1'ff'+f^2(-k_1^3-k_1k_2^2+k_1''+k_1\rho_1+\beta\rho_3\cos\theta_1)=0\\ 4k_1k_2f'+f(2k_2fk_1'+k_1k_2'+\beta\sin\theta_1\rho_3)=0\end{cases}$$

provided $r=3$.

If ρ_1, ρ_3, and the first two curvatures of σ are constants, then the following result is obtained:

Corollary 4.22. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3\}$. Then, σ is a bi-f-harmonic Legendre curve if and only if f is defined by one of the form given in (4.10), (4.11), or (4.12) and

$$f(s) = e^{\frac{\rho_3}{4}\int \sin\theta_1 \cos\theta_1 \, ds}$$

where $s \in I$.

Let $r = 2$. Then, $\varsigma \in \text{span}\{V_2\}$ which implies $\varsigma = \pm V_2$ by taking $\theta_1 \in \{0, \pi\}$ and $\theta_2 = 0$.

Theorem 4.23. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma = \pm V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (3f''f + 2(f')^2)k_1 + 4k_1' f f' + f^2(-k_1^3 + k_1'' + k_1\rho_1 + \beta\rho_3\cos\theta_1) = 0 \end{cases}$$

provided $r = 2$.

If ρ_1, ρ_3, and the first curvature of σ are constants, then the following result is obtained:

Corollary 4.24. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 = 0$, $\rho_3 = \text{cons} \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma = \pm V_2$. Then, σ is a bi-f-harmonic curve if and only if f is defined by one of the form given in (4.10), (4.11), or (4.12).

CASE IV. Let $\rho_2 \neq 0$, $\rho_3 = 0$, and $V_2 \perp \dot{\vartheta}T$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1)f^2 + 4k_1' f f' + 3k_1 f'' f + 2(f')^2 k_1 = 0 \\ 4k_1k_2 f' + (2k_2k_1' + k_1k_2')f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

Corollary 4.25. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an (α,β)-TSGSSF, satisfying $\rho_2 \neq 0$, $\rho_3 = 0$, and $V_2 \perp \dot{\vartheta}T$.

Note that because the conditions obtained in Cases I and IV are the same, it is not necessary to investigate the subcases for Case IV.

CASE V: Let $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $m = \min\{r, 4\} = 4$, which implies $r \geq 4$. Then,

$$\dot{\vartheta}T = \cos a_1 V_2 + \sin a_1 \cos a_2 V_3 + \sin a_1 \sin a_2 V_4 \tag{4.17}$$

which implies

$$g(\dot{\vartheta}T, V_2) = \cos a_1$$

$$g(\dot{\vartheta}T, V_3) = \sin a_1 \cos a_2$$

and

$$g(\dot{\vartheta}T, V_4) = \sin a_1 \sin a_2 \tag{4.18}$$

Here, $a_1 : I \to \mathbb{R}$ denotes the angle function between $\dot{\vartheta}T$ and V_2 and $a_2 : I \to \mathbb{R}$ is the angle function between V_3 and the orthogonal projection of $\dot{\vartheta}T$ onto $\text{span}\{V_3, V_4\}$. Thus, the following result is obtained:

Theorem 4.26. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, V_4\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2)\, f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 = 0 \\ f\ (2k_2 f\, k_1' + k_1 f\, k_2' + 4k_1k_2 f') + 3\rho_2 k_1 f^2 \cos a_1 \cos a_2 \sin a_1 \ = 0 \\ k_1k_2k_3 + 3\rho_2 k_1 \sin a_1 \sin a_2 \cos a_1 = 0 \end{cases}$$

If the first three curvatures are constants, the following result is obtained:

Theorem 4.27. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if k_1, k_2, and k_3 satisfy the following differential equations

$$\begin{cases} (ff'')' - 4k_1^2 f f' = 0 \\ (-k_1^2 - k_2^2 + \rho_1 + 3\rho_2(\cos a_1)^2) f^2 + 3f'' f \ + 2(f')^2 = 0 \end{cases}$$

where

$$f(s) = e^{\frac{k_3}{4}\int \cot a_2\, ds}$$

and a_1 and a_2 are constants.

Let $r = 3$. Therefore,

$$\dot{\vartheta}T = \cos a_1 V_2 + \sin a_1 V_3$$

Hence, $g(\dot{\vartheta}T, V_2) = \cos a_1$, $g(\dot{\vartheta}T, V_3) = \sin a_1$, $a_2 = 0$, and $k_3 = 0$.

Theorem 4.28. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3\}$, and $g(\dot{\vartheta}T, V_2) \neq 0$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2) f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 = 0 \\ f\ (2k_2 f\, k_1' + k_1 f\, k_2' + 4k_1k_2 f') + 3\rho_2 k_1 f^2 \cos a_1 \sin a_1 \ = 0 \end{cases}$$

provided $r = 3$.

Let $r = 2$. Therefore, $\dot{\vartheta}T = \pm V_2$. Hence, $g(\dot{\vartheta}T, V_2) = \pm 1$, $g(\dot{\vartheta}T, V_3) = 0$, $a_1 = a_2 = 0$, and $k_2 = k_3 = 0$.

Theorem 4.29. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α,β)-TSGSSF with $\rho_2 \neq 0$, $\rho_3 = 0$, and $\dot{\vartheta}T = \pm V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 + k_1'' + k_1\rho_1 \pm 3k_1\rho_2) f^2 + 4f\, k_1' f' + 3f'' f\, k_1 + 2(f')^2 k_1 \ = 0 \end{cases}$$

provided $r = 2$.

