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A Study On Geometry of Spatial Kinematics in Lorentzian Space

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Abstract: In this paper, we study geometry of mappings of spatial kinematics in Lorentzian space with the aid of dual number and split quaternion. Also, we give orthogonal rotation matrix **A** with respect to the Lorentzian Rodrigues parameters and the Lorentzian Euler parameters in such a space. Then, Study's soma is developed and used to define the mapping of spatial kinematics into points of a dual Lorentzian projective space.

Lorentz Uzayında Uzaysal Kinematiklerin Geometrisi Üzerine Bir Çalışma

Anahtar Kelimeler
Dual sayı,
Euler parametreleri,
Lorentz uzay,
Uzaysal kinematik,
Split kuaterniyon

Özet: Bu çalışmada, dual sayı ve split kuaterniyon yardımıyla Lorentz uzayında uzaysal kinematiklerin dönüşümlerinin geometrisi ele alınmıştır. Ayrıca, bu uzayda bir **A** ortogonal dönüşüm matrisi, Lorentziyen Rodrigues ve Euler parametrelerine göre verilmiştir. Son olarak, Study'nin "soma" olarak isimlendirdiği dönüşüm uzayı geliştirilmiş ve bu yapı bir dual Lorentz projektif uzayın noktaları içinde uzaysal kinematiklerin dönüşümünü tanımlamak için kullanılmıştır.

1. Introduction and Notations

We know that in Euclidean 3−space, a displacement of a spatial rigid body motion is explained by six parameters. So, the geometry of general displacement is considered as six dimensional. This idea brings of mapping a set of displacements in to the points of a six dimensional space. Important contributions to the theory of spatial displacement have been made by Study [10]. Study has used eight homogeneous coordinate instead of six inhomogeneous coordinates. Obtained eight homogeneous coordinates divide into two sets of four, each of them represents a vector in a Euclidean 4−space. There are many papers in shed light on this elegant notion [3, 5, 6, 11, 12]. Especially, spatial kinematics of points and lines in Euclidean space have been studied by Ravani and Roth [9]. Inspired by their paper, we get the corresponding of the study in Lorentzian space. So, we have a different geometric representation for Study soma. Therefore, we can use four homogeneous coordinates corresponding to a point of a dual Lorentzian space instead of eight homogeneous coordinates.

This idea showed that a spatial displacement can be mapped into a point of a dual Lorentzian projective space. So, we have important tool for studying spatial displacements and mentions in this paper.

Now, we give some basic concepts and notations in Lorentzian space.

Let E_1^3 be Lorentzian 3−space with the scalar product

$$\langle \vec{u}, \vec{v} \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

and the vector product

$$\vec{u} \times \vec{v} = \left(- \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right),$$

where $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in E_1^3$ [7].

Definition 1.1. A vector $\vec{u}$ in E_1^3 is said to be spacelike if $\langle \vec{u}, \vec{u} \rangle > 0$ or $\vec{u} = 0$, timelike if $\langle \vec{u}, \vec{u} \rangle < 0$, lightlike or null if $\langle \vec{u}, \vec{u} \rangle = 0$ and $\vec{u} \neq 0$. The norm of a vector $\vec{u} \in E_1^3$ is defined by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle|}$, [7].

Definition 1.2. The signature matrix S in Lorentzian 3-space $\mathbb{E}_1^3$ is the diagonal matrix whose diagonal entries are $s_1 = -1$ and $s_2 = s_3 = +1$. A matrix A is said to be a skew-symmetric matrix in Lorentzian 3-space if its transpose satisfies the equation $A^t = -SAS$, and a matrix **A** is said to be an orthogonal matrix in Lorentzian 3-space if its transpose satisfies the equation $A^t = SA^{-1}S$, [7].

