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RESEARCH PAPER

Bifurcation and hysteresis analysis of the non-degenerate Euler beam problem

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Abstract

In this work, the bifurcation and hysteresis phenomena of the Euler beam problem are focused on from the singularity theory viewpoint. Confirming the continuity of the problem is a necessary condition for performing a bifurcation and hysteresis analysis. A bifurcation problem is transformed from an infinite dimension to a finite dimension by applying the Lyapunov equations. A suitable central force minimizes our considered model and makes the problem stable. Moreover, we perform numerical investigations and interpret the results obtained from the bifurcation and hysteresis analysis geometrically with suitable values of the new unfolding parameters and with different lengths.

Keywords: Bifurcation; non-degeneracy; Euler beam; hysteresis

AMS 2020 Classification: 34C23; 37G10; 34C55

1 Introduction

Golubetsky and Schaeffer showed how singularity [1, 2] works to investigate the variational problems or bifurcation solutions of the partial differential equations. They presented several important cases, including bifurcations from a single eigenvalue and double eigenvalue problem. They discussed the bifurcations that were subjected to tiny perturbations using the theory of singularities. They introduced the fundamental bifurcation problem [3]: a unit-strength torsional spring balances the compressive force applied to two rigid rods of equal length connected by frictionless pins. In 1994, Jerrod and Marsden considered a beam free to move in a plane, distorted

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from its natural state by the application of a compressive force. They noticed that the beam buckles into one of two conceivable states after a critical load is achieved, despite initially compressing somewhat. When the compressive force is less than the Euler's critical load then nothing happens. The stability has been moved from the initial trivial solution to the stable buckled solutions, although the compressed state is still present but is now unstable.

This work is concerned with generating a new bifurcation model and bifurcation analysis with the unfolding parameters, whereas we can also discuss the hysteresis. We examine the Euler buckling problem [1], a well-known pitchfork bifurcation scenario.

A mathematical expression of the perturbed energy functional

$$E(v,\lambda,\alpha,\beta) = \frac{1}{2}\int_0^{\pi}\left[\frac{v''}{(1-v'^2)^{\frac{1}{2}}}-\alpha\right]^2 ds + \lambda\int_0^{\pi}\left(1-v'^2\right)^{\frac{1}{2}} ds + \beta v\left(\frac{\pi}{2}\right). \tag{1}$$

For $\alpha = \beta = 0$, the idealized set-up is when the rod is perfectly straight and is only subjected to the compressive force λ in its unstressed state. We intend to observe an appropriate central force α to minimize the problem.

By minimizing the energy problem, we can find solutions that correspond to stable or ideal states, which allows us to better understand the equilibrium positions, stable configurations, or optimal solutions that lead to efficient, cost-effective, or high-performance outcomes. We find the restricted tangent space and the tangent space of the problem considered, and we calculate the codimension to be 2 on the basis of the restricted tangent space. Co-dimension 2 ensures the minimum unfolding parameter is two. The changes in the unfolding parameters presented the imperfection of the problem.

Our considered problem is as follows:

$$E : V \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad v \mapsto E(v,\lambda,\alpha), \quad \alpha = (\alpha,\beta),$$

where $V = \{v \in X : \|v'\|_\infty \le 1-\epsilon\}$ and $\mathcal{X}$ is the Sobolev space

$$\mathcal{X} = \{v \in H^2[0,l] : v(0) = v(l) = 0\},$$

defined by

$$E(v,\lambda,\alpha,\beta) = \frac{1}{2}\int_0^{l}\left(\frac{v''}{(1-(v')^2)^{1/2}}-\alpha\mathcal{F}\right)^2 ds + \lambda\int_0^{l}\sqrt{1-(v')^2}ds + \beta\delta(v), \tag{2}$$

where $\delta(v) = v(l/2)$. Here, the function $\mathcal{F}$ is defined by [4, 5]

$$\mathcal{F} = \frac{1}{\sqrt{l/2}}\left[a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\frac{2k\pi s}{l} + b_k\sin\frac{2k\pi s}{l}\right)\right], \tag{3}$$

with $\|\mathcal{F}_0\|_\infty < \infty$. Since $v \in H^2[0,l]$, we can choose that v is C^1, and there is a constant ε_1, $0 < \varepsilon_1 < 1$, so that $|v'(s)| < 1-\varepsilon_1$.

