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Research Article

A Study of *q*-Deformed Bosons, and Their Implications to Quantum Optics

Mustafa Senay

Bartın University, Vocational School of Health Services, Department of Medical Services and Techniques, Bartın, Türkiye, msenay@bartin.edu.tr

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ABSTRACT

In this study, we investigate three types of *q*-deformed boson oscillators, focusing on their mathematical frameworks and thermodynamic properties. We calculate key thermodynamic quantities, such as internal energy and entropy, as functions of the deformation parameter *q*. Our results reveal that these oscillators are eigenstates of specific deformed boson annihilation operators. We also analyze their unique characteristics and implications in deformed quantum optics. Furthermore, we examine the impact of *q*-deformation on qutrit logic gates, including cycle, self-shift, controlled cycle, controlled self-shift, Feynman, ternary Toffoli, and Fredkin gates, highlighting their altered computational properties. This research contributes to a deeper understanding of *q*-deformed systems and their applications in quantum computing. Overall, it opens new avenues for exploring the interplay between deformation parameters and quantum information processing.

1. Introduction

The spin-statistics theorem is a fundamental principle in quantum field theory that bridges quantum mechanics and statistical mechanics [1]. It establishes an important connection between the symmetry of particles (especially their spin) and the statistical behavior they exhibit in many-body systems. This theorem states that particles with integer spin (bosons) have symmetric wave functions, allowing multiple particles to occupy the same quantum state simultaneously. In contrast, particles with half-integer spin (fermions) have antisymmetric wave functions, enforcing the Pauli exclusion principle that no two fermions can occupy the same quantum state. These symmetrization or antisymmetrization requirements are expressed through commutation or anticommutation relations in the second quantization framework, which govern how the creation and annihilation operators work for bosons and fermions, respectively. This distinction in quantum statistical behavior directly affects the number of possible states a system can use; It affects the collective statistical mechanical description of the system by determining the set of occupancy numbers and thus shaping macroscopic properties such as thermal conductivity and specific heat capacity.

In recent years, there has been increasing research into quantum statistics, which departs from the traditional classifications of bosons and fermions. Building on the fundamental contributions of Gentile and Green [2, 3], researchers have developed various extensions beyond these standard statistics. These include parastatistics, fractional statistics, quon statistics, anion statistics, and quantum group theory, which have attracted great attention due to their potential applications in various fields of physics. In condensed matter physics, these statistics have proven vital in elucidating phenomena such as the fractional quantum Hall effect and the behavior of anionic particles that are neither fermions nor bosons, but exhibit unique statistical properties [4].

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There are two important approaches to investigate the statistics and thermodynamics of intermediate states, each offering unique perspectives and methodologies: One method uses Tsallis' non-extensive statistics [5] and generalized entropies. Tsallis statistics generalize standard Boltzmann-Gibbs statistics by introducing a parameter q that modifies the entropy formula [6]. The second method involves the use of deformed quantum algebras [7-14], leading to deformed thermostatistics functions [15-39]. Deformed quantum algebras, such as quantum groups or q-deformed algebras, introduce modifications to the standard commutation relations of quantum mechanics. These changes affect the formulation of statistical mechanics, changing the distribution functions and thermodynamic properties of the particles.