Definition 1.3. A split quaternion q is given of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k,$$

where a_0, a_1, a_2 and a_3 are real numbers and i, j, k are split quaternionic units which satisfy the non-commutative multiplications rules, [4],

$$i^2 = -1, \quad j^2 = k^2 = 1$$
$$ij = -ji = k, \quad jk = -kj = -i \text{ and } ki = -ik = j.$$

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If $r = a_0 + a_1 i + a_2 j + a_3 k$ and $p = b_0 + b_1 i + b_2 j + b_3 k$ be the split quaternions and $r = pq$, then r is given by

$$\begin{aligned} r &= \mathrm{Re}(p)\,\mathrm{Re}(q) + \langle \mathrm{Im}(p), \mathrm{Im}(q) \rangle + \mathrm{Re}(p)\,\mathrm{Im}(q) \\ &\quad + \mathrm{Re}(q)\,\mathrm{Im}(p) + \mathrm{Im}(p) \times \mathrm{Im}(q) \end{aligned}$$

where

$$\begin{aligned} \mathrm{Re}(p) &= a_0, \quad \mathrm{Im}(p) = a_1 i + a_2 j + a_3 k, \\ \mathrm{Re}(q) &= b_0, \quad \mathrm{Im}(q) = b_1 i + b_2 j + b_3 k. \end{aligned}$$

Now, we can give Lorentzian spatial displacements for points and lines. In here, we will be obtained orthogonal rotation matrix **A** with respect to the Lorentzian Rodrigues and the Lorentzian Euler parameters in the Lorentzian space.

2. Lorentzian spatial displacements and Lorentzian Euler parameters

Let Σ be a fixed space and σ be a moving space with cartesian Lorentzian coordinate systems $\{X, Y, Z\}$ and $\{x, y, z\}$ embedded in Σ and σ, respectively. So, we can write the following linear transformations between the Lorentzian coordinates of points and lines of σ and their Lorentzian coordinates in Σ, [2]:
For points

$$\mathbf{X} = \mathbf{A}x + d; \tag{1}$$

For lines

$$W = \mathbf{A}w \tag{2}$$

$$W' = \mathbf{A}w' + \mathbf{AD}w$$

where $\mathbf{X}$ and x are vectors whose scalar components are the Lorentzian coordinates of a point in σ as measured in Σ and σ, respectively; (W, W') and (w, w') are Plücker vectors of a line of σ as measured in Σ and σ, respectively. **A** is the Lorentzian orthogonal rotation matrix and **D** is the Lorentzian skew-symmetric matrix as

$$\mathbf{D} = \begin{bmatrix} 0 & d_3 & d_2 \\ d_3 & 0 & -d_1 \\ d_2 & d_1 & 0 \end{bmatrix},$$

where d_i, $\{i = 1, 2, 3\}$ are the Lorentzian cartesian components in Σ of d, the displacement vector of the origin of the coordinate system $\{x, y, z\}$.

Theorem 2.1. *The Lorentzian rotation matrix* **A** *with respect to Lorentzian Rodrigues parameters is*

$$\mathbf{A} = \Delta^{-1} \begin{bmatrix} 1 + d_1^2 + d_2^2 + d_3^2 & 2(d_3 + d_1 d_2) & 2(d_2 - d_1 d_3) \\ 2(d_3 - d_1 d_2) & 1 - d_1^2 - d_2^2 + d_3^2 & 2(d_2 d_3 - d_1) \\ 2(d_2 + d_1 d_3) & 2(d_1 + d_2 d_3) & 1 - d_1^2 + d_2^2 - d_3^2 \end{bmatrix},$$

where $\Delta = 1 + d_1^2 - d_2^2 - d_3^2$.

Theorem 2.2. *The Lorentzian rotation matrix* **A** *with respect to the Lorentzian Euler parameters is*

$$\mathbf{A} = S^{-2} \begin{bmatrix} c_0^2 + c_1^2 + c_2^2 + c_3^2 & 2(c_0 c_3 + c_1 c_2) & 2(c_0 c_2 - c_1 c_3) \\ 2(c_0 c_3 - c_1 c_2) & c_0^2 - c_1^2 - c_2^2 + c_3^2 & 2(c_2 c_3 - c_0 c_1) \\ 2(c_0 c_2 + c_1 c_3) & 2(c_0 c_1 + c_2 c_3) & c_0^2 - c_1^2 + c_2^2 - c_3^2 \end{bmatrix},$$

where $c_i = c_0 d_i$, $\{i = 1, 2, 3\}$, *and*

$$S^2 = c_0^2 + c_1^2 - c_2^2 - c_3^2.$$

If c_i's are not all zero, the Lorentzian Euler parameters c_i can be normalized by

$$S^2 = c_0^2 + c_1^2 - c_2^2 - c_3^2 = 1$$

where $c_0 = \cosh\theta/2$ and $c_i = \sinh\theta/2$, for $\{i = 1, 2, 3\}$, [8].
So, we can give following corollary.