Variational formulation of (1) gives [1, 6, 7],

$$\Phi_{(v,\lambda,\alpha,\beta)}\cdot\xi = (dE)_{(v,\lambda,\alpha,\beta)}\cdot\xi.$$

We must verify that Φ is smooth and demonstrate that Φ is C^∞. The Lyapunov-Schmidt reduction can be used to make the problem finite-dimensional if Φ is continuous. According to the Lyapunov Schmidt reduction, the zero of $\Phi(x, \lambda, \alpha, \beta)$ describes the bifurcation for zero of $\Phi(v, \lambda, \alpha, \beta)$ as mentioned below

$$\Phi(x, \lambda, \alpha, \beta) = P\Phi(xv_n + \mathcal{W}(x, \lambda, \alpha, \beta), \lambda, \alpha, \beta). \tag{4}$$

From the bifurcation equation, $\Phi = 0$ and its Taylor coefficients help us to discuss the bifurcation solutions. First, second, and third-order derivatives of Φ we can discuss the bifurcation set and hysteresis set and the set of the zeros of

$$\Phi = \frac{x^3}{6}\tilde{\Phi}_{xxx} + \tilde{\Phi}_{x\lambda}\lambda x + \tilde{\Phi}_{\alpha}\alpha + \tilde{\Phi}_{\beta}\beta + \frac{x^2}{2}\ell(\alpha, \beta) + xQ(\alpha, \beta) + C(\alpha, \beta) + O(4),$$

where $\ell(\alpha, \beta) = \tilde{\Phi}_{xx\alpha}\alpha + \tilde{\Phi}_{xx\beta}\beta$, $Q(\alpha, \beta) = \frac{1}{2}(\tilde{\Phi}_{x\alpha\alpha}\alpha^2 + 2\tilde{\Phi}_{x\alpha\beta}\alpha\beta + \tilde{\Phi}_{x\beta\beta}\beta^2)$, and $C(\alpha, \beta) = \frac{1}{6}(\tilde{\Phi}_{\alpha\alpha\alpha}\alpha^3 + 3\tilde{\Phi}_{\alpha\alpha\beta}\alpha^2\beta + 3\tilde{\Phi}_{\alpha\beta\beta}\alpha\beta^2 + \tilde{\Phi}_{\beta\beta\beta}\beta^3)$.

Using the numerical results to discuss the graphical representation, we can observe the change of bifurcation and hysteresis according to the suitable values of the central load, which depend on the unfolding parameters. Our considered model is static degenerate, i.e., the system has innumerable solutions, and they remain constant under minor disruption due to the system's built-in regularities. The static degenerate system also has a few advantages in the field of singularity theory and bifurcation analysis; it has the ability to simplify complex systems, come up with stability and robustness, and offer deep apprehension into the underlying uniformity and structures of these systems. Systems revealing static degeneracy have solutions that are immutable under small perturbations, leading to robustness in applications where stability is crucial.

This paper is organized as follows: In Section 2, we recall some fundamentals on Sobolev space, state degeneracy and nondegeneracy, tangent space and restricted tangent space, and codimension. In Section 3, we discuss versal unfolding, and in Section 4, we consider the Euler beam problem, along with the basic bifurcation theorem and perturbation theory. We talk about the smoothness of the considered problem in Section 5. In Section 6, we apply the Taylor coefficients of the Φ and Lyapunov equations with the projections of the bifurcation function. We present the numerical analysis and the graphical representation in Section 7, and the conclusion is in Section 8.

2 Preliminaries

In this section, we recall some fundamentals of classics which are essential for our work.

Sobolev space

Suppose $\Phi[0, l]$ is the set of functions $[0, l] \to \mathbb{R}$ modulo the equivalence relation [8, 9] with the ith order distributional derivatives of v is $D^i v$ and the Sobolev space is

$$\|v\|_{W^{k,p}[0,l]} = \left(\sum_{i=0}^{k} \binom{k}{i} \|D^i v\|^p_{L^p[0,l]} \right)^{\frac{1}{p}},$$

$$\|v\|_p = \begin{cases} \left(\int_0^l |v|^p ds \right)^{1/p}, & 1 \leq p < \infty, \\ \max\{|v(s)| : s \in [0, l]\}, & p = \infty. \end{cases}$$

We consider $L^p[0,l]$ the set $\{v : [0,l] \to \mathbb{R} : \|v\|_p < \infty\}$ and for $W^{k,2}[0,l]$, we find $H^k[0,l]$ which constitutes a Hilbert space under the following the inner product

$$\langle v_m, v_j \rangle_k = \begin{cases} (1 + \frac{\pi^2 m^2}{l^2})^k, & \text{if } m = j, \\ 0, & \text{if } m \neq j, \end{cases} \tag{5}$$

where $v_m = \sqrt{\frac{2}{l}} \sin(\frac{m\pi s}{l})$.