In recent work, q-deformed theory has found applications in a wide range of physics disciplines and has contributed to various fields such as: In ref [15], generalized thermodynamics with q-deformed bosons and fermions has been used to describe systems with non-trivial quantum deformations, which are conventional has worked in ways that extend beyond statistical mechanics, Ref. [16]; q-deformed theory has been applied to study Bose–Einstein condensation, offering new insights into the behavior of particles at extreme temperatures and densities; Ref. [17], investigations into the thermodynamic geometry of deformed bosons and fermions have provided geometric insights into the statistical properties of quantum systems and phase transitions; Ref. [20] used thermosize effects in models of q-deformed fermion gases, investigating how quantum deformations affect the thermodynamic properties and size scaling of fermionic systems. Moreover, it has been studied a two-parameter deformed boson gas model based on commuting Fibonacci oscillators, leading to a generalized Fibonacci energy spectrum in [23,24]. Friedmann equations and Einstein field equations incorporating the deformation parameter q were derived in [25,26]. A quantum Otto cycle was analyzed using q-deformed oscillators as the working substance alongside classical thermal baths in [29]. Further, q-deformed harmonic oscillators were utilized to construct qubits and quantum gates in [30,31]. Investigations into a single particle's q-deformed harmonic oscillator have focused on how deformation influences statistical complexity, including Shannon information entropy and disequilibrium in [32]. A modified cosmological scenario was proposed, featuring q-deformed Friedmann and Raychaudhuri equations that introduce effective dark energy components in [33].

On the other hand, q-deformed qubits are generalizations of traditional qubits in which the algebra governing quantum states is replaced by a deformation parameter q derived from quantum groups [30,31]. These generalized states introduce a deformation that changes how quantum states interact and evolve. This leads to changes in the energy spectra, quantum gate operations, and overall dynamics of the qubits. Such changes can create new avenues for quantum state manipulation and lead to more generalized forms of quantum systems with potential applications in fields such as quantum optics and condensed matter physics. In quantum information theory, q-deformed qubits can be used as a tool to better understand non-standard entanglement properties and quantum correlations.

Our aim in this work is to perform an analysis of quantum algebraic properties associated with three types of q-deformed boson oscillator algebra models: the Arik-Coon (AC) model, the Biedenharn-Macfarlane (BM) model, and the Quesne model. These models represent different mathematical formulations that incorporate quantum deformations into the standard bosonic oscillator algebra. We will also examine some thermostatic properties of AC and BM models. This will include the study of statistical mechanics aspects such as their partition functions, entropy formulations and thermodynamic behavior. Understanding these properties is crucial for applications in various physical systems where quantum deformations play an important role. As an application, we will investigate the effect of the Quesne model in quantum optics. Quantum optics deals with the interaction of light and matter at the quantum level [40-42]. For instance, in Ref. [40], photonic structural designs have been emphasized to enhance light-matter interaction in two dimensional material based optoelectronic devices.

Moreover, understanding how q-deformed patterns affect photon statistics, coherence properties, and other optical phenomena can provide valuable information about new regimes of light-matter interactions. For example, in Ref. [43], it was focused on the study of a multi-level atom excited by a laser pulse in the form of a q-deformed hyperbolic function. The dynamic properties of a two-level system excited by a q-deformed laser beam were analyzed. Also, the optical properties of a three-level atom system interacting with two electromagnetic fields were investigated.

2. AC-type Oscillators Model

In this section, we investigate the quantum statistical mechanical properties of q-deformed boson oscillators, focusing specifically on Arik-Coon (AC)-oscillators. The algebraic framework of quantum AC-oscillators are characterized by the q-deformed Heisenberg algebra, where the creation and destruction operators are denoted a^* and a, respectively. The q-deformed Heisenberg algebra associated with AC-oscillators is expressed as [7, 44]

$$aa^* - qa^*a = 1$$

$$[\hat{N}, a^*] = a^*, \quad [\hat{N}, a] = -a, \tag{2.1}$$

where $\hat{N}$ is the total number operator and q is the deformation paramater. Also, the basic number is given as

$$[x] = \frac{q^x - 1}{q - 1}. \tag{2.2}$$

Furtheremore, the Jackson derivative (JD) operator for the AC-oscilators is expressed as

$$\partial_x^{(q)} f(x) = \frac{1}{x}\left[\frac{f(qx) - f(x)}{q - q^{-1}}\right], \tag{2.3}$$

which reduces to the ordinary derivative when q goes to unity. The mean occupation number of AC oscillators is defined as [44]

$$n_i = \frac{1}{\log q} \ln\left(\frac{z^{-1}e^{\beta\varepsilon_i} - 1}{z^{-1}e^{\beta\varepsilon_i} - q}\right) \tag{2.4}$$

where $z = \exp(\mu/k_B T)$ is the fugacity and $\beta = 1/k_B T$. Following the standard procedure [45], one can esaily find

