CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. SI, pp. 93-113
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Maxwell orthogonal polynomials

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. In the framework of the theory of semiclassical linear functionals in this contribution, we deal with the sequence of orthogonal polynomials associated with the linear functional $\langle L, p\rangle = \int_0^\infty p(x)e^{-x^2}dx$, where $p \in \mathbb{P}$, the linear space of polynomials with complex coefficients. The class of L is one and we deduce a differential/difference equation (structure relation) for the sequence of orthogonal polynomials. The Laguerre-Freud equations that the coefficients of the three term recurrence relation satisfy are deduced. The connection with discrete Painlevé IV equations is emphasized. Finally, we analyze the lowering and raising operators (ladder operators) for such polynomials in order to find a second order linear differential equation they satisfy. As a consequence, an electrostatic interpretation of their zeros is formulated.

Keywords: Maxwell linear functional, Stieltjes function, Pearson equation, Ladder operators, Laguerre-Freud equations, electrostatic interpretation, Painlevé equations.

2020 Mathematics Subject Classification: 42C05, 33C50.

1. INTRODUCTION

Let consider the sequence of orthogonal polynomials associated with a linear functional L defined from a weight function $w(x) = e^{-x^2}$ supported on the positive real semi-axis by $\langle L, p\rangle = \int_0^\infty p(x)e^{-x^2}dx$. The linear functional belongs to a wide class called semiclassical ([13]). Indeed, they are semiclassical of class $s = 1$. The concept of class allows to introduce a hierarchy of linear functionals that constitutes an alternative to the Askey tableau based on the hypergeometric character of the corresponding sequence of orthogonal polynomials. Semiclassical orthogonal polynomials appear in the seminal paper [16], where weight functions whose logarithmic derivatives are rational functions were considered. A second order linear differential equation that the corresponding sequence of orthogonal polynomials is obtained. The theory of semiclassical orthogonal polynomials has been developed by P. Maroni (see [12, 13]) and combine techniques of functional analysis, distribution theory, z and Fourier transforms, ordinary differential equations, among others, in order to deduce algebraic and structural properties of such polynomials. Many of the most popular families of orthogonal polynomials coming from Mathematical Physics as Hermite, Laguerre, Jacobi and Bessel are semiclassical (of class $s = 0$) but other families appearing in the literature are also semiclassical. For instance, semiclassical families of class $s = 1$ are described in [1]. As a sample in [7] truncated Hermite polynomials associated with the normal distribution supported on a symmetric interval of the real line have been considered. They are semiclassical of class $s = 2$.

Received: 09.07.2024; Accepted: 03.11.2024; Published Online: 16.12.2024
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DOI: 10.33205/cma.1513303

The coefficients of the three term relation satisfied by orthogonal polynomials associated with a semiclassical linear functional satisfy coupled non linear difference equations (Laguerre-Freud equations, see [2]) which are related to discrete Painlevé equations, see [11, 17] among others. On the other hand, ladder operators (lowering and raising) are associated with semiclassical orthogonal polynomials. Their construction is based on the so called first structure relation , see [13], they satisfy. As a consequence, you can deduce a second order linear differential equation that the polynomials satisfy. Notice that the coefficients are polynomials with degrees depending of the class.

In the framework of random matrices, several authors have considered Gaussian unitary ensembles with one or two discontinuities (see [10]). Therein it is proved that the logarithmic derivative of the Hankel determinants generated by the normal (Gaussian) weight with two jump discontinuities in the scaled variables tends to the hamiltonian of a coupled Painlevé II system and it satisfies a second order PDE. The asymptotics of the coefficients of the three term recurrence relation for the corresponding orthogonal polynomials is deduced. They are related to the solutions of the coupled Painlevé II system. The techniques are based on the analysis of ladder operators associated with such orthogonal polynomials. For more information, see [3, 4].

Integrable systems also provide nice examples of semiclassical orthogonal polynomials. They are related to time perturbations of a weight function (Toda, Langmuir lattices). For more information, see [18].

It is important to highlight that, according to S. Belmehdi and A. Ronveaux (see [2]), the MOPS associated with the linear functional L defined as above is known as the family of Maxwell polynomials. Nevertheless, in the literature, the name Maxwell polynomials is quoted when you deal with the weight function $\omega(x) = x^2 e^{-x^2}$, supported on the positive real semi-axis (see, for instance, [6]). They appear in many problems of kinetic theory since involve the evaluation of averages over a Maxwellian distribution function $f^M(c)$, where c is the particle speed. The equilibrium average value of a function $F(c)$ is defined by

$$\overline{F} = \int f^M(c)F(c)dc = \frac{4}{\sqrt{\pi}} \int_0^\infty x^2 e^{-x^2} F(x)dx.$$

Here $f^M(c) = (m/2\pi kT)^{3/2} e^{-mc^2/2kT}$, m is the mass and $x = (m/2kT)^{1/2}$ is the dimesionless speed. They are called speed polynomials.

In the literature, see [15], among others, Gaussian quadrature rules are analyzed for weight functions $w(x) = x^p e^{-x^2}$ supported on the positive real semi-axis. They constitute an useful tool in the solution of the Boltzman and/or Fokker-Planck equations. It is important to point out that the coefficients of the three term recurrence relations for the corresponding sequences of orthogonal polynomials satisfy nonlinear equations which are numerically unstable. In [15] extended precision arithmetic is used to generate the recurrence coefficients to high order. As a consequence, the polynomials and associated quadrature weights and nodes are deduced.

Maxwell polynomials are used in various fields for different purposes, such as in pseudo-spectral collocation schemes. Among the numerous characteristics that make these polynomials desirable for pseudo-spectral discretization schemes for the velocity variable, one may emphasize their capability to handle semi-infinite intervals, the convenient distribution of their zeros that balances clustering around zero, where increased resolution is often needed, with the sampling at increasingly larger distances from the origin, and their optimal location for the computation of integrals involving a Maxwell-Boltzmann distribution. For further details, see [9, 14].

On the other hand, strong asymptotic properties of orthogonal polynomials with respect to weights $\omega(x) = x^{\alpha}e^{-Q(x)}$, where $\alpha > -1$ and Q is a polynomial of degree m with positive leading coefficient, have been analyzed in [19]. An application to prove universality in random matrix theory is also provided therein.

The presentation is structured as follows. In Section 2, a basic background concerning linear functionals, orthogonal polynomials is given. The semiclassical case is emphasized. Section 3 deals with the Pearson equation associated with the Maxwell linear functional. As a consequence you get the first order linear differential equation that the Stieltjes function satisfy. Section 4 focuses the attention on the Laguerre-Freud equations satisfied by the coefficients of the three-term recurrence relation associated with the Maxwell polynomials. The connection with discrete IV Painlevé equation is stated. Finally, in Section 5, the expressions of the ladder operators are given as well as the second order linear differential equation these polynomials satisfy. The electrostatic interpretation of their zeros is deduced.

2. Basic background

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathbb{P}_n$ be the $(n+1)$-dimensional subspace of polynomials of degree less than or equal to n. Let $L : \mathbb{P} \to \mathbb{C}$ be a linear functional and denote by μ_n the moments of L on the monomial basis:

$$L[x^n] = \langle L, x^n \rangle := \mu_n. \tag{2.1}$$

A sequence $\{p_n\}_{n\geq 0}$, with $p_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ for all $n \geq 0$ ($\mathbb{P}_{-1} = \emptyset$), is said to be an orthogonal polynomial sequence (OPS) with respect to L if

$$\langle L, p_k p_n \rangle = h_n \delta_{k,n}, \quad k, n \in \mathbb{N} \cup \{0\}, \quad h_n \neq 0, \tag{2.2}$$

where $\delta_{k,n}$ denotes the Kronecker delta symbol. We will denote by $\{P_n\}_{n\geq 0}$ the OPS with P_n monic, for all $n \geq 0$ (MOPS) and by $\{\tilde{p}_n\}_{n\geq 0}$ the orthonormal family (ONPS) i.e., $h_n = 1$ in (2.2), for all $n \geq 0$ (this family is unique under the condition that all the leading coefficients are positive).

We recall (see [5]) that a linear functional L defined on $\mathbb{P}$ is said to be quasi-definite if an OPS with respect to L exists. A quasi-definite linear functional L is positive-definite when $h_n > 0$ in (2.2), $n \geq 0$. In this case, L can be represented in terms of a positive Borel measure μ supported on the real line as $\langle L, p \rangle = \int_{\mathbb{R}} p(x) d\mu(x)$ for $p \in \mathbb{P}$, L has real moments and thus, MOPS and ONPS have real coefficients. Also, the zeros of p_n are real, distinct, located in the interior of the convex hull of the support of μ and they interlace with those of p_{n+1}.

It is very well known that MOPS and ONPS satisfy three term recurrence relations (TTRR).