Degeneracy and non-degeneracy

Suppose f is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at x_0 if differential vanishes at x_0 that is $df_{x_0} = 0$ critical point is also known as a singular point [10]. The symmetric matrix $n \times n$ of second order partial derivatives is the Hessian matrix of f at x_0 if x_0 is a critical point at $F : \mathbb{R}^n \to \mathbb{R}$, $F_{x_0} = d^2 H_f(x_0) = (h_{ij})$ in this case, $h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)$. If $\det H_f(x_0) \neq 0$, then a critical point x_0 of f is non-degenerate; if not, then degenerate. The number of negative eigenvalues of the Hessian matrix counting multiplicity is the index of a non-degenerate critical point of the smooth function f if $x_0 \in \mathbb{R}^n$.

Tangent space and restricted tangent space

The tangent space [2] to a germ f in $\mathcal{E}_{x,\lambda}$, represented as $T(F)$, is composed of all germs of the form $Af + Bf_x + Cf_\lambda$, where $A, B \in \mathcal{E}_{x,\lambda}$. We use tangent space to compute the universal unfolding and to demonstrate a versal unfolding of F of a germ f using tangent space. The expression $C(\lambda)f_\lambda(x,\lambda)$ presents a challenge because multiplying it by any arbitrary germ $\mathcal{E}_\lambda(x,\lambda)$ does not maintain its form. This finding touches on an issue with $T(f)$ computations, which is the presumption that coordinate changes in λ are independent of x. We are therefore interested in the restricted tangent space.

The restricted tangent space of a germ f is the set of all germs p that may be represented in the way shown below

$$p(x,\lambda) = A(x,\lambda)f(x,\lambda) + B(x,\lambda)f_x(x,\lambda),$$

where $A, B \in \mathcal{E}_{x,\lambda}$ and $B(0,0) = 0$. "Restricted" refers to a creation with strong equivalency.

If $RT(f + tp) = RT(f) \forall t \in [0,1]$, the $f + tp$ is strongly equivalent to $f \forall t \in [0,1]$, [key : $p \in RT(f)$]. The restricted tangent space $RT(f)$, which is also made up of all the mappings p such that, for small t, $f + tp$ is strongly similar to h, is necessary in order to construct the tangent space. A linear combination of lowest-order monomials is indicated by h, and p represents the perturbation in this case.

Codimension

The versal unfolding of F of f is based on the fewest possible parameters, provided that all other unfoldings of f factor through F. The value with the lowest codimension is f.

Bifurcation set and hysteresis set

Consider the versal unfolding of a germ as $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ of $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The bifurcation set is represented by B [1], which is the set of values of the parameters that can be used to represent the qualitative behavior in the system. A bifurcation point termed as a singular point on the bifurcation diagram can occur only α belongs to the bifurcation set

$$\begin{aligned} B &= \{\alpha | f_\alpha \text{ has a bifurcation point}\} \\ &= \{\alpha \in \mathbb{R}^k : \exists (x,\lambda) \in \mathbb{R}\times\mathbb{R}, \Phi(x,\lambda,\alpha)=0, \Phi_x(x,\lambda,\alpha)=0, \Phi_\lambda(x,\lambda,\alpha)=0\}. \end{aligned}$$

The symbol H [1] indicates a hysteresis point, which is a point on the bifurcation diagram having a vertical tangent; thus α belongs to the hysteresis set

$$\begin{aligned} H &= \{\alpha | f_\alpha \text{ has a hysteresis point}\} \\ &= \{\alpha \in \mathbb{R}^k : \exists (x,\lambda) \in \mathbb{R}\times\mathbb{R}, \Phi(x,\lambda,\alpha)=0, \Phi_x(x,\lambda,\alpha)=0, \Phi_{xx}(x,\lambda,\alpha)=0\}. \end{aligned}$$

3 Versal unfolding

If $F(x,\lambda)$ is the original function of the modified energy function $E(x,\lambda,\alpha,\beta)$ [2]. The tangent space of $F(x,\lambda)$ is as follows:

$$T(F(x,\lambda)) = \left(\mathcal{M}^3 + \left\langle \lambda^2 \right\rangle\right) \oplus R\{F, F_x, xF_x, F_\lambda, \lambda F_\lambda\}.$$

Here, the restricted tangent space is

$$RT(F) = \mathcal{M}^3 + \left\langle \lambda^2 \right\rangle,$$

and the codimension of F is 3 for the basis $\langle 1, x, x\lambda \rangle$. For the pitchfork bifurcation codimension of $RT(F) = \text{codim}\,\{F\} - 1 = 2$ where $\mathcal{M}^k = \{f \in \mathcal{E}_n \left(\frac{\partial}{\partial v}\right)^\alpha f(0) = 0 \text{ for } |\alpha| \leq k-1\}$ since $\mathcal{M}^k \subset RT(F)$.