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} h_{5/2}(z, q), \tag{2.5}$$

$$\frac{N}{V} = \frac{1}{\lambda^3} h_{3/2}(z, q), \tag{2.6}$$

where $\lambda = h/(2\pi m k_B T)^{1/2}$ is the thermal wavelength and the generalized Bose Einstein functions $h_n(z, q)$ are defined as

$$h_n(z, q) = \frac{1}{\log q}\left[\sum_{l=1}^{\infty} \frac{(zq)^l}{l^{n+1}} - \sum_{l=1}^{\infty} \frac{(z)^l}{l^{n+1}}\right]. \tag{2.7}$$

The internal energy and the entropy of the AC-oscillators gas can be found as [44]

$$U = \frac{3}{2} \frac{k_B T V}{\lambda^3} h_{5/2}(z, q). \tag{2.8}$$

and

$$\frac{S}{N k_B} = \frac{5}{2} \frac{h_{5/2}(z,q)}{h_{3/2}(z,q)} - \log z. \tag{2.9}$$

respectively. Also, from the thermodynamic relation $F = \mu N - PV$, the Helmholtz free energy can be derived as

$$F = N k_B T \left[\log z - \frac{h_{5/2}(z,q)}{h_{3/2}(z,q)}\right]. \tag{2.10}$$

3. BM-type Oscillators Model

The symmetric q-deformed algebraic structure of BM-type oscillators is characterized by the q-deformed Heisenberg algebra, with the creation operator c^*, the annihilation operator and c, and the total number operator $\hat{N}$ playing central roles. In this framework, the BM-oscillators algebra is defined as [8, 9, 15]

$$cc^* - qc^*c = q^{-N}$$

$$[\hat{N}, c^*] = c^*, \quad [\hat{N}, c] = -c, \tag{3.1}$$

where q is the real deformation parameter. Also, the operators have the following relations [15]

$$c^*c = [\hat{N}], \quad c^*c = [1 + \hat{N}]. \tag{3.2}$$

The basic q-deformed quantum number is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (3.3)$$

Moreover, the Jackson derivative (JD) operator for the system is given as

$$D_x^{(q)} f(x) = \frac{1}{x}\left[\frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}\right], \quad (3.4)$$

for any function $f(x)$.

To investigate the high-temperature properties of the BM-oscillators model, we examine the logarithm of grand partition function:

$$\ln Z = -\sum_i \ln(1 - ze^{-\beta\varepsilon_i}) \quad (3.5)$$

The mean occupation number is expressed by the following form

$$n_i = \frac{1}{q - q^{-1}} \ln\left(\frac{z^{-1}e^{\beta\varepsilon_i} - q^{-1}}{z^{-1}e^{\beta\varepsilon_i} - q}\right). \quad (3.6)$$

The pressure of the BM-oscillators model is given as

$$P = \frac{k_B T}{V} \sum_i \ln(1 - ze^{-\beta\varepsilon_i}). \quad (3.7)$$

This relation can be written by using Eq. (3.4)

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} g_{5/2}(z, q), \quad (3.8)$$

where q-deformed $h_n(z, q)$ function is defined as

$$g_n(z, q) = \frac{1}{q - q^{-1}}\left[\sum_{l=1}^{\infty} \frac{(zq)^l}{l^{n+1}} - \sum_{l=1}^{\infty} \frac{(zq^{-1})^l}{l^{n+1}}\right]. \quad (3.9)$$

One can also derive the following thermodynamic functions for a gas of BM-oscillators: the particle density, internal energy, and entropy, respectively

$$\frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z, q), \quad (3.10)$$

$$U = \frac{3}{2}\frac{k_B T V}{\lambda^3} g_{5/2}(z, q), \quad (3.11)$$

$$\frac{S}{N k_B} = \frac{5}{2}\frac{g_{5/2}(z,q)}{g_{3/2}(z,q)} - \ln z. \quad (3.12)$$