Corollary 2.3. *The matrix form of equation* (1) *such that*

$$\beta_i = \beta_0 S^2 d_i \ (\beta_0 \neq 0) \ i = 1, 2, 3$$

is

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} \varphi_1 & \varphi_4 & \varphi_7 & \beta_1 \\ \varphi_2 & \varphi_5 & \varphi_8 & \beta_2 \\ \varphi_3 & \varphi_6 & \varphi_9 & \beta_3 \\ 0 & 0 & 0 & S^2\beta_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix},$$

where

$$\begin{aligned} \varphi_1 &= \beta_0(c_0^2 + c_1^2 + c_2^2 + c_3^2), & \varphi_4 &= 2\beta_0(c_0 c_3 + c_1 c_2), \\ \varphi_2 &= 2\beta_0(c_0 c_3 - c_1 c_2), & \varphi_5 &= \beta_0(c_0^2 - c_1^2 - c_2^2 + c_3^2), \\ \varphi_3 &= 2\beta_0(c_0 c_2 + c_1 c_3), & \varphi_6 &= 2\beta_0(c_0 c_1 + c_2 c_3), \end{aligned}$$

$$\begin{aligned} \varphi_7 &= 2\beta_0(c_0 c_2 - c_1 c_3), \\ \varphi_8 &= 2\beta_0(c_2 c_3 - c_0 c_1), \\ \varphi_9 &= \beta_0(c_0^2 - c_1^2 + c_2^2 - c_3^2), \end{aligned}$$

and (X, Y, Z, W), (x, y, z, w) *are homogeneous coordinates in* Σ *and* σ, *respectively.*

A Lorentzian spatial displacements can be represented by the sets of homogeneous quadruples of numbers $(c_i{:}\beta_i)$ $\{i = 1, 2, 3\}$. Also, for the scalar number λ, $(\lambda c_i{:}\lambda\beta_i)$ would represent the same displacement as $(c_i{:}\beta_i)$.

3. Representation of Study's Soma in Lorentzian Space

In this section, we get representation of Study's soma and their relations between the components of the displacement vector and the Lorentzian Euler parameters in Lorentzian space.

Theorem 3.1. *The Study's eight homogeneous parameters in Lorentzian space are*

$$\begin{aligned} g_0 &= d_1 c_1 + d_2 c_2 + d_3 c_3, \\ g_1 &= d_1 c_0 + d_3 c_2 - d_2 c_3, \\ g_2 &= d_2 c_0 - d_3 c_1 - d_1 c_3, \\ g_3 &= d_3 c_0 + d_2 c_1 + d_1 c_2, \end{aligned} \tag{3}$$

where c_i *are the Lorentzian Euler parameters and* d_i *are the components of the displacement vector of the origin of* σ.

From equation (3), we see that all $(c_i{:}g_i)$, $\{i = 0, 1, 2, 3\}$, pairs are not independent but satisfy the relation

$$c_0 g_0 - c_1 g_1 - c_2 g_2 - c_3 g_3 = 0. \tag{4}$$

The eight homogeneous parameters $(c_i{:}g_i)$, $\{i = 0, 1, 2, 3\}$, divide into two sets of four, each of which can represent a vector in a four-dimensional space. These two vectors are referred to as "Study vectors" and are analogous to the

Plücker vectors which in three dimensions describe a line. In this way a spatial displacements is mapped into the set of two Study vectors in the four-dimensional space, [9]. The eight homogeneous parameters $(c_i{:}g_i)$, $\{i=0,1,2,3\}$, can also be used to represent the homogeneous coordinates of a point in a seven-dimensional space. Equation (4) is then a quadratic equation with respect to the point coordinates of this space and symbolize a quadric. A more compact geometric results represent if a displacement is mapped into a point of a dual Lorentzian projective space.