Theorem 2.1. *Let $\{P_n\}_{n\geq 0}$ and $\{\tilde{p}_n\}_{n\geq 0}$ be the MOPS and ONPS with respect to the linear functional L given by (2.1), respectively. Define $\alpha_n = \frac{\langle L, xP_n^2 \rangle}{\langle L, P_n^2 \rangle}$, $\forall n \geq 0$, and $\beta_m = \frac{\langle L, P_m^2 \rangle}{\langle L, P_{m-1}^2 \rangle} > 0$, $\forall m \geq 1$. Setting $P_{-1}(x) \equiv \tilde{p}_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$ and $\tilde{p}_0(x) \equiv \frac{1}{\sqrt{\mu_0}}$, then it holds for all $n \geq 0$,*

$$P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x), \tag{2.3}$$

$$x\tilde{p}_n(x) = \sqrt{\beta_{n+1}}\tilde{p}_{n+1}(x) + \alpha_n \tilde{p}_n(x) + \sqrt{\beta_n}\tilde{p}_{n-1}(x). \tag{2.4}$$

We consider also the reproducing kernel $K_n(x, y) = \sum_{k=0}^{n} \tilde{p}_k(x)\tilde{p}_k(y)$, that satisfies the reproducing property $\langle L_y, K_n(x, y)q(y) \rangle = \int K_n(x, y)q(y)\, d\mu(y) = q(x)$, for all $q \in \mathbb{P}_n$. The well known Christoffel-Darboux and confluent formulas are

$$K_n(x,y) = \sqrt{\beta_{n+1}} \cdot \frac{\tilde{p}_{n+1}(x)\tilde{p}_n(y) - \tilde{p}_n(x)\tilde{p}_{n+1}(y)}{x-y}, \qquad x \neq y, \tag{2.5}$$

$$K_n(x,x) = \sqrt{\beta_{n+1}} \cdot \left[\tilde{p}'_{n+1}(x)\tilde{p}_n(x) - \tilde{p}'_n(x)\tilde{p}_{n+1}(x)\right] > 0, \tag{2.6}$$

respectively.

The following definition of D-semiclassical linear functionals (with respect to the derivative operator) is a natural generalization of the D-classical functionals.

Definition 2.1. *A quasi-definite functional L is said to be semiclassical if there exist nonzero polynomials ϕ and ψ, where ϕ is monic, $\deg\phi \geq 0$, and $\deg\psi \geq 1$, such that L satisfies the distributional Pearson equation*

$$D(\phi L) = \psi L. \tag{2.7}$$

Furthermore, an OPS associated with L is called a semiclassical sequence of orthogonal polynomials. Recall that the adjoints of the derivative and multiplication operators are $\langle DL, p\rangle = -\langle L, p'\rangle$ and $\langle xL, p\rangle = \langle L, xp\rangle$, respectively (see [13]).

One can observe that $\deg\psi \geq 1$. Indeed, if $\psi(x) = \psi_0 \neq 0$, then

$$0 = -\langle \phi L, 0\rangle = \langle D(\phi L), 1\rangle = \langle \psi L, 1\rangle = \langle L, \psi\rangle = \psi_0\mu_0 \quad \text{with} \quad \mu_0 = \langle L, 1\rangle .$$

Thus, $\mu_0 = 0$, contradicting the quasi-definiteness of L. Clearly, we also need the polynomial ϕ to be nonzero in order to guarantee that L is quasi-definite. To prevent any potential conflict arising from the quasi-definite nature of the semiclassical functional L, a new requirement has to be imposed: if $\phi(x) = \sum_{k=0}^{r} a_k x^k$ and $\psi(x) = \sum_{k=0}^{t} b_k x^k$ ($r \geq 0$, $t \geq 1$), then for any $n \geq 0$ it must hold $na_r + b_{r-1} \neq 0$ when $r = t+1$, to ensure that all the moments are well defined. Indeed, from (2.7) we can write $\langle D(\phi L), x^n\rangle = \langle \psi u, x^n\rangle$, which implies $\langle u, n\phi x^{n-1} + \psi x^n\rangle = 0$ for all $n \geq 0$. So, the Pearson equation is equivalent for $n \geq 0$ in this case to

$$n \cdot \sum_{k=0}^{r} a_k \mu_{n+k-1} + \sum_{k=0}^{r-1} b_k \mu_{n+k} = 0,$$

implying

$$(na_r + b_{r-1})\mu_{n+r-1} = -\sum_{k=0}^{r-2}(na_{k+1} + b_k)\mu_{n+k} - na_0\mu_{n-1}, \quad n \geq 0.$$

Consequently, if there is n_0 such that $n_0 a_r + b_{r-1} = 0$, we see that μ_{n_0+r-1} cannot be computed from the previous identity. This circumstance may potentially give inconsistencies with the quasi-definiteness of L. When L is semiclassical, it is possible that (2.7) not to be minimal due to the non-uniqueness of the polynomials ϕ and ψ. Specifically, for any non-zero polynomial q, one can see that

$$D(q\phi L) = q'\phi L + qD(\phi L) = q'\phi L + q\psi L = (q'\phi + q\psi)L.$$

Therefore, L satisfies $D(\tilde{\phi}L) = \tilde{\psi}L$, with $\tilde{\phi} = q\phi$ and $\tilde{\psi} = q'\phi + q\psi$. Moreover,

$$\deg(q\phi) = \deg q + \deg\phi \quad \text{and} \quad \deg(q'\phi + q\psi) = \deg q + \max\{\deg\phi - 1, \deg\psi\}.$$

Therefore, a semiclassical linear functional exhibits various Pearson equations based on the choices of polynomials ϕ and ψ. Defining the class of a semiclassical linear functional L stems from the minimum degree choices of ϕ and ψ, given by

$$s(L) = \min\max\{\deg\phi - 2, \deg\psi - 1\},$$

where the minimum is taken among all pairs of polynomials ϕ and ψ such that (2.7) holds. It is noteworthy that D-classical linear functionals, such as Hermite, Laguerre, Jacobi, and Bessel polynomials, fall under the category of semiclassical ones with class $s = 0$. In this respect, a proof of the following two results can be found in [13].

Theorem 2.2. *For any semiclassical linear functional L, the polynomials ϕ and ψ in (2.7) such that $s(L) = \max\{\deg\phi - 2, \deg\psi - 1\}$ are unique up to a constant factor.*

Proposition 2.1. *Let L be a semiclassical linear functional and let ϕ and ψ be nonzero polynomials, with $\deg\phi = r$ and $\deg\psi = t$, such that (2.7) is satisfied. Consider $s = \max\{r-2, t-1\}$. Then $s = s(L)$, if and only if*

$$\prod_{c:\ \phi(c)=0} \left(|\psi(c) - \phi'(c)| + \left|\langle L, \theta_c\psi(x) - \theta_c^2\phi(x)\rangle\right|\right) > 0,$$

where

$$\theta_c q(x) := \frac{q(x) - q(c)}{x - c}, \quad q \in \mathbb{P}.$$

We can consider from the moment sequence its z-transform, that yields a formal power series

$$S(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}} \tag{2.8}$$

that is referred in the literature as Stieltjes function (see [8, 13]). Semiclassical functionals are introduced alongside various characterizations. It is worth noting that some of these characterizations naturally extend the criteria used for defining class functions. We refer to [8, 13] for the proof of the following:

Proposition 2.2. *Let L be a quasi-definite functional, $\{P_n\}_{n\geq 0}$ its associated MOPS and s a nonnegative integer. The following statements are equivalent:*

(1) L is semiclassical of class s.

(2) There exist two non-zero polynomials ϕ and ψ, with $\deg\phi = r \geq 0$ and $\deg\psi = t \geq 1$, such that the Stieltjes function S associated with L given by (2.8) satisfies $\phi\mathcal{S}' + (\phi' - \psi)\mathcal{S} = C$, with $C = L\cdot\theta_0(\phi' - \psi) + DL\cdot\theta_0\phi$, where ϕ, $\phi' - \psi$ and C are coprime (they do not share common zeros). Notice that

$$L \cdot q(x) := \langle L, \frac{xq(x) - yq(y)}{x - y}\rangle, \quad q \in \mathbb{P}.$$

(3) There exist a monic polynomial ϕ, $\deg\phi = r$, with $0 \leq r \leq s+2$, such that

$$\phi(x)P'_{n+1}(x) = \sum_{k=n-s}^{n+r} \lambda_{n+1,k}P_k(x), \quad n \geq s, \quad \lambda_{n+1,n-s} \neq 0. \tag{2.9}$$

If $s \geq 1$, $r \geq 1$, and $\lambda_{s+1,0} \neq 0$, then s is the class of L.