CASE VI. Let $\rho_2 \neq 0, \rho_3 \neq 0, V_2 \perp \dot{\vartheta}T$, and $V_2 \perp \varsigma$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic Legendre curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (k_1'' - k_1^3 - k_1k_2^2 + k_1\rho_1)f^2 + 4k_1' f f' + 3k_1 f'' f + 2k_1(f')^2 = 0 \\ 4k_1k_2f' + 2k_2k_1'f + k_1k_2'f = 0 \\ k_1k_2k_3 = 0 \end{cases}$$

Corollary 4.30. There is no any bi-f-harmonic Legendre curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 \neq 0, \rho_3 \neq 0, V_2 \perp \dot{\vartheta}T$, and $V_2 \perp \varsigma$.

Note that because the conditions obtained in Cases I and VI are the same, it is not necessary to investigate the subcases for Case VI.

CASE VII. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $V_2 \perp \dot{\vartheta}T$, $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$, and $\dot{\eta}(V_2) \neq 0$. Then, from (4.3), $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ is a proper bi-f-harmonic curve if and only if

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + \beta\rho_3(\cos\theta_1)^2)f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 = 0 \\ 4k_1k_2ff' + f^2(2k_2k_1' + k_1k_2' + \beta\rho_3 \sin\theta_1 \cos\theta_2) = 0 \\ k_1k_2k_3 + \beta\rho_3k_1 \sin\theta_1 \sin\theta_2 = 0 \end{cases}$$

Corollary 4.31. There is no any bi-f-harmonic curve of osculating order $r \geq 4$ in an α-Sasakian generalized Sasakian space form, satisfying $\rho_2 \neq 0$, $\rho_3 \neq 0$, $V_2 \perp \dot{\vartheta}T$, $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$, and $\dot{\eta}(V_2) \neq 0$.

Note that because the conditions obtained in Cases III and VII are the same, we omit to investigate the subcases for Case VII.

CASE VIII. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $\varsigma \perp V_2$. Then, from (4.17) and (4.18), the following result is obtained:

Theorem 4.32. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an α-Sasakian generalized Sasakian space form with $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, and $\varsigma \perp V_2$. Then, σ is a bi-f-harmonic Legendre curve if and only if k_1, k_2, and k_3 satisfy the following differential equations:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2)f^2 + 4f k_1' f' + 3f'' f k_1 + 2(f')^2 k_1 = 0 \\ f\ (2k_2 f k_1' + k_1 f k_2' + 4k_1k_2f') + 3\rho_2k_1f^2 \cos a_1 \cos a_2 \sin a_1 + \beta\rho_3\dot{\eta}(V_3) = 0 \\ k_1k_2k_3 + 3\rho_2k_1 \sin a_1 \sin a_2 \cos a_1 + \beta\rho_3\dot{\eta}(V_4) = 0 \end{cases} \tag{4.19}$$

If $r = 3$, then the first three equations of the (4.19) are satisfied, taking $a_2 = 0$.

If $r = 2$, then the first two equations of the (4.19) are satisfied, taking $a_1 \in \{0, \pi\}$.

CASE IX. Let $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, the following result is obtained:

Theorem 4.33. Let $\sigma : I \to (\mathbb{M}, \dot{\vartheta}, \varsigma, \dot{\eta}, g)$ be a curve parameterized by its arclenght on an (α, β)-trans-Sasakian generalized Sasakian space form with $\rho_2 \neq 0$, $\rho_3 \neq 0$, $\dot{\vartheta}T \in \text{span}\{V_2, V_3, \cdots, V_m\}$, $g(\dot{\vartheta}T, V_2) \neq 0$, $\dot{\eta}(V_2) \neq 0$, and $\varsigma \in \text{span}\{V_2, V_3, \cdots, V_m\}$. Then, σ is a bi-f-harmonic curve if and only if k_1, k_2, and k_3 satisfy the following differential equations:

$$\begin{cases} (ff'')' - 3k_1k_1'f^2 - 4k_1^2 f f' = 0 \\ \begin{cases} (-k_1^3 - k_1k_2^2 + k_1'' + k_1\rho_1 + 3k_1\rho_2(\cos a_1)^2 + \beta\rho_3\cos\theta_1)f^2 + 4f k_1' f' \\ +(3f''f + 2(f')^2)k_1 = 0 \end{cases} \\ 4k_1k_2ff' + f^2(2k_2k_1' + k_1k_2' + 3\rho_2k_1\cos a_1\cos a_2\sin a_1 + \beta\rho_3\sin\theta_1\cos\theta_2) = 0 \\ k_1k_2k_3 + 3\rho_2k_1\sin a_1\sin a_2\cos a_1 + \beta\rho_3\sin\theta_1\sin\theta_2 = 0 \end{cases} \tag{4.20}$$

If $r = 3$, then the first three equations of the (4.20) are satisfied, taking $a_2 = 0$ and $\theta_2 = 0$.

If $r = 2$, then the first two equations of the (4.20) are satisfied, taking $\theta_1 \in \{0, \pi\}$ and $a_1 \in \{0, \pi\}$.

5. Conclusion

This study has obtained the necessary and sufficient conditions for a curve to be bi-f-harmonic Legendre in the (α,β)-trans-Sasakian generalized Sasakian space form. While conducting this investigation, the functions from the manifold's curvature tensor, curvature and torsion of the curve, and the relative positions of the basis vectors have been considered. Future studies could focus on different curves, such as Slant, in the (α,β)-trans-Sasakian generalized Sasakian space form. Additionally, research can be conducted on special cases of the (α,β)-trans-Sasakian manifold, including α-Sasakian, Sasakian, β-Kenmotsu, Kenmotsu, and cosymplectic manifold types.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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