4. Mappings of Spatial Kinematics in Lorentzian Space

In this section, we will use Study's eight homogeneous parameters to define the mapping.

Theorem 4.1. *Let $\hat{\Sigma}'$ is a dual Lorentzian projective space and a point of $\hat{\Sigma}'$ be*

$$\left(\hat{X}_1,\, \hat{X}_2,\, \hat{X}_3,\, \hat{X}_4\right),$$

where $\hat{X}_i = X_i + \varepsilon X_i'$ $\{i=1,2,3,4\}$ are dual numbers. Here, ε is the dual operator and given with $\varepsilon^2=0$. A spatial displacement can be mapped into a point of the dual Lorentzian space $\hat{\Sigma}'$ by

$$\begin{aligned}\hat{X}_1 &\rightarrow (-c_1+\frac{\varepsilon}{2}g_1),\\ \hat{X}_2 &\rightarrow (c_2+\frac{\varepsilon}{2}g_2),\\ \hat{X}_3 &\rightarrow (c_3+\frac{\varepsilon}{2}g_3),\\ \hat{X}_4 &\rightarrow (c_0+\frac{\varepsilon}{2}g_0),\end{aligned} \tag{5}$$

where c_i, $\{i=0,1,2,3\}$, are the Lorentzian Euler parameters and g_i, $\{i=0,1,2,3\}$, are as defined in equation (3).

From the equations (5) and (3), the real parts of the $\hat{\Sigma}'$ space coordinates are Lorentzian Euler parameters, c_i, and the dual parts are quadratic homogeneous functions of c_i and d_i, respectively. $\left(X_i,\, X_i'\right)$, $\{i=1,2,3,4\}$, are not all independent but satisfy the fundamental relation

$$X_1X_1'-X_2X_2'-X_3X_3'+X_4X_4'=0. \tag{6}$$

If the quantity $\hat{X}_1^2-\hat{X}_2^2-\hat{X}_3^2+\hat{X}_4^2$ is a real number, we can normalized the set of coordinates of a point in $\hat{\Sigma}'$ space. Then, $\hat{X}_i = X_i+\varepsilon X_i'$ with $\varepsilon^2=0$, so we have

$$\begin{aligned}\hat{X}_1^2-\hat{X}_2^2-\hat{X}_3^2+\hat{X}_4^2 \;=\;& (X_1^2-X_2^2-X_3^2+X_4^2)\\ &+2\varepsilon(X_1X_1'-X_2X_2'-X_3X_3'+X_4X_4').\end{aligned}$$

From equation (6), the right-hand side of this last equation is a real number. So, a point representing a spatial displacement has normalized coordinates in $\hat{\Sigma}'$ space. Also, all c_i's are not zero for a finite displacement in such a space and a point representing a finite displacement are not all pure duals.

Let we define $\mathbf{X}$, $\mathbf{X}'$ and $\mathbf{d}$ as the three split quaternions

$$\begin{aligned}\mathbf{X} &= X_4-X_1\mathbf{i}+X_2\mathbf{j}+X_3\mathbf{k},\\ \mathbf{X}' &= X_4'+X_1'\mathbf{i}+X_2'\mathbf{j}+X_3'\mathbf{k},\\ \mathbf{d} &= d_1\mathbf{i}+d_2\mathbf{j}+d_3\mathbf{k}\end{aligned}$$

with the basis $\{1,i,j,k\}$ where $X_i' = \frac{g_i}{2}$ $\{i=1,2,3\}$ and $X_4' = \frac{g_0}{2}$. Considering the structure of equations (3) and the split quaternion multiplication, we have

$$2\mathbf{X}' = \mathbf{dX} \tag{7}$$

and inverse relationship of equation (7) is as

$$2\mathbf{d} = S^{-2}\mathbf{X}'\tilde{\mathbf{X}}. \tag{8}$$

Here, $\tilde{\mathbf{X}}$ is the conjugate split quaternion of $\mathbf{X}$ and

$$S^2 = X_1^2-X_2^2-X_3^2+X_4^2 = c_0^2+c_1^2-c_2^2-c_3^2$$

with c_i $\{i=1,2,3,4\}$ are the Lorentzian Euler parameters. So, if the coordinates of a point are given with normalized coordinates $\hat{X}_i$, $\{i=1,2,3,4\}$, the corresponding displaced position of the real space σ is uniquely determined in $\hat{\Sigma}'$ space. The rotation part of the displacement only depends on the real part of $\hat{X}_i$ $\{i=1,2,3,4\}$.