4. Quesne-type Oscillators Model

Quesne-oscillators algebra is defined as [46, 47]

$$q a_q a_q^* - a_q^* a_q = I, \quad a_q a_q^* - a_q^* a_q = q^{-N-1}. \quad (4.1)$$

where a_q^* and a_q are deformed creation and annihilation operators, respectively, and they satisfy in the following forms

$$a_q|n >= \sqrt{[n]}|n - 1 >, \quad (4.2)$$

$$a_q^*|n >= \sqrt{[n+1]}|n + 1 >, \quad (4.3)$$

where $a_q|0 >= 0$ and the q-basic number is given as

$$[x] = \frac{1 - q^{-x}}{q - 1}. \quad (4.4)$$

Moreover, q-deformed coherent states are expressed as

$$|z >= [E_q((1 - q)q|z|^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]}} \sqrt{[n]}|n > \quad (4.5)$$

where q-exponentials are $E_q(z) = \prod_{k=0}^{\infty}(1 + q^k z)$.

Now, we investigate the geometric and physical properties of the new q-deformable coherent state |z⟩. To achieve this, we need to compute the expectation values of various Hermitian monomials involving the boson creation and annihilation operators a^* and a. These expectation values can be expressed in terms of derivatives of the function $\mathcal{N}_q(x)$ [25]

$$< (a^*)^r a^r >= \frac{x^r}{\mathcal{N}_q(x)} \frac{d^r \mathcal{N}_q(x)}{dx^r}, \ r=0,1,2\ldots. \quad (4.6)$$

The last equation can be rewritten as

$$< (a^*)^r a^r >= (z^*)^p z^r S_q^{(p,r)}(x) \quad (4.7)$$

where

$$S_q^{(p,r)}(x) = \frac{1}{\mathcal{N}_q(x)} \sum_{n=0}^{\infty} \left(\frac{(n+p)!(n+r)!}{[n+p]![n+r]!}\right)^{1/2} \frac{x^n}{n!} \quad (4.8)$$

with $r, p = 0,1,2, \ldots$

Moreover, in two dimensional surface, the metric factor can be defined as [25]

$$w_q(x) = \frac{d}{dx} < N >= \left(\frac{x\mathcal{N}_q(x}{\mathcal{N}_q(x}\right)' \quad (4.9)$$

where primes indicate the number of times differentiation is performed with respect to the variable x. For $x \ll 1$, one can easliy reach

$$w_q(x) \approx q\left[1 - \frac{2q(1-q)}{1+q}x + \cdots\right]. \quad (4.10)$$

On the other hand, the variance of the number operator N is equal to its mean, which can be evaluated using deviations from the Poisson statistics [46]

$$Q_q(x) = \frac{(\Delta N)_q^2 - <N>_q}{<N>_q}. \quad (4.11)$$

From Eq. (4.6), it can be found

$$Q_q(x) = x\left(\frac{\mathcal{N}_q''(x)}{\mathcal{N}_q'(x)} - \frac{\mathcal{N}_q'(x)}{\mathcal{N}_q(x)}\right). \quad (4.12)$$

For $x \ll 1$, we reach

$$Q_q(x) \approx -\frac{q(1-q)}{1+q}x + \cdots. \quad (4.13)$$

5. Possible application of Quesne-type Oscillators Model to Quantum Optics

Unlike binary systems, ternary logic gates offer more information carrying capacity in quantum computing. Qutrits (three-level systems) can encode more information than classical qubits, which makes quantum circuits more efficient. In quantum computers and quantum optics, qutrit systems offer advantages such as lower error rates, lower energy consumption, and more efficient use of physical resources. In addition, qutrit logic and ternary gates are an innovative research area in quantum information processing and are less studied in the current literature.