Universal unfolding theorem

Let F be a k-parameter versal unfolding of f, where f is a germ in $\mathcal{E}_{x,\lambda}$ if and only if [2]

$$\mathcal{E}_{x,\lambda} = T(f) + \mathbb{R}\left(\frac{\partial F}{\partial \alpha_1}(x,\lambda,0), \ldots, \frac{\partial F}{\partial \alpha_k}(x,\lambda,0)\right).$$

The minimum number of codimensions is the number of versal unfolding parameters.

4 Euler beam problem

To perform a bifurcation and hysteresis analysis of the Euler beam problem, we need to apply some well-known mathematical tools and techniques [11, 12] under the singularity theory. A brief description of such tools is presented in the following sections. Let $v(s)$ represent the beam's deflection that is perpendicular to a reference line. This can be represented as a function of the arc length towards the beam.

We determine the curvature of an element in the following form

$$\mathcal{F} = \frac{d}{ds}\sin^{-1} v' = v''(1-v'^2)^{-\frac{1}{2}}.$$

We have the strain energy and the potential energy of the system, respectively.

$$S = \frac{1}{2}\int_0^l \mathcal{F}^2 ds = \frac{1}{2}\int_0^l v''^2(1-v'^2)^{-1} ds, \tag{6}$$

$$T = \int_0^l (1-v'^2)^{1/2} ds. \tag{7}$$

The total energy E is

$$E = S + \lambda T = \frac{1}{2}\int_0^l \left[\frac{v''}{(1-v'^2)^{1/2}}\right]^2 ds + \lambda \int_0^l \sqrt{1-(v')^2} ds,$$

on $V = \{v \in \mathcal{X} : \|v'\|_\infty \le 1-\epsilon\}$, $0 < \epsilon \ll 1$, while $\mathcal{X}$ represents a Sobolev space defined as

$$\mathcal{X} = \{v \in H^2[0,l] : v(0) = v(l) = 0\}.$$

Here, α stands for the beam's initial (constant) curvature, and the strain energy functional for the problem under consideration is obtained

$$S = \frac{1}{2}\int_0^l (\mathcal{F}-\alpha)^2 ds = \frac{1}{2}\int_0^l \left[\frac{v''}{(1-v'^2)^{1/2}} - \alpha\right]^2 ds.$$

In this idealized situation, β denotes a central load and α denotes the (constant) initial curvature of the beam for the perturbed energy functional. Here we set $V = \delta(v) = v(\frac{l}{2})$.

Now the total energy is the summation of the strain energy, potential energy and central force for the Euler buckling problem

$$E = S + \lambda T + \beta V = \frac{1}{2}\int_0^l (\mathcal{F}-\alpha)^2 ds + \lambda \int_0^l (1-v'^2)^{1/2} ds + \beta\delta(v).$$

Eq. (1), which is the perturbed energy function, is the modified Euler buckling problem of Golubitsky and Schaeffer's technique. If an external force acts on the beam, however, the beam does not deform as the compressive force λ is zero.
The reducing idealized problem's initial solution $v = 0$ shows a supercritical bifurcation at $\lambda = \frac{n^2\pi^2}{l^2}$ for $\alpha = \beta = 0$. It is our goal to demonstrate an idealized model presented by the versal unfolding parameters α and β. We assume that the curvature is not constant initially while the strain energy is minimized as follows:

$$S = \frac{1}{2}\int_0^l \left(\frac{v''}{(1-(v')^2)^{1/2}} - \alpha\mathcal{F}\right)^2 ds,$$

where $\mathcal{F}$ is a function that can be defined as

$$\mathcal{F} = \frac{1}{\sqrt{l/2}}\left[a_0 + \sum_{i=1}^{\infty}\left(a_i \cos\frac{2i\pi s}{l} + b_i \sin\frac{2i\pi s}{l}\right)\right].$$