Theorem 4.2. *A displacement of points in terms of the point coordinates in $\hat{\Sigma}'$ is expressed*

$$\begin{bmatrix} X\\ Y\\ Z\\ W\end{bmatrix} = \begin{bmatrix}\psi_1 & \psi_4 & \psi_7 & \psi_{10}\\ \psi_2 & \psi_5 & \psi_8 & \psi_{11}\\ \psi_3 & \psi_6 & \psi_9 & \psi_{12}\\ 0 & 0 & 0 & \psi_{13}\end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w\end{bmatrix}, \tag{9}$$

where

$$\begin{aligned}&\psi_1 = (X_4^2+X_1^2+X_2^2+X_3^2), && \psi_4 = 2(X_4X_3+X_1X_2),\\ &\psi_2 = 2(X_4X_3-X_1X_2), && \psi_5 = (X_4^2-X_1^2-X_2^2+X_3^2),\\ &\psi_3 = 2(X_4X_2+X_1X_3), && \psi_6 = 2(X_4X_1+X_2X_3),\\ &\psi_7 = 2(X_4X_2-X_1X_3), && \psi_{10} = 2(X_4X_1'+X_1X_4'-X_2X_3'+X_3X_2'),\\ &\psi_8 = 2(X_2X_3-X_4X_1), && \psi_{11} = 2(X_4X_2'-X_2X_4'+X_1X_3'+X_3X_1'),\\ &\psi_9 = (X_4^2-X_1^2+X_2^2-X_3^2), && \psi_{12} = 2(X_4X_3'-X_3X_4'-X_1X_2'-X_2X_1'),\\ & && \psi_{13} = (X_4^2+X_1^2-X_2^2-X_3^2).\end{aligned}$$

Here, $(X,\, Y,\, Z,\, W)$ and $(x,\, y,\, z,\, w)$ are homogeneous point coordinates in the fixed and moving spaces Σ and σ, respectively. X_i and X_i', $\{i=1,2,3,4\}$, are the real and the dual parts of point coordinates in $\hat{\Sigma}'$ space.

Theorem 4.3. *The transformation for displacement of lines is*

$$\begin{bmatrix}\hat{X}\\ \hat{Y}\\ \hat{Z}\end{bmatrix} = \begin{bmatrix}\rho_1 & \rho_4 & \rho_7\\ \rho_2 & \rho_5 & \rho_8\\ \rho_3 & \rho_6 & \rho_9\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}, \tag{10}$$

where

$$\begin{aligned}&\rho_1 = \hat{X}_4^2+\hat{X}_1^2+\hat{X}_2^2+\hat{X}_3^2, && \rho_4 = 2(\hat{X}_4\hat{X}_3+\hat{X}_1\hat{X}_2),\\ &\rho_2 = 2(\hat{X}_4\hat{X}_3-\hat{X}_1\hat{X}_2), && \rho_5 = \hat{X}_4^2-\hat{X}_1^2-\hat{X}_2^2+\hat{X}_3^2,\\ &\rho_3 = 2(\hat{X}_4\hat{X}_2+\hat{X}_1\hat{X}_3), && \rho_6 = 2(\hat{X}_4\hat{X}_1+\hat{X}_2\hat{X}_3),\end{aligned}$$

$$\begin{aligned}\rho_7 &= 2(\hat{X}_4\hat{X}_2-\hat{X}_1\hat{X}_3),\\ \rho_8 &= 2(\hat{X}_2\hat{X}_3-\hat{X}_4\hat{X}_1),\\ \rho_9 &= \hat{X}_4^2-\hat{X}_1^2+\hat{X}_2^2-\hat{X}_3^2,\end{aligned}$$

and

$$\begin{aligned}\hat{X} = W_1+\varepsilon W_1', \quad \hat{Y} = W_2+\varepsilon W_2', \quad \hat{Z} = W_3+\varepsilon W_3',\\ \hat{x} = w_1+\varepsilon w_1', \quad \hat{y} = w_2+\varepsilon w_2', \quad \hat{z} = w_3+\varepsilon w_3'.\end{aligned}$$