Therefore, the study of ternary gates within the scope of the article is important for the advancement of quantum technologies, both theoretically and practically. In particular, studying the effects of q-deformation on such systems can lead to new discoveries.

In this section, we follow the Ref. [31] to investigate the impact of q-deformation on qutrit logic gates such as cycle, self-shift, controlled cycle, controlled self-shift, Feynman, ternary Toffoli, and Fredkin gates. These gates can be expressed in the framework of angular momentum states, where their operations correspond to transformations in a quantum system characterized by total angular momentum. For example, the cycle gate effectively permutes the qutrit states, while the self-shift gate applies a shifting operation. Utilizing the Schwinger representation allows us to depict these gates in terms of ladder operators acting on the angular momentum states. This perspective provides valuable insights into how q-deformation alters the algebraic structure and functional properties of these gates.
These gates can be represented in the context of angular momentum states as

$$\boldsymbol{C_n}|i >= |n + i > \quad (5.1)$$

$$\boldsymbol{S_n}|i >= |2i + n > \quad (5.2)$$

$$\boldsymbol{CC_n}|ij >= \frac{i(i-1)}{2}|i\ j + n > + \frac{(2-i)(i+1)}{2}|ij > \quad (5.3)$$

$$\boldsymbol{CS_n}|ij >= \frac{i(i-1)}{2}|i\ 2j + n > + \frac{(2-i)(i+1)}{2}|ij > \quad (5.4)$$

$$\boldsymbol{FG}|ij >= |i\ i + j > \quad (5.5)$$

$$\boldsymbol{TTG}|ijk >= |i\ j\ \ i.j + k > \quad (5.6)$$

$$\boldsymbol{TFG}|ijk >= \frac{i(3-i)}{2}|i\ j\ k > + \frac{(2-i)(i-1)}{2}|i\ j\ k > \quad (5.7)$$

In the context of qubits, quantum gates are represented as unitary transformations acting on a three-dimensional Hilbert space. A qutrit is a quantum system with three possible states, and can be represented $|0 >$, $|1 >$ and $|2 >$. The operators on the left side of equations (5.1) – (5.7) above operate on these three quantum states to reach a different quantum state. Also, $|ij >$ represents a two-qutrit state, which can be expressed as $|ij > = |i > |j >$. The connection between q-deformed operators and conventional operators is specified as

$$a_q = a\sqrt{\frac{\phi_1 - q^{-N}\phi_2}{N(q-1)}} \tag{5.5}$$

$$a_q^* = \sqrt{\frac{\phi_1 - q^{-N}\phi_2}{N(q-1)}}\,a^* \tag{5.6}$$

where ϕ_1 and ϕ_2 represent arbitrary constants, a and a^* indicate the standard annihilation and creation operators, respectively. To create a qutrit, the total angular momentum number j must be equal to $j = 1$. Therefore, since $m = -j, \dots, 0, \dots + j$, there are three different possible states $(|1-1>=|0>, (|10>=|1>, (|11> = |2>)$. So, the q-deformed qutrit states can be expressed using the creation operators from the q-deformed algebra

$$|0>_q = \frac{(a_2^*)_q^2}{\sqrt{[2]!}}|\tilde{0}_1\tilde{0}_2> \tag{5.7}$$

$$|1>_q = \frac{(a_1^*)_q(a_1^*)_q}{\sqrt{[1]!}}|\tilde{0}_1\tilde{0}_2> \tag{5.8}$$

$$|2>_q = \frac{(a_1^*)_q^2}{\sqrt{[2]!}}|\tilde{0}_1\tilde{0}_2> \tag{5.9}$$

where $\tilde{0}_1$ and $\tilde{0}_2$ are ground states of j and m, respectively, and $[n]! = [1][2] \dots [n]$. A general formulation of q-deformed qutrits is

$$|x>_q = \frac{(a_1^*)_q^x(a_2^*)_q^{(2-x)}}{\sqrt{[x]![(2-x)]!}}|\tilde{0}_1\tilde{0}_2> \tag{5.10}$$