Now, using the TTRR (2.3) for MOPS, (2.9) can be rewritten in a shorter form (see [13]):

Theorem 2.3. *Let L be a semiclassical functional of class s and $\{P_n\}_{n\geq 0}$ its corresponding MOPS. Then it holds (lowering operator)*

$$\left[\phi(x)D_x - \frac{C_{n+1}(x) - C_0(x)}{2}\right] P_{n+1}(x) = -D_{n+1}(x)P_n(x), \quad n \geq 0, \tag{2.10}$$

where $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *satisfy*

$$C_0(x) = -\psi(x) + \phi'(x), \quad C_{n+1}(x) = -C_n(x) + \frac{2D_n(x)}{\beta_n}(x-\alpha_n), \quad n \geq 0, \tag{2.11}$$

with $D_{-1} = 0$, $D_0 = (L \cdot \theta_0 \phi)' - (L \cdot \theta_0 \psi)$ *and*

$$D_{n+1}(x) = -\phi(x) + \frac{\beta_n}{\beta_{n-1}} D_{n-1}(x) + (x-\alpha_n)^2 \frac{D_n(x)}{\beta_n} - (x-\alpha_n)C_n(x), \quad n \geq 0.$$

The expressions of the previous theorem lead to the so-called ladder operators associated with the semiclassical functional L. Indeed, using (2.11) and the TTRR (2.9), we can deduce from (2.10) that the raising operator is given by

$$\left[\phi(x)D_x + \frac{C_{n+2}(x)+C_0(x)}{2}\right] P_{n+1}(x) = \frac{D_{n+1}(x)}{\beta_{n+1}} P_{n+2}(x), \quad n \geq 0.$$

This relation, together with (2.11) are essential to deduce a second-order linear differential equation satisfied by the polynomials $\{P_n\}_{n\geq 0}$ (see [8, 13]):

$$J(x,n)P''_{n+1}(x) + K(x,n)P'_{n+1}(x) + M(x,n)P_{n+1}(x) = 0, \quad n \geq 0,$$

where, for $n \geq 0$,

$$J(x,n) = \phi(x)D_{n+1}(x),$$
$$K(x,n) = (\phi'(x) + C_0(x))D'_{n+1}(x) - \phi(x)D'_{n+1}(x),$$
$$M(x,n) = \frac{C_{n+1}(x)-C_0(x)}{2}D'_{n+1}(x) - \frac{C'_{n+1}(x)-C'_0(x)}{2}D_{n+1}(x) - D_{n+1}(x)\sum_{k=0}^{n}\frac{D_k(x)}{\beta_k}.$$

Remark 2.1. *Observe that the degrees of the polynomials* J, K, M *are at most* $2s+2$, $2s+1$, *and* $2s$, *respectively, where* s *denotes the class of the linear functional.*

In [3, 4], assuming that the linear functional is positive-definite, it is associated with a weight function, and under some integrability conditions, raising and lowering operators for orthogonal polynomials with respect to a weight function supported on the real line are derived, as well as a second-order linear differential equation satisfied by these polynomials (particularly, in [4], the case of orthogonal polynomials with discontinuous weights is studied). Let $-\infty < a < b < \infty$ and $\{P_n\}_{n\geq 0}$ be the MOPS associated with the linear functional L defined over $\mathbb{P}$ such that $\langle L, p\rangle = \int_a^b p(x)\omega(x)dx$, with $\omega(x) := e^{-v(x)}$, where v is a twice continuously differentiable and convex function for $x \in [a,b]$. If h_n is given by (2.2) and α_n, β_n are the coefficients of the TTRR (2.3), then the actions of the ladder operators on P_n and P_{n-1} are (see [3]):

$$\left(\frac{d}{dx} + B_n(x)\right) P_n(x) = \beta_n A_n(x) P_{n-1}(x), n \geq 1, \tag{2.12}$$

$$-\left(\frac{d}{dx} - B_n(x) - v'(x)\right) P_{n-1}(x) = A_{n-1}(x)P_n(x), n \geq 1, \tag{2.13}$$

where

$$A_n(x) := \frac{1}{h_n}\int_a^b \frac{v'(x)-v'(y)}{x-y} P_n^2(y)\omega(y)\,dy + \left.\frac{P_n^2(y)\omega(y)}{h_n(y-x)}\right|_{y=a}^{y=b}, \tag{2.14}$$

$$B_n(x) := \frac{1}{h_{n-1}}\int_a^b \frac{v'(x)-v'(y)}{x-y} P_{n-1}(y)P_n(y)\omega(y)\,dy + \left.\frac{P_n(y)P_{n-1}(y)\omega(y)}{h_{n-1}(y-x)}\right|_{y=a}^{y=b}. \tag{2.15}$$

Observe that $A_n > 0$ due to the convexity of v. Thus, the lowering and raising operators are given, respectively, by

$$L_n = \frac{d}{dx} + B_n(x) \quad \text{and} \quad R_n = -\left(\frac{d}{dx} - B_n(x) - v'(x)\right).$$

In addition, the coefficient functions in the ladder operators, $A_n(z)$ and $B_n(z)$, satisfy

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) - v'(x),$$
$$B_{n+1}(x) - B_n(x) = \frac{\beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x) - 1}{x - \alpha_n}.$$

Remark 2.2. *The raising and lowering operators are adjoint within the context of a Hilbert space endowed with the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$. Note that the adjoint relationship between raising and lowering operators persists even within the framework of classical orthogonal polynomials, which adds an intriguing layer to their general setting.*

Furthermore, the combination of the raising and lowering operators leads to a second-order linear differential equation.

Theorem 2.4 ([3]). *Under the above conditions, the MOPS $\{P_n\}_{n\geq 0}$ associated with the linear functional L satisfy*

$$R_n\left[\frac{1}{A_n(x)}\left(L_n P_n(x)\right)\right] = \frac{h_{n-1}}{h_n}A_{n-1}(x)P_n(x), \quad n \geq 1,$$

or equivalently,

$$P_n''(x) + S_n(x)P_n'(x) + Q_n(x)P_n(x) = 0, \quad n \geq 1,$$

where

$$S_n(x) = -\left[v'(x) + \frac{A_n'(x)}{A_n(x)}\right]$$

and

$$Q_n(x) = A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)' - B_n(x)[v'(x) + B_n(x)] + A_n(x)A_{n-1}(x)\frac{h_{n-1}}{h_n}$$
$$= -B_n'(x) - B_n(x)\frac{A_n'(x)}{A_n(x)} - B_n(x)[v'(x) + B_n(x)] + \frac{h_{n-1}}{h_n}A_n(x)A_{n-1}(x).$$

3. Pearson equation, moments and Stieltjes function

Throughout the rest of the paper we will focus our attention on the study of the truncated Hermite positive-definite linear functional L_M, defined by

$$L_M[p] = \langle L_M, p\rangle = \int_0^\infty p(x)e^{-x^2}\,dx, \quad p \in \mathbb{P} = \mathbb{R}[x]. \tag{3.16}$$

We will recognize L_M as a D-semiclassical functional of class $s = 1$. To prove this, we begin by examining the associated Pearson equation. We will then derive a second-order linear recurrence equation for its moments. Using the z-transform of the moment sequence (the Stieltjes function), we will obtain a first-order linear differential equation satisfied by this function. Subsequently, we will derive the Laguerre-Freud equations. Finally, we will construct the ladder operators and explore the electrostatic interpretation of the zeros of the corresponding orthogonal polynomials.

The Pearson equation representing the linear functional (3.16) can be expressed as follows.

Proposition 3.3. *Let $p \in \mathbb{R}[x]$, $\phi(x) = x$ and $\psi(x) = 1 - 2x^2$. Then, the linear functional L_M defined by (3.16) satisfies the Pearson equation established in (2.7).*

Proof. For $p \in \mathbb{R}[x]$ it follows that

$$\begin{aligned}\langle D(\phi L_M), p\rangle &= -\langle L_M, \phi p'\rangle = -\int_0^\infty \phi(x)p'(x)e^{-x^2}dx \\ &= \int_0^\infty p(x)\left(1-2x^2\right)e^{-x^2}dx - \underbrace{\left[\phi(x)p(x)e^{-x^2}\right]_0^\infty}_{=0} = \langle L_M, \psi p\rangle \\ &= \langle \psi L_M, p\rangle .\end{aligned}$$

□

From Proposition 3.3 we see that L_M is a D-semiclassical functional whose class does not exceed $s = 1$. In Remark 3.4 we will show that the class is indeed $s = 1$, see further.

Remark 3.3. *Since $\psi(x) = \phi'(x) - 2x\phi(x)$, the Pearson equation from Proposition 3.3 can be rewritten as*

$$-\langle D(\phi L_M), p\rangle = \langle L_M, \phi p'\rangle = -\langle L_M, (\phi' - 2x\phi)p\rangle = \langle L_M, (2x\phi - \phi')p\rangle . \tag{3.17}$$

Regarding the corresponding moments of the linear functional (3.16), by setting $s = x^2$ we can write for all $n \geq 0$,

$$\mu_{2n} = \langle L_M, x^{2n}\rangle = \int_0^\infty x^{2n}e^{-x^2}dx = \frac{1}{2}\int_0^\infty s^{n-\frac{1}{2}}e^{-s}ds = \frac{1}{2}\cdot\Gamma\left(n+\frac{1}{2}\right), \tag{3.18}$$

$$\mu_{2n+1} = \langle L_M, x^{2n+1}\rangle = \int_0^\infty x^{2n+1}e^{-x^2}dx = \frac{1}{2}\int_0^\infty s^n e^{-s}ds = \frac{1}{2}\cdot\Gamma(n+1) = \frac{n!}{2}. \tag{3.19}$$

To deduce a second-order homogeneous difference equation that characterizes the moments associated with the linear functional L_M, we can make use of the Pearson equation given by (3.17):

Proposition 3.4. *Let $\mu_n = \langle L_M, x^n\rangle$, $n \geq 0$. Then, $\{\mu_n\}_{n\geq 0}$ satisfies*

$$2\mu_{n+2} - (n+1)\mu_n = 0, \quad n \geq 0, \tag{3.20}$$

with initial conditions $\mu_0 = \frac{\sqrt{\pi}}{2}$ and $\mu_1 = \frac{1}{2}$.