$(W_i,\, W_i')$ and $(w_i,\, w_i')$ are Lorentzian Plücker coordinates of a line of σ as measured in Σ and σ, respectively.

Corollary 4.4. *If the determinant of equation* (10) *is*

$$\Delta = (\hat{X}_1^2 - \hat{X}_2^2 - \hat{X}_3^2 + \hat{X}_4^2)^3 \neq 0,$$

then we have inverse transformation as

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_4 & \lambda_7 \\ \lambda_2 & \lambda_5 & \lambda_8 \\ \lambda_3 & \lambda_6 & \lambda_9 \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{bmatrix}, \qquad (11)$$

where

$$\begin{array}{ll} \lambda_1 = \hat{X}_4^2 + \hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2, & \lambda_4 = 2(\hat{X}_1\hat{X}_2 - \hat{X}_4\hat{X}_3), \\ \lambda_2 = 2(-\hat{X}_4\hat{X}_3 - \hat{X}_1\hat{X}_2), & \lambda_5 = \hat{X}_4^2 - \hat{X}_1^2 - \hat{X}_2^2 + \hat{X}_3^2, \\ \lambda_3 = 2(\hat{X}_1\hat{X}_3 - \hat{X}_4\hat{X}_2), & \lambda_6 = 2(\hat{X}_2\hat{X}_3 - \hat{X}_4\hat{X}_1), \end{array}$$

$$\begin{array}{l} \lambda_7 = 2(-\hat{X}_4\hat{X}_2 - \hat{X}_1\hat{X}_3), \\ \lambda_8 = 2(\hat{X}_2\hat{X}_3 + \hat{X}_4\hat{X}_1), \\ \lambda_9 = \hat{X}_4^2 - \hat{X}_1^2 + \hat{X}_2^2 - \hat{X}_3^2. \end{array}$$

Meanwhile, when

$$\Delta = (\hat{X}_1^2 - \hat{X}_2^2 - \hat{X}_3^2 + \hat{X}_4^2)^3 = 0,$$

the transformation (10) is called singular. So, the transformation then has no meaning. So, we exclude the point $(0,0,0,0)$ and the points $(\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4)$ which are satisfy the equation $\hat{X}_1^2 + \hat{X}_4^2 = \hat{X}_2^2 + \hat{X}_3^2$ in $\hat{\Sigma}'$ space.
If we consider normalized Lorentzian Euler's parameters and use split quanternions instead of matrices, we can give following corollary.

Corollary 4.5. *If we define*

$$\hat{\mathbf{R}} = \hat{X}i + \hat{Y}j - \hat{Z}k \text{ and } \hat{\mathbf{r}} = \hat{x}i + \hat{y}j - \hat{z}k$$

where $\hat{X}, \hat{Y}, \hat{Z}, \hat{x}, \hat{y}, \hat{z}$ *as defined in equations* (10) *and* (11) *and use split quanternions instead of matrices, equations* (10) *and* (11) *become, respectively,*

$$\hat{\mathbf{R}} = \hat{\mathbf{X}}\hat{\mathbf{r}}\tilde{\hat{\mathbf{X}}},$$
$$\hat{\mathbf{r}} = \tilde{\hat{\mathbf{X}}}\hat{\mathbf{R}}\hat{\mathbf{X}}.$$

From the last two equations, we can get the relationship between the mapping for a displacement and that of the inverse displacement.

5. Conclusion

This work gives and develops spatial displacements and mappings of spatial kinematics in Lorentzian spaces. So, this study may shed light on future work about kinematic mappings to study spatial motions and help some unsolved problems of spatial kinematics in Lorentzian spaces.

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