Using Eq. (5.6), Eq. (5.10) can be re-written as

$$|x>_q = \left(\sqrt{\frac{\phi_1 - q^{-N_1}\phi_2}{N_1(q-1)}}\,a_1^*\right)^x \left(\sqrt{\frac{\phi_3 - q^{-N_2}\phi_4}{N_2(q-1)}}\,a_2^*\right)^{(2-x)} \frac{1}{\sqrt{[x]![(2-x)]!}}|\tilde{0}_1\tilde{0}_2> \tag{5.11}$$

Eqs. (5.1)-(5.7) can be re-derived by using Eq. (5.11). To meet the requirements of these expressions, it's essential to establish the arbitrary parameters. Let's demonstrate how to find these parameters using an example

$$\boldsymbol{C_1}|0>_q = |1>_q \tag{5.12}$$

In Eq. (5.11), if we put $\boldsymbol{x = 0}$ to find $|\boldsymbol{0}>_{\boldsymbol{q}}$ and $\boldsymbol{x = 1}$ to find $|\boldsymbol{1}>_{\boldsymbol{q}}$, we reach

$$C_1\left(\sqrt{\frac{\phi_3 - q^{-N_2}\phi_4}{N_2(q-1)}}\,a_2^*\right)\left(\sqrt{\frac{\phi_3 - q^{-N_2}\phi_4}{N_2(q-1)}}\,a_2^*\right)\frac{1}{\sqrt{[2]!}}|\tilde{0}_1\tilde{0}_2> =$$

$$\left(\sqrt{\frac{\phi_1 - q^{-N_1}\phi_2}{N_1(q-1)}}\,a_1^*\right)\left(\sqrt{\frac{\phi_3 - q^{-N_2}\phi_4}{N_2(q-1)}}\,a_2^*\right)|\tilde{0}_1\tilde{0}_2> \tag{5.13}$$

From the relation $a^*|n>=\sqrt{n+1}|n+1>$, one can find $\phi_1 = \phi_2 = \phi_3 = \phi_4$. Similarly, arbitrary parameters can be obtained using Eqs. (5.2)-(5.7) for other gates.

Following Ref. [30], it can be determined qutrit gates by utilizing q-deformed three-level quantum states

$$\boldsymbol{C_{nq}} = \sum_{j=0}^{2}|n+j>_{q\ q}<j| \tag{5.14}$$

$$\boldsymbol{S_{nq}} = \sum_{j=0}^{2}|2j+n>_{q\ q}<j| \tag{5.15}$$

$$\boldsymbol{CC_{nq}} = \sum_{i,j=0}^{2}\frac{i(i-1)}{2}|i\ j+n>_{q\ q}<j\ i| + \sum_{i,j=0}^{2}\frac{(2-i)(i+1)}{2}|i\ j>_{q\ q}<i\ j| \tag{5.16}$$

$$\boldsymbol{CS_{nq}} = \sum_{i,j=0}^{2}\frac{i(i-1)}{2}|i\ 2j+n>_{q\ q}<j\ i| + \sum_{i,j=0}^{2}\frac{(2-i)(i+1)}{2}|i\ j>_{q\ q}<i\ j| \tag{5.17}$$

$$\boldsymbol{FG}_q = \sum_{i,j=0}^{2}|i\ i+j>_{q\ q}<j\ i| \tag{5.18}$$

$$\boldsymbol{TTG}_q = \sum_{i,j=0}^{2}|i\ j\ \ i.j+k>_{q\ q}<k\ j\ i| \tag{5.19}$$

$$\boldsymbol{TFG}_q = \sum_{i,j=0}^{2}\frac{i(3-i)}{2}|i\ k\ j>_{q\ q}<k\ j\ i| + \sum_{i,j=0}^{2}\frac{(2-i)(i-1)}{2}|i\ j\ k>_{q\ q}<k\ i\ j| \tag{5.20}$$