Proof. From (3.17) with $p(x) = x^n$, we get

$$\langle L_M, n\phi(x)x^{n-1}\rangle = \langle L_M, (2x\phi(x) - \phi'(x))x^n\rangle . \tag{3.21}$$

Since $\phi(x) = x$ and $2x\phi(x) - \phi'(x) = 2x^2 - 1$, we can write $\langle L_M, n\phi(x)x^{n-1}\rangle = n\langle L_M, x^n\rangle = n\mu_n$ and $\langle L_M, (2x\phi(x) - \phi'(x))x^n\rangle = 2\langle L_M, x^{n+2}\rangle - \langle L_M, x^n\rangle = 2\mu_{n+2} - \mu_n$. Therefore, (3.21) is equivalent to (3.20). The initial conditions follows directly by taking $n = 0$ in (3.18)-(3.19). □

Concerning the corresponding Stieltjes function (2.8), we can prove the following

Proposition 3.5. *Let $\mu_n = \langle L_M, x^n\rangle$, $n \geq 0$. Then, the Stieltjes function $\mathcal{S}(z)$ associated with the linear functional L_M satisfies the first-order non-homogeneous linear ordinary differential equation*

$$z\mathcal{S}'(z) + 2z^2\mathcal{S}(z) = 2(\mu_1 + z\mu_0), \tag{3.22}$$

or equivalently,

$$\phi(z)\mathcal{S}'(z) + \left[\phi'(z) - \psi(z)\right]\mathcal{S}(z) = 2(\mu_1 + \phi(z)\mu_0),$$

where $\phi(z) = z$ and $\psi(z) = 1 - 2z^2$.

Proof. If we apply the z-transform to (3.20), we get

$$2\sum_{n=0}^{\infty}\frac{\mu_{n+2}}{z^{n+2}}-\sum_{n=0}^{\infty}(n+1)\frac{\mu_n}{z^{n+2}}=0.$$

Since

$$\sum_{n=0}^{\infty}\frac{\mu_{n+2}}{z^{n+2}}=z\left[\mathcal{S}(z)-\frac{\mu_0}{z}-\frac{\mu_1}{z^2}\right]\quad\text{and}\quad\sum_{n=0}^{\infty}(n+1)\frac{\mu_n}{z^{n+2}}=-D_z\mathcal{S}(z),$$

it follows that

$$2z\left[\mathcal{S}(z)-\frac{\mu_0}{z}-\frac{\mu_1}{z^2}\right]+\mathcal{S}'(z)=0$$

and we conclude (3.22). □

Remark 3.4. *Observe that from Proposition 2.2 and (3.22), we get*

$$C(z)=2(\mu_1+z\mu_0). \tag{3.23}$$

Taking into account that $\phi(z)=z$, $\psi(z)=1-2z^2$, $\phi(0)=(\phi'-\psi)(0)=0$ and $C(0)=2\mu_1\neq 0$, the Stieltjes function associated with L_M, satisfies

$$\phi(z)\mathcal{S}'(z)+[\phi'(z)-\psi(z)]\,\mathcal{S}(z)=C(z),$$

where C is given by (3.23) and ϕ, $\phi'-\psi$, and C are coprime. Therefore, from Proposition 2.2 we conclude that L_M is a D-semiclassical functional of class $s=1$.

4. Laguerre-Freud equations

The equations governing the coefficients of the TTRR (2.3), which are typically nonlinear in nature, are recognized in the literature as Laguerre-Freud equations (see, for instance, [2]). It is noteworthy that these equations can be viewed as discrete analogues of the well-established Painlevé equations (see [11]). The aim of this section is to obtain the Laguerre-Freud equations for the Maxwell linear functional given by (3.16). We start with a structure relation satisfied by Maxwell polynomials:

Theorem 4.5. *Let $\{P_n\}_{n\geq 0}$ be the MOPS associated with the linear functional defined in (3.16) and α_n, β_n the corresponding coefficients in the TTRR (2.3). Then the polynomials P_n satisfy the structure relation*

$$\phi(x)P'_{n+1}(x)=(n+1)P_{n+1}(x)+\lambda_{n+1,n}P_n(x)+\lambda_{n+1,n-1}P_{n-1}(x),\quad n\geq 1, \tag{4.24}$$

where $\lambda_{n+1,n}=2\beta_{n+1}\left(\alpha_{n+1}+\alpha_n\right)$ and $\lambda_{n+1,n-1}=2\beta_{n+1}\beta_n$.

Proof. From Proposition 2.2 with $s=1$ and $r=\deg\phi=1$, the structure relation satisfied by the MOPS with respect to the semiclassical functional defined in (3.16) becomes for all $n\geq 1$,

$$\phi(x)P'_{n+1}(x)=(n+1)P_{n+1}(x)+\lambda_{n+1,n}P_n(x)+\lambda_{n+1,n-1}P_{n-1}(x),$$

where

$$\lambda_{n+1,n}=\frac{\left\langle L_M,\phi P'_{n+1}P_n\right\rangle}{\|P_n\|^2}\qquad\text{and}\qquad\lambda_{n+1,n-1}=\frac{\left\langle L_M,\phi P'_{n+1}P_{n-1}\right\rangle}{\|P_{n-1}\|^2}.$$

Indeed, one can observe that if we write $xP'_{n+1} = (n+1)P_{n+1} + \sum_{k=0}^{n} \lambda_{n+1,k}P_k$, then

$$\begin{aligned}\lambda_{n+1,k} &= \frac{\langle L_M, xP'_{n+1}P_k\rangle}{\|P_k\|^2} = \frac{\langle L_M, x\left[(P_{n+1}P_k)' - P_{n+1}P'_k\right]\rangle}{\|P_k\|^2} \\ &= \frac{\langle xL_M, (P_{n+1}P_k)' - P_{n+1}P'_k\rangle}{\|P_k\|^2} = \frac{-\langle D(xL_M), P_{n+1}P_k\rangle - \langle xL_M, P_{n+1}P'_k\rangle}{\|P_k\|^2} \\ &= \frac{-\langle (1-2x^2)L_M, P_{n+1}P_k\rangle - \langle L_M, xP_{n+1}P'_k\rangle}{\|P_k\|^2},\end{aligned}$$

where we have used (3.17) in the last equality. Hence,

$$\lambda_{n+1,k} = \begin{cases} 0, & \text{if } k < n-1, \\ 2\frac{\|P_{n+1}\|^2}{\|P_{n-1}\|^2}, & \text{if } k = n-1, \\ 2\frac{\langle L_M, x^2P_{n+1}P_n\rangle}{\|P_n\|^2}, & \text{if } k = n. \end{cases}$$

From the TTRR (2.3) we can write

$$\begin{cases} xP_{n+1}(x) = P_{n+2}(x) + \alpha_{n+1}P_{n+1}(x) + \beta_{n+1}P_n(x), \\ xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \end{cases}$$

which implies that

$$\langle L_M, x^2P_{n+1}P_n\rangle = \langle L_M, xP_{n+1}xP_n\rangle = \alpha_{n+1}\|P_{n+1}\|^2 + \alpha_n\beta_{n+1}\|P_n\|^2.$$

Therefore, for all $n \geq 1$ we get

$$\begin{aligned}\lambda_{n+1,n} &= 2(\alpha_{n+1}\beta_{n+1} + \alpha_n\beta_{n+1}) = 2\beta_{n+1}(\alpha_{n+1} + \alpha_n), \\ \lambda_{n+1,n-1} &= 2\beta_{n+1}\beta_n,\end{aligned}$$

and the proof is concluded. □

Theorem 4.6. *The coefficients α_n and β_n of the TTRR (2.3) for the linear functional defined in (3.16) satisfy for all $n \geq 1$ the Laguerre-Freud equations*

$$\alpha_n^2 + \beta_{n+1} + \beta_n = n + \frac{1}{2}, \tag{4.25}$$

$$2\beta_{n+2}\alpha_{n+2} + \alpha_{n+1}(2\beta_{n+2} - 2\beta_{n+1} - 1) - 2\alpha_n\beta_{n+1} = 0, \tag{4.26}$$

with initial conditions $\beta_1 = \frac{1}{2} - \frac{1}{\pi}$, $\beta_2 = \frac{\pi(\pi-3)}{(\pi-2)^2}$ and $\alpha_1 = \frac{2}{(\pi-2)\sqrt{\pi}}$.