Basic bifurcation theorem

Suppose $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$; we discuss a basic bifurcation theorem [13, 14] for this function. Using the bifurcation theorem, we can identify the types of global and different types of local bifurcation. For $x = 0$, the theorem mentioned below concerns a trivial solution [$f(0, \lambda) = 0 \ \forall \ \lambda$, so $\left(\frac{\partial f}{\partial \lambda}\right)(0, \lambda_0) = 0$], where F is symmetric that ensures $F_{xx}(0, \lambda) = 0$ and concerns the simplest case in which $(0, \lambda_0)$ might be a bifurcation point [so $\left(\frac{\partial F}{\partial x}\right)(0, \lambda_0)$ must vanish]. From the bifurcation theorem [2] we can discuss the followings:
If the following criteria are met and $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth mapping:

$$(i) \ \ F(x_0, \lambda_0) = 0, \ \ F_x(x_0, \lambda_0) = 0, \ \ F_\lambda(x_0, \lambda_0) = 0, \ \text{ and } \ F_{xx}(x_0, \lambda_0) = 0, \ \text{ and}$$
$$(ii) \ \ F_{xxx}(x_0, \lambda_0) \neq 0, \ \text{ and } \ F_{x\lambda}(x_0, \lambda_0) \neq 0.$$

Then (x_0, λ_0) is a bifurcation point, and the neighborhood of (x_0, λ_0) has a smooth coordinate change in the following form

$$x = \xi(\tilde{x}, \lambda) \ \ with \ \ \xi(0, \lambda_0) = x_0,$$

(here bar indicates evaluate at the critical point) and a smooth function $\mathcal{T}(\tilde{x}, \lambda)$ with $\mathcal{T}(0, \lambda_0) = +1$ such that

$$\mathcal{T}(\tilde{x}, \lambda) F(\xi(\tilde{x}, \lambda), \lambda) = \tilde{x}^3 \pm \lambda \tilde{x},$$

according to the sign of $[F_{x\lambda}(x_0, \lambda_0) \cdot F_{xxx}(x_0, \lambda_0)]$. This follows that $\mathcal{T}(\tilde{x}, \lambda) F(\xi(\tilde{x}, \lambda), \lambda)$ and $\tilde{x}^3 \pm \lambda \tilde{x}$ are $\mathcal{P} - \mathcal{K}$-equivallent.

Perturbation theory

As shown [1, 13], we have since

$$\tilde{\xi} = \tilde{\Phi}_x = \tilde{\Phi}_{xx} = \tilde{\Phi}_\lambda = 0, \quad \tilde{\Phi}_{xxx} \neq 0, \qquad \tilde{\Phi}_{x\lambda} \neq 0,$$

the bifurcation of $\xi(x, \lambda) = 0$ at $(0, \lambda^*)$, where $\xi(x, \lambda) = \Phi(x, \lambda, 0, 0)$, is a pitchfork.

Definition 1 *$\mathcal{P} - \mathcal{K}$ -versality:*
*An unfolding $\Phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, (0, \lambda^*, 0, 0)) \rightarrow (\mathbb{R}, 0)$, $(x, \lambda, \alpha, \beta) \mapsto \Phi(x, \lambda, \alpha, \beta)$, of $f : (\mathbb{R} \times \mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, $(x, \lambda) \mapsto f(x, \lambda)$, is known as **p-$\mathcal{K}$-versal**, if*

$$\mathcal{E}_{x,\lambda}\Phi + \mathcal{E}_{x,\lambda}\Phi_x + \mathcal{E}_\lambda \Phi_\lambda + \langle \Phi_i|_{(x,\lambda,\alpha,\beta)=(0,\lambda^*,0,0)} : i = 1, \ldots, k \rangle_{\mathbb{R}} = \mathcal{E}_{x,\lambda}.$$

Here germs $\mathcal{E}_{x,\lambda}$, and $\mathcal{E}_\lambda$ denoting the ring of C^∞-function on $(\mathbb{R}^2, (0, \lambda^*))$, and $(\mathbb{R}, \lambda^*)$ of the variables (x, λ), and λ accordingly.
Here we compute