To build the q-deformed quantum ternary gates, we must identify the parameters ϕ. In the Eq. (5.14), if we take $n = 1$, we get

$$\boldsymbol{C_{1q}} = |1>_{q\ q}<0| + |2>_{q\ q}<1| + {}_q|0>_{q\ q}<2| \tag{5.21}$$

where we use the orthogonality relation $<j|i> = \delta_{ij}$. Now, we consider that $\boldsymbol{C_{1q}}$ act on the state $|1>_q$, such that

$$\boldsymbol{C_{1q}}|1>_q = |2>_q \tag{5.22}$$

Based on the orthogonality relation, it becomes clear that the only remaining term is the second term of Eq. (5.21). Thus, the parameters can be obtained from the orthogonality condition

$$
{}_q\!< 1|1 >_q = < \tilde{0}_1\tilde{0}_2| a_2 \left(\sqrt{\frac{\phi_7 - q^{-N_2}\phi_8}{N_2(q-1)}}\right) a_1 \left(\sqrt{\frac{\phi_5-q^{-N_1}\phi_6}{N_1(q-1)}}\right) \left(\sqrt{\frac{\phi_1-q^{-N_1}\phi_2}{N_1(q-1)}}\right) a_1^* \left(\sqrt{\frac{\phi_3-q^{-N_2}\phi_4}{N_2(q-1)}}\right) a_2^* |\tilde{0}_1\tilde{0}_2 > \quad (5.23)
$$

In the last Eq., one can find $\phi_1 = \phi_2$, $\phi_3 = \phi_4$, $\phi_5 = \phi_6$, and $\phi_7 = \phi_8$. Similary, for cases $\boldsymbol{C_{1q}}|2>_q = |0>_q$ and $\boldsymbol{C_{1q}}|0>_q = |1>_q$, ${}_q\!< 2|2>_q$ and ${}_q\!< 0|0>_q$ can be obtained.

5. Conclusion

In this article, we present a comprehensive analysis of q-deformed boson oscillator algebras, focusing particularly on their thermo-statistical properties and their relationship to quantum optics. In the second section, we examine AC oscillator algebras. This analysis examines the thermo-statistical properties of these algebras in detail and investigates how these algebras modify the thermal behavior of quantum systems. In this context, we provide a comprehensive assessment of how thermo-statistical parameters are modified by these algebras, examining how different thermal properties and temperature-related behaviors of quantum systems are affected. In the third section, we focus on the BM oscillator model.

This section examines the thermo-statistical properties of the BM model in detail. In this context, we delve into various thermo-statistical quantities, such as internal energy and entropy, and investigate how these quantities differ from classical systems. The fourth section is devoted to Quesne oscillator algebras, which have not received sufficient attention in the literature. In this section, we discuss the theoretical framework and practical applications of Quesne algebras, especially in the context of quantum optics. We discuss the mathematical properties of Quesne algebras, their effects on quantum states, and their potential applications in quantum optical systems. We also evaluate possible application scenarios on how these algebras can be used in quantum optics and innovative approaches in this field. In the last section, we construct q-deformed qubits by exploiting q-deformed angular momentum states, which allows us to investigate their distinct characteristics. We study the effects of logical qubit gates on these q-deformed qubits to understand how their deformation influences the performance of quantum operations.

Although there are similarities in structure between the arbitrary parameters obtained with Reference 31 and the parameters obtained in Section 5, the main difference lies in the deformation algebra used. The arbitrary parameters obtained in Section 5 depend on the q-deformation parameter and the properties of these parameters differ from the parameters obtained in Reference 31. This difference arises due to the use of quantum algebraic properties of the Quesne oscillator system in our work. However, in Reference 31, different deformed quantum algebra is used. Our findings aim to contribute to the broader landscape of quantum information processing, offering new pathways for enhancing computational efficiency and enabling innovative quantum technologies.

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