Proof. Let $\{P_n\}_{n\geq 0}$ be the MOPS with respect to the linear functional L_M given by (3.16). Taking $p = P_n^2$ in (2.7) with $\phi(x) = x$ and $\psi(x) = 1 - 2x^2$, we get

$$-2\langle L_M, P_n\phi P'_n\rangle = \langle L_M, \psi P_n^2\rangle, \quad n \geq 1. \tag{4.27}$$

The right side of (4.27) can be rewritten as

$$\langle L_M, \psi P_n^2\rangle = \langle L_M, (1-2x^2)P_n^2\rangle = \langle L_M, P_n^2\rangle - 2\langle L_M, x^2P_n^2\rangle, \quad n \geq 1. \tag{4.28}$$

From (2.2) we have that $\langle L_M, P_n^2\rangle = h_n$ and

$$\begin{aligned} x^2P_n(x) &= x\left[P_{n+1}(x)+\alpha_nP_n(x)+\beta_nP_{n-1}(x)\right] \\ &= xP_{n+1}(x)+\alpha_nxP_n(x)+\beta_nxP_{n-1}(x) \\ &= P_{n+2}(x)+\alpha_{n+1}P_{n+1}(x)+\beta_{n+1}P_n(x)+\alpha_nP_{n+1}(x)+\alpha_n^2P_n(x) \\ &+\alpha_n\beta_nP_{n-1}(x)+\beta_nP_n(x)+\alpha_{n-1}\beta_nP_{n-1}(x)+\beta_n\beta_{n-1}P_{n-2}(x) \\ &= P_{n+2}(x)+(\alpha_{n+1}+\alpha_n)P_{n+1}(x)+(\alpha_n^2+\beta_{n+1}+\beta_n)P_n(x) \\ &+\beta_n(\alpha_n+\alpha_{n-1})P_{n-1}(x)+\beta_n\beta_{n-1}P_{n-2}(x), \end{aligned}$$

so (4.28) is equivalent to

$$\langle L_M, \psi P_n^2\rangle = h_n - 2\left(\alpha_n^2+\beta_{n+1}+\beta_n\right)h_n = \left[1-2\left(\alpha_n^2+\beta_{n+1}+\beta_n\right)\right]h_n, \quad n\ge 1. \tag{4.29}$$

In addition, from (4.24) we have that

$$\phi(x)P_n'(x) = nP_n(x)+2\beta_n\left(\alpha_n+\alpha_{n-1}\right)P_{n-1}(x)+2\beta_n\beta_{n-1}P_{n-2}(x), \quad n\ge 1,$$

so the left side of (4.27) becomes

$$\begin{aligned} -2\langle L_M, P_n\phi P_n'\rangle &= -2\langle L_M, P_n\left[nP_n+2\beta_n\left(\alpha_n+\alpha_{n-1}\right)P_{n-1}+2\beta_n\beta_{n-1}P_{n-2}\right]\rangle \\ &= -2nh_n. \end{aligned} \tag{4.30}$$

Finally, equating (4.29) and (4.30) we obtain $-2n = 1-2\left(\alpha_n^2+\beta_{n+1}+\beta_n\right)$ and (4.25) is deduced.

On the other hand, since $xP_{n+1}' = (xP_{n+1})' - P_{n+1}$ and $\langle L_M, P_{n+1}P_n\rangle = 0$, we see that

$$\begin{aligned} \lambda_{n+1,n} &= \frac{\langle L_M, xP_{n+1}'P_n\rangle}{\|P_n\|^2} = \frac{\langle L_M, (xP_{n+1})'P_n\rangle}{\|P_n\|^2} \\ &= \frac{\langle L_M, \left(P_{n+2}'+\alpha_{n+1}P_{n+1}'+\beta_{n+1}P_n'\right)P_n\rangle}{\|P_n\|^2} = \frac{\langle L_M, P_{n+2}'P_n\rangle}{\|P_n\|^2} + \alpha_{n+1}(n+1). \end{aligned}$$

Let us write

$$P_{n+2}(x) = x^{n+2}+\sum_{k=0}^{n+1}\delta_{n+2,k}x^k = x^{n+2}+\delta_{n+2,n+1}x^{n+1}+\dots,$$

so

$$P_{n+2}'(x) = (n+2)x^{n+1}+\sum_{k=1}^{n+1}k\delta_{n+2,k}x^{k-1} = (n+2)x^{n+1}+(n+1)\delta_{n+2,n+1}x^n+\dots$$

It follows that

$$\frac{\langle L_M, P_{n+2}'P_n\rangle}{\|P_n\|^2} = \frac{(n+2)\langle L_M, x^{n+1}P_n\rangle}{\|P_n\|^2} + (n+1)\delta_{n+2,n+1}.$$

Since $x^{n+1} = P_{n+1}(x) - \delta_{n+1,n}x^n - \dots$, we have $\langle L_M, x^{n+1}P_n\rangle = -\delta_{n+1,n}\|P_n\|^2$. Then,

$$\lambda_{n+1,n} = -(n+2)\delta_{n+1,n}+(n+1)\delta_{n+2,n+1}+(n+1)\alpha_{n+1}. \tag{4.31}$$

Alternatively, identifying coefficients of x^n in (4.24), we can observe that

$$\lambda_{n+1,n}+(n+1)\delta_{n+1,n} = n\delta_{n+1,n} \quad\Rightarrow\quad \lambda_{n+1,n} = -\delta_{n+1,n}.$$

Since $xP_{n+1}(x) = P_{n+2}(x) + \alpha_{n+1}P_{n+1}(x) + \beta_{n+1}P_n(x)$ we get $\delta_{n+1,n} = \delta_{n+1,n+1} + \alpha_{n+1}$, and this allows us to recover (4.31). Therefore, we can write

$$\begin{cases} 2\beta_{n+1}(\alpha_{n+1} + \alpha_n) & = -\delta_{n+1,n}, \\ 2\beta_{n+2}(\alpha_{n+2} + \alpha_{n+1}) & = -\delta_{n+2,n+1}, \end{cases}$$

and this implies

$$\alpha_{n+1} = 2\beta_{n+2}(\alpha_{n+2} + \alpha_{n+1}) - 2\beta_{n+1}(\alpha_{n+1} + \alpha_n), \quad n \geq 1.$$

which is equivalent to (4.26). The initial values α_1, β_1 and β_2 can be obtained by direct computation (see Remark 4.5 and Theorem 4.8 further). $\square$

A second recurrence relation for the parameters of the TTRR (2.3) which is of interest can be obtained as follows. Let $\{\tilde{p}_n\}_{n\geq 0}$ be the ONPS associated with the linear functional defined in (3.16), with TTRR given by (2.4) and $\tilde{p}_n = \frac{1}{\sqrt{h_n}} P_n$ with h_n defined in (2.2). If we multiply by $x^2 e^{-x^2}$ in the confluent formula (2.6) and we integrate over $[0, \infty)$, it follows that

$$\sum_{k=0}^{n} \int_0^\infty x^2 \tilde{p}_k^2(x) e^{-x^2} dx = \sqrt{\beta_{n+1}} \left[\int_0^\infty \tilde{p}'_{n+1}(x) x^2 \tilde{p}_n(x) e^{-x^2} dx - \int_0^\infty \tilde{p}_{n+1}(x) \tilde{p}'_n(x) x^2 e^{-x^2} dx \right]. \tag{4.32}$$

The integration by parts in the first member of (4.32) yields

$$\begin{aligned} \int_0^\infty x \tilde{p}_k^2(x) x e^{-x^2} dx &= \frac{1}{2} \int_0^\infty \left[\tilde{p}_k^2(x) + 2x \tilde{p}_k(x) \tilde{p}'_k(x) \right] e^{-x^2} dx \\ &= \frac{1}{2} + \int_0^\infty x \tilde{p}_k(x) \tilde{p}'_k(x) e^{-x^2} dx = \frac{1}{2} + k. \end{aligned}$$

Therefore, we have that

$$\sum_{k=0}^{n} \int_0^\infty x^2 \tilde{p}_k^2(x) e^{-x^2} dx = \sum_{k=0}^{n} \frac{1}{2} + k = \frac{1}{2}(n+1)^2. \tag{4.33}$$

Regarding the right-hand side of (4.32), notice that

$$\begin{aligned} &\int_0^\infty \tilde{p}'_{n+1}(x) x^2 \tilde{p}_n(x) e^{-x^2} dx - n \int_0^\infty \tilde{p}_{n+1}(x) \frac{\sqrt{h_{n+1}}}{\sqrt{h_n}} \tilde{p}_{n+1}(x) x^2 e^{-x^2} dx \\ &= -n\sqrt{\beta_{n+1}} - \int_0^\infty \tilde{p}_{n+1}(x) \left(2x \tilde{p}_n(x) e^{-x^2} \right) dx - \int_0^\infty \tilde{p}_{n+1}(x) x^2 \tilde{p}'_n(x) e^{-x^2} dx \\ &+2 \int_0^\infty \tilde{p}_{n+1}(x) \tilde{p}_n(x) x^3 e^{-x^2} dx \\ &= -n\sqrt{\beta_{n+1}} - 2\sqrt{\beta_{n+1}} - n\sqrt{\beta_{n+1}} \\ &= -2(n+1)\sqrt{\beta_{n+1}} + 2 \int_0^\infty \tilde{p}_{n+1}(x) \tilde{p}_n(x) x^3 e^{-x^2} dx. \end{aligned}$$

Now, from (2.4) we get

$$\begin{aligned}x^2\tilde{p}_n(x) =&\sqrt{\beta_{n+1}}\left[\sqrt{\beta_{n+2}}\tilde{p}_{n+2}(x)+\alpha_{n+1}\tilde{p}_{n+1}(x)+\sqrt{\beta_{n+1}}\tilde{p}_n(x)\right]\\ &+\alpha_n\left[\sqrt{\beta_{n+1}}\tilde{p}_{n+1}(x)+\alpha_n\tilde{p}_n(x)+\sqrt{\beta_n}\tilde{p}_{n-1}(x)\right]\\ &+\sqrt{\beta_n}\left[\sqrt{\beta_n}\tilde{p}_n(x)+\alpha_{n-1}\tilde{p}_{n-1}(x)+\sqrt{\beta_{n-1}}\tilde{p}_{n-2}(x)\right]\\ &=\sqrt{\beta_{n+1}\beta_{n+2}}\tilde{p}_{n+2}(x)+\left(\sqrt{\beta_{n+1}}\alpha_{n+1}+\alpha_n\sqrt{\beta_{n+1}}\right)\tilde{p}_{n+1}(x)\\ &+\left(\beta_{n+1}+\alpha_n^2+\beta_n\right)\tilde{p}_n(x),\end{aligned}$$

and by making use of the TTRR (2.4) with n replaced by $n+1$ it follows that

$$\int_0^\infty x^3\tilde{p}_{n+1}(x)\tilde{p}_n(x)e^{-x^2}dx=\sqrt{\beta_{n+1}}\left[\beta_{n+2}+\alpha_{n+1}\left(\alpha_{n+1}+\alpha_n\right)+\beta_{n+1}+\alpha_n^2+\beta_n\right],$$

which implies that the right-hand side of (4.32) reads as

$$2\beta_{n+1}\left[\beta_{n+2}+\alpha_{n+1}\left(\alpha_{n+1}+\alpha_n\right)+\beta_{n+1}+\alpha_n^2+\beta_n-(n+1)\right]. \tag{4.34}$$

If we equate (4.33) and (4.34),

$$\begin{aligned}(n+1)^2 &= 4\beta_{n+1}\left[\beta_{n+2}+\alpha_{n+1}\left(\alpha_{n+1}+\alpha_n\right)+\beta_{n+1}+\alpha_n^2+\beta_n-(n+1)\right]\\ &=4\beta_{n+1}\left[n+\frac{3}{2}+\alpha_n\left(\alpha_n+\alpha_{n+1}\right)+\beta_n-n-1\right]\\ &=4\beta_{n+1}\left[\frac{1}{2}+\beta_n+\alpha_n\left(\alpha_n+\alpha_{n+1}\right)\right].\end{aligned}$$

Therefore, we have proved the following

Theorem 4.7. *The coefficients α_n and β_n of the TTRR (2.3) for the linear functional defined in* (3.16) *satisfy the Laguerre-Freud equations* (4.25) *and*

$$4\beta_{n+1}\left[\frac{1}{2}+\beta_n+\alpha_n\left(\alpha_n+\alpha_{n+1}\right)\right]=(n+1)^2,\quad n\geq 1, \tag{4.35}$$

with initial conditions $\beta_1=\frac{1}{2}-\frac{1}{\pi}$, $\beta_2=\frac{\pi(\pi-3)}{(\pi-2)^2}$ and $\alpha_1=\frac{2}{(\pi-2)\sqrt{\pi}}$.

Moreover, one can easily see that (4.35) can be rewritten as

$$(n+1)^2=4\beta_{n+1}\left[\frac{1}{2}+\alpha_n\alpha_{n+1}+n+\frac{1}{2}-\beta_{n+1}\right]=4\beta_{n+1}\left[n+1-\beta_{n+1}+\alpha_n\alpha_{n+1}\right],$$

which implies that

$$4\beta_{n+1}\alpha_n\alpha_{n+1}=(n+1)^2-4(n+1)\beta_{n+1}+4\beta_{n+1}^2=(n+1-2\beta_{n+1})^2,$$

or equivalently,

$$16\beta_{n+1}^2\alpha_n^2\alpha_{n+1}^2=(n+1-2\beta_{n+1})^4. \tag{4.36}$$

On the other hand, from (4.25) we have that $2\beta_{n+1}+2\beta_n+2\alpha_n^2=2n+1$ and hence, $2\alpha_n^2=n+1-2\beta_{n+1}+n-2\beta_n$. Thus, (4.36) becomes

$$4\beta_{n+1}^2(n+1-2\beta_{n+1}+n-2\beta_n)(n+2-2\beta_{n+2}+n+1-2\beta_{n+1})=(n+1-2\beta_{n+1})^4,$$

or, alternatively,

$$\beta_{n+1}^2\left(\frac{n+1}{2}-\beta_{n+1}+\frac{n}{2}-\beta_n\right)\left(\frac{n+2}{2}-\beta_{n+2}+\frac{n+1}{2}-\beta_{n+1}\right)=\left(\frac{n+1}{2}-\beta_{n+1}\right)^4.$$

Setting $g_n = \frac{n}{2} - \beta_n$, the previous equation turns into

$$g_{n+1}^4 = \left(\frac{n+1}{2} - g_{n+1}\right)^2 (g_{n+1}g_n)(g_{n+2} + g_{n+1}), \quad n \geq 1,$$

which is a discrete Painlevé IV equation (denoted in the literature as $d - P_{IV}$). We refer to [17, p. 10] and [18, Section 4] for more details.

Finally, one can observe that

$$\alpha_n^2 = n + \frac{1}{2} - \beta_n - \beta_{n+1} = n + \frac{1}{2} + \left(g_n - \frac{n}{2}\right) = g_n + g_{n+1}.$$

Remark 4.5. *Concerning the initial conditions, from the relation* $\beta_n = \frac{h_n}{h_{n-1}}$ $(n \geq 1)$ *that follows from Theorem 2.1, where* $h_n = \|P_n\|^2 = \langle L_M, P_n^2 \rangle$*, and from the explicit expressions for the moments (3.18)-(3.19) it is an easy exercise to check that*

$$P_0(x) \equiv 1, \qquad P_1(x) = x - \frac{1}{\sqrt{\pi}}, \qquad P_2(x) = x^2 - \frac{\sqrt{\pi}}{\pi - 2}x + \frac{4 - \pi}{2(\pi - 2)},$$

which implies $\beta_1 = \frac{1}{2} - \frac{1}{\pi}$ *and* $\beta_2 = \frac{\pi(\pi-3)}{(\pi-2)^2}$*. Therefore, we get*

$$g_1 = \frac{1}{2} - \beta_1 = \frac{1}{\pi}, \qquad g_2 = 1 - \beta_2 = 1 - \frac{\pi(\pi-3)}{(\pi-2)^2} = \frac{4-\pi}{(\pi-2)^2}. \tag{4.37}$$

Thus, the following result has been proved.

Theorem 4.8. *Let* α_n *and* β_n *be the coefficients of the TTRR (2.3) for the linear functional defined in (3.16) and let* $\{g_n\}_{n\geq 0}$ *be defined by* $g_n = \frac{n}{2} - \beta_n$*. Then,* $\{g_n\}_{n\geq 0}$ *satisfy the* $d - P_{IV}$ *equation*

$$g_{n+1}^4 = \left(\frac{n+1}{2} - g_{n+1}\right)^2 (g_{n+1} + g_n)(g_{n+2} + g_{n+1}), \quad n \geq 1,$$

with initial conditions given by (4.37). In addition, for all $n \geq 1$ *we have* $\alpha_n^2 = g_n + g_{n+1}$*.*

To conclude this section, we will study the asymptotic behavior of the coefficients α_n and β_n. Our aim is to find the leading behavior of an asymptotic series solution for the recurrence relations discussed previously, that is, the first term in the expansion. Since α_n and β_n are all positive and approach infinity as $n \to \infty$, we can write

$$\alpha_n = n^r \tilde{\alpha}_n \quad \text{and} \quad \beta_n = n^s \tilde{\beta}_n,$$

where $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ approach positive constants A and B respectively, as $n \to \infty$. Hence, let us consider $\alpha_n \sim An^r$ and $\beta_n \sim Bn^s$ as $n \to \infty$, where r and s are unknown positive constants and $\sim$ means *asymptotic to*. Substituting these asymptotic forms into the recurrence relation (4.25) yields

$$B^2 n^{2r} + 2An^s = n + \frac{1}{2}.$$

We require that this equation hold for all $n > 0$ so that $s = 1$ and $2r = 1$. Then, the coefficients A and B satisfy the equation

$$B^2 + 2A = 1. \tag{4.38}$$

On the other hand, if we replace the asymptotic expressions in (4.35), then

$$4A(n+1)\left[\frac{1}{2} + An + Bn^{\frac{1}{2}}\left(Bn^{\frac{1}{2}} + B(n+1)^{\frac{1}{2}}\right)\right] = (n+1)^2,$$

which is equivalent to

$$4A\left[\frac{1}{2}+An+B^2n\left(1+\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\right)\right]=(n+1).$$

Again, since this equation also hold for all $n>0$, we have

$$4A\left(A+2B^2\right)=1. \tag{4.39}$$

Now, we need to solve the nonlinear system of equations formed by (4.38) and (4.39). Note that we can write $2B^2=2-4A$ due to (4.38), so (4.39) can be turned into $4A\left(2-3A\right)=1 \Leftrightarrow 12A^2-8A+1=0$, whose solutions are $A_1=\frac{1}{2}\Rightarrow B_1=0$ and $A_2=\frac{1}{6}\Rightarrow B_2=\sqrt{\frac{2}{3}}$. Given that α_n and β_n are positive for all n, so A and B must both be positive. Thus, $A=\frac{1}{6}$ and $B_2=\sqrt{\frac{2}{3}}$ is the only possible solution, so the leading behavior is given by

$$\alpha_n\sim\frac{1}{6}n \quad \text{and} \quad \beta_n\sim\sqrt{\frac{2n}{3}}.$$

5. Ladder operators and electrostatic interpretation of the zeros

In this section, we construct first the ladder operators for the semiclassical functional of class $s=1$ defined by (3.16). Then, a second-order linear differential equation (in x) for the Maxwell polynomial P_{n+1} will be obtained, as well as the electrostatic interpretation of the zeros of these polynomials.

Theorem 5.9. *Let the lowering operator L_{n+1} be defined by $L_{n+1}=A_{n+1}(x)D_x-B_{n+1}(x)$ for all $n\geq 0$, where*

$$A_{n+1}(x)=\frac{\phi(x)}{2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}} \quad \text{and} \quad B_{n+1}(x)=\frac{n+1-2\beta_{n+1}}{2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}},$$

with $\phi(x)=x$ and $\lambda_{n+1,n}=2\beta_{n+1}(\alpha_{n+1}+\alpha_n)$. Then,

$$L_{n+1}P_{n+1}=P_n, \qquad n\geq 0. \tag{5.40}$$

Proof. From (4.24) we have that

$$\phi(x)P'_{n+1}(x)=(n+1)P_{n+1}(x)+\lambda_{n+1,n}P_n(x)+2\beta_{n+1}\beta_nP_{n-1}(x).$$

Using the TTRR (2.3), we can see that $\beta_nP_{n-1}(x)=(x-\alpha_n)P_n(x)-P_{n+1}(x)$, so the previous expression is equivalent to

$$\begin{aligned}\phi(x)P'_{n+1}(x)&=(n+1)P_{n+1}(x)+\lambda_{n+1,n}P_n(x)+2\beta_{n+1}\left[(x-\alpha_n)P_n(x)-P_{n+1}(x)\right]\\&=\left(n+1-2\beta_{n+1}\right)P_{n+1}(x)+\left[2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}\right]P_n(x).\end{aligned} \tag{5.41}$$

Therefore, we can write

$$\left[\phi D_x-(n+1-2\beta_{n+1})\right]P_{n+1}(x)=\left[2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}\right]P_n(x),$$

and (5.40) follows. □

Theorem 5.10. *Let the raising operator R_{n+1} be defined by $R_{n+1} = C_{n+1}(x)D_x + E_{n+1}(x)$, for all $n \geq 0$, where*

$$C_{n+1}(x) = -\frac{\phi(x)}{2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}},$$

$$E_{n+1}(x) = \frac{n+1-2\beta_{n+1} + \left(2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}\right)(x-\alpha_{n+1})}{2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}},$$

with $\phi(x) = x$ and $\lambda_{n+1,n} = 2\beta_{n+1}(\alpha_{n+1} + \alpha_n)$. Then,

$$R_{n+1}P_{n+1} = P_{n+2}, \qquad n \geq 0. \tag{5.42}$$

Proof. Using the TTRR (2.3), it is clear that $P_n = \frac{1}{\beta_{n+1}}\left[(x-\alpha_{n+1})P_{n+1} - P_{n+2}\right]$, where $\beta_{n+1} > 0$, so (5.41) can be rewritten as

$$\begin{aligned}\phi P'_{n+1} &= (n+1-2\beta_{n+1})\,P_{n+1} + \frac{1}{\beta_{n+1}}\left[2\beta_{n+1}(x-\alpha_n) + \lambda_{n+1,n}\right]\left[(x-\alpha_{n+1})P_{n+1} - P_{n+2}\right] \\ &= \left[n+1-2\beta_{n+1} + \left(2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}\right)(x-\alpha_{n+1})\right]P_{n+1} \\ &\quad - \left(2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}\right)P_{n+2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left[\phi D_x - n - 1 + 2\beta_{n+1} - \left(2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}\right)(x-\alpha_{n+1})\right]P_{n+1} \\ &= -\left(2(x-\alpha_n) + \frac{\lambda_{n+1,n}}{\beta_{n+1}}\right)P_{n+2},\end{aligned}$$

and (5.42) follows. □

Considering the definition of the lowering operator L_{n+1} in Theorem 5.9, we can derive a second-order linear differential equation in terms of x for P_{n+1}.

Theorem 5.11. *Let the second-order linear differential operator $\mathcal{D}_{n+1}$ be defined for all $n \geq 0$ by*

$$\begin{aligned}\mathcal{D}_{n+1} =& \beta_n A_n(x)A_{n+1}(x)D_x^2 \\ &+ \left[\beta_n\left(A_n(x)\big(A'_{n+1}(x) - B_{n+1}(x)\big) - A_{n+1}(x)B_n(x)\right) + \alpha_n A_{n+1}(x) - xA_{n+1}(x)\right]D_x \\ &+ \beta_n\left[B_n(x)B_{n+1} - A_n(x)B'_{n+1}(x)\right] + xB_{n+1}(x) - \alpha_n B_{n+1}(x) + 1.\end{aligned}$$

Then, $\mathcal{D}_{n+1}P_{n+1} = 0$, for all $n \geq 0$.

Proof. By (2.3), we have that $xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$, which can be rewritten in terms of (5.40) as

$$xL_{n+1}P_{n+1}(x) = P_{n+1}(x) + \alpha_n L_{n+1}P_{n+1}(x) + \beta_n L_n L_{n+1}P_{n+1}(x).$$

For a function f of x, we can observe that

$$\begin{aligned}L_nL_{n+1}f(x) &= \left[A_n(x)D_x - B_n(x)\right]\cdot\left[A_{n+1}(x)f'(x) - B_{n+1}(x)f(x)\right] \\ &= A_n(x)\left[A'_{n+1}(x)f'(x) + A_{n+1}(x)f''(x) - B'_{n+1}(x)f(x) - B_{n+1}(x)f'(x)\right] \\ &\quad - B_n(x)\left[A_{n+1}(x)f'(x) - B_{n+1}(x)f(x)\right].\end{aligned}$$

Hence, for $f = P_{n+1}$ it follows that

$$\begin{aligned}
0 &= [\beta_n L_n L_{n+1} - x L_{n+1} + \alpha_n L_{n+1} + 1] f(x) \\
&= \beta_n A_n(x) A_{n+1}(x) f''(x) \\
&+ \beta_n \left[A_n(x) \left(A'_{n+1}(x) - B_{n+1}(x) \right) - A_{n+1}(x) B_n(x) \right] f'(x) \\
&+ [\alpha_n A_{n+1}(x) - x A_{n+1}(x)] f'(x) \\
&+ \beta_n \left[B_n(x) B_{n+1} - A_n(x) B'_{n+1}(x) \right] f(x) \\
&+ [x B_{n+1}(x) - \alpha_n B_{n+1}(x) + 1] f(x),
\end{aligned}$$

and the result is obtained. □

Let us consider the ladder operators discussed in [3, 4] so that we can obtain the electrostatic interpretation of the zeros in a simpler way. Let P_n be the monic orthogonal polynomial of degree n associated with the linear functional in (3.16). Recalling (2.12) and (2.13), we have

$$\frac{1}{\beta_n \tilde{A}_n(x)} \left(\frac{d}{dx} + \tilde{B}_n(x) \right) P_n(x) = P_{n-1}(x), \quad n \geq 1,$$

and

$$-\frac{1}{\tilde{A}_{n-1}(x)} \left(\frac{d}{dx} - \tilde{B}_n(x) - v'(x) \right) P_{n-1}(x) = P_n(x), \quad n \geq 1,$$

where $\tilde{A}_n(x)$ and $\tilde{B}_n(x)$ are given by (2.14) and (2.15), respectively. In the case of the linear functional L_M defined in (3.16), we observe that $a = 0$, $b = \infty$, $v(x) = x^2$ and $\omega(x) = e^{-x^2}$. Therefore, since $\langle L_M, P_n^2 \rangle = h_n$ and $\langle L_M, P_n P_{n-1} \rangle = 0$, we have

$$\begin{aligned}
\tilde{A}_n(x) &= \frac{2}{h_n} \int_0^\infty P_n^2(y) e^{-y^2} dy + \left. \frac{P_n^2(y) e^{-y^2}}{h_n (y-x)} \right|_{y=0}^{y=\infty} = \frac{2}{h_n} \langle L_M, P_n^2 \rangle + \frac{1}{h_n} \cdot \frac{P_n^2(0)}{x} \\
&= 2 + \frac{1}{h_n} \cdot \frac{P_n^2(0)}{x}, \\
\tilde{B}_n(x) &= \frac{2}{h_{n-1}} \int_0^\infty P_{n-1}(y) P_n(y) e^{-y^2} dy + \left. \frac{P_n(y) P_{n-1}(y) e^{-y^2}}{h_{n-1}(y-x)} \right|_{y=0}^{y=\infty} \\
&= \frac{2}{h_{n-1}} \langle L_M, P_n P_{n-1} \rangle + \frac{1}{h_{n-1}} \cdot \frac{P_n(0) P_{n-1}(0)}{x} = \frac{1}{h_{n-1}} \cdot \frac{P_n(0) P_{n-1}(0)}{x}.
\end{aligned}$$

Then, the lowering and raising operators are also determined, respectively, by

$$\tilde{L}_n = \frac{1}{\beta_n \tilde{A}_n(x)} \left(\frac{d}{dx} + \tilde{B}_n(x) \right), \quad n \geq 1, \tag{5.43}$$

$$\tilde{R}_n = -\frac{1}{\tilde{A}_{n-1}(x)} \left(\frac{d}{dx} - \tilde{B}_n(x) - 2x \right), \quad n \geq 1. \tag{5.44}$$

Moreover, from Theorem 2.4, we have that P_n satisfies the second-order linear ordinary differential equation

$$P''_n(x) - \left(\frac{\tilde{A}'_n(x)}{\tilde{A}_n(x)} - 2x \right) P'_n(x) + Q_n(x) P_n(x) = 0, \quad n \geq 1,$$

where

$$\frac{\tilde{A}'_n(x)}{\tilde{A}_n(x)} = -\frac{P_n^2(0)}{x(2h_n x + P_n^2(0))}$$

and

$$
\begin{aligned}
Q_n(x) &= -\tilde{B}_n'(x) - \tilde{B}_n(x)\frac{\tilde{A}_n'(x)}{\tilde{A}_n(x)} - \tilde{B}_n(x)\left[\tilde{B}_n(x) - 2x\right] + \frac{h_{n-1}}{h_n}\tilde{A}_n(x)\tilde{A}_{n-1}(x) \\
&= \frac{P_n^3(0)P_{n-1}(0)}{h_{n-1}x^2(2h_nx + P_n^2(0))} - \frac{P_n^2(0)P_{n-1}^2(0) + h_{n-1}P_n(0)P_{n-1}(0)}{h_{n-1}^2x^2} \\
&+ \frac{2P_n(0)P_{n-1}(0)}{h_{n-1}} + 4\frac{h_{n-1}}{h_n} + \frac{2}{h_n}\frac{P_{n-1}^2(0)}{x} + 2\frac{h_{n-1}}{h_n^2}\frac{P_n^2(0)}{x} \\
&+ \frac{1}{h_nh_{n-1}}\frac{P_n^2(0)P_{n-1}^2(0)}{x^2}, \quad n \geq 1.
\end{aligned} \tag{5.45}
$$

Therefore, we have proved the following

Theorem 5.12. *Let the lowering and the raising operators be defined by (5.43) and (5.44), respectively. Then, $\tilde{R}_nP_n = P_{n+1}$ and $\tilde{L}_nP_n = P_{n-1}$, for $n \geq 1$. Moreover, if we define the second-order linear differential operator $\tilde{\mathcal{D}}_n$ as*

$$
\tilde{\mathcal{D}}_n = D_x^2 + \left(\frac{P_n^2(0)}{x(2h_nx + P_n^2(0))} + 2x\right)D_x + Q_n(x), \quad n \geq 1, \tag{5.46}
$$

where Q_n is given by (5.45), then $\tilde{\mathcal{D}}_nP_n = 0$, for all $n \geq 1$.

To end, we can obtain an electrostatic interpretation of the zeros of the corresponding OPS by means of Theorem 5.12. Indeed, let us denote by $\{x_{n,k}\}_{k=1}^n$ the zeros of P_n in increasing order, i.e.,

$$
P_n(x_{n,k}) = 0,\ 1 \leq k \leq n, \quad \text{and} \quad x_{n,1} < x_{n,2} < \cdots < x_{n,n}.
$$

Evaluating the operator $\tilde{\mathcal{D}}_n$ given by (5.46) at $x = x_{n,k}$, we see that

$$
\frac{P_n''(x_{n,k})}{P_n'(x_{n,k})} = -\frac{P_n^2(0)}{x(2h_nx + P_n^2(0))} - 2x_{n,k} = D_x\left[\ln(\tilde{A}_n(x_{n,k}))\right] - 2x_{n,k}, \quad n \geq 1, \tag{5.47}
$$

where $\tilde{A}_n$ is given by

$$
\tilde{A}_n(x) = 2 + \frac{1}{h_n}\cdot\frac{P_n^2(0)}{x} = \frac{2h_nx + P_n^2(0)}{h_nx}. \tag{5.48}
$$

Theorem 5.13. *The zeros of $P_n(x)$ correspond to the equilibrium positions of n unit-charged particles distributed within the interval $(0,\infty)$ under the influence of the potential*

$$
V_n(x) = x^2 + \ln|x| - \ln\left|x + \frac{P_n^2(0)}{2h_n}\right|. \tag{5.49}
$$

Proof. If we write $P_n(x) = \prod_{k=1}^n(x - x_{n,k})$, then according to [8, Ch. 10],

$$
\left.\frac{P_{n+1}''(x)}{P_{n+1}'(x)}\right|_{x=x_{n,k}} = \sum_{j=1,j\neq k}^{n}\frac{2}{x_{n,k} - x_{n,j}},
$$

and so, (5.47) implies that

$$
\begin{aligned}
&\sum_{j=1,j\neq k}^{n}\frac{2}{x_{n,k} - x_{n,j}} + 2x_{n,k} + \frac{P_n^2(0)}{x_{n,k}(2h_nx_{n,k} + P_n^2(0))} \\
&= \sum_{j=1,j\neq k}^{n}\frac{2}{x_{n,k} - x_{n,j}} + 2x_{n,k} - D_x\left[\ln(\tilde{A}_n(x_{n,k}))\right] = 0,
\end{aligned}
$$

or equivalently,

$$\frac{\partial E_n}{\partial x_{n,k}} = 0, \quad k = 1, \ldots, n,$$

where the total energy of the system, $E_n := E_n(x_{n,1}, \ldots, x_{n,n})$ is given by

$$E_n = -2 \sum_{1 \le j < k \le n} \ln |x_{n,k} - x_{n,j}| + \sum_{k=1}^{n} \left[x_{n,k}^2 - \ln \left(\tilde{A}_n(x_{n,k}) \right) \right]. \tag{5.50}$$

Due to (5.48), it is clear that

$$\begin{aligned} \frac{\tilde{A}'_n(x)}{\tilde{A}_n(x)} &= D_x \left[\ln \left(\tilde{A}_n(x) \right) \right] = D_x \left[\ln \left(2h_n x + P_n^2(0) \right) - \ln \left(h_n x \right) \right] \\ &= \frac{2h_n}{2h_n x + P_n^2(0)} - \frac{1}{x} = \frac{1}{x + \frac{P_n^2(0)}{2h_n}} - \frac{1}{x}. \end{aligned}$$

Therefore, we can rewrite (5.50) as

$$E_n = -2 \sum_{1 \le j < k \le n} \ln |x_{n,k} - x_{n,j}| + \sum_{k=1}^{n} \left[x_{n,k}^2 + \ln |x_{n,k}| - \ln \left| x_{n,k} + \frac{P_n^2(0)}{2h_n} \right| \right]$$

and consequently, the external potential V_n is expressed as (5.49). □

Remark 5.6. *Regarding the value of $P_n(0)$, it is important to note that it can be generated iteratively. Indeed, by the TTRR (2.3), one has that*

$$P_{n+1}(0) + \alpha_n P_n(0) + \beta_n P_{n-1}(0) = 0, \quad n \ge 1, \qquad P_0(0) = 1, \quad P_1(0) = -\frac{1}{\sqrt{\pi}}.$$

Moreover, considering that $\frac{P_n^2(0)}{2h_n} > 0$, we have an extra charge located at $-\frac{P_n^2(0)}{2h_n} < 0$. Additionally, it can be observed that there is a negative charge at the origin, which attracts the positive ones.

6. Concluding remarks

In this paper, we have analyzed a semiclassical linear functional of class 1 defined by the weight function $w(x) = e^{-x^2}$ supported in the positive real semi-axis. We deduce the Laguerre-Freud equations for the coefficients of the three term recurrence relation that the corresponding sequence of orthogonal polynomials satisfy. These coefficients are given in terms of a sequence satisfying a discrete Painlevé IV equation. In a next step the ladder operators associated with such a sequence of orthogonal polynomials are obtained. As a consequence, we get a second order linear differential equation with polynomial coefficients that such orthogonal polynomials satisfy. Thus an electrostatic interpretation of their zeros is discussed.

7. Acknowledgements

Authors thank the valuable comments by the referees. They have contributed to improve the presentation of the manuscript. The work of Francisco Marcellán has been supported by the research project [PID2021- 122154NB-I00] *Ortogonalidad y Aproximación con Aplicaciones en Machine Learning y Teoría de la Probabilidad* funded by MICIU/AEI/10.13039/501100011033 and by "ERDF A Way of making Europe".

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