$$\begin{vmatrix} \tilde{\Phi}_x & \tilde{\Phi}_{xx} & \tilde{\Phi}_{xxx} & \tilde{\Phi}_{x\lambda} \\ \tilde{\Phi}_\lambda & \tilde{\Phi}_{x\lambda} & \tilde{\Phi}_{xx\lambda} & \tilde{\Phi}_{\lambda\lambda} \\ \tilde{\Phi}_\alpha & \tilde{\Phi}_{x\alpha} & \tilde{\Phi}_{xx\alpha} & \tilde{\Phi}_{\lambda\alpha} \\ \tilde{\Phi}_\beta & \tilde{\Phi}_{x\beta} & \tilde{\Phi}_{xx\beta} & \tilde{\Phi}_{\lambda\beta} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \tilde{\Phi}_{xxx} & \tilde{\Phi}_{x\lambda} \\ 0 & \tilde{\Phi}_{x\lambda} & \tilde{\Phi}_{xx\lambda} & \tilde{\Phi}_{\lambda\lambda} \\ \tilde{\Phi}_\alpha & 0 & \tilde{\Phi}_{xx\alpha} & 0 \\ \tilde{\Phi}_\beta & 0 & \tilde{\Phi}_{xx\beta} & 0 \end{vmatrix} = (\tilde{\Phi}_{x\lambda})^2 \begin{vmatrix} \tilde{\Phi}_\alpha & \tilde{\Phi}_{xx\alpha} \\ \tilde{\Phi}_\beta & \tilde{\Phi}_{xx\beta} \end{vmatrix} \neq 0,$$

which concludes Φ is $\mathcal{P}$-$\mathcal{K}$-versal when n is odd [1].

5 Smoothness

Smoothness of Φ or differentiability of Φ [4] then we have,

$$dE = \Phi(v,\lambda,\alpha,\beta)\cdot\xi = \int_0^l \left[\frac{v''}{(1-v'^2)^{\frac{1}{2}}} - \mathcal{F}_0\alpha\right]\left[\frac{\xi''}{(1-v'^2)^{\frac{1}{2}}} + \frac{v'v''\xi'}{(1-v'^2)^{\frac{3}{2}}}\right]ds + \beta\xi\left(\frac{l}{2}\right)$$
$$= ((\Psi)_v - \lambda(\Lambda)_v)\cdot\xi - \alpha(\mathcal{F})_v\cdot\xi + \beta\xi\left(\frac{l}{2}\right)$$

Lemma 1 *Φ is C^1 where the image of Φ is in $\mathcal{X}'$ [5].*

Remarks:

- $|\Phi_{(v,\lambda,\alpha,\beta)}\cdot\xi| \le \|\xi\|_{2,2}$.
- Here $\Phi_{(v,\lambda,\alpha,\beta)}\cdot\xi = ((\psi)_v - \lambda(\Lambda)_v)\cdot\xi - \alpha(\mathcal{F})_v\cdot\xi + \beta\xi(\frac{l}{2})$, with the following calculation

$$|((\psi_1)_{v_1} - (\psi_1)_{v_2})[u]\cdot\xi| \le \epsilon(\|u'\|_2\|\xi''\|_2 + \|u''\|_2\|\xi''\|_2 + \|u'\|_2\|\xi\|_2 + \|u''\|_2\|\xi'\|_2),$$
$$|((\Lambda_1)_{v_1} - (\Lambda_2)_{v_2})[u]\cdot\xi| \le \epsilon\|u'\|_2\|\xi'\|_2,$$
$$|((\mathcal{F}_1)_{v_1} - (\mathcal{F}_2)_{v_2})[u]\cdot\xi| \le (\|u'\|_2\|\xi''\|_2 + \|u'\|_2\|\xi\|_2 + \|u'\|_2\|\xi'\|_2 + \|u''\|_2\|\xi'\|_2),$$

where $\psi_1, \Lambda_1, \mathcal{F}_1$ indicates the first derivatives of $\psi, \Lambda, \mathcal{F}$.

Lemma 2 *Φ is C^∞ [5].*

Remark: If $j + i_1 + \cdots + i_{\mathcal{F}} \le k+2$, $j \ge 2$ then we get

$$\left|\int_0^l \mathcal{A}(v')(v'')^j {v_1}^{(i_1)}\cdots v_k^{(i_k)}ds\right| \le C\|\mathcal{A}(v')\|_\infty\|v\|_{2,2}^j\|v_1\|_{2,2}\cdots\|v_k\|_{2,2},$$

and if $j + i_1 + \cdots + i_k > k+2$, we need to substitute $\|\cdot\|_{2,2}$ by $\|\cdot\|_{3,2}$.

6 Taylor coefficients of Φ and Lyapunov equations

Here $(\Phi_k)_v$ is the k-th order differential coefficient at v of $(\Phi)_v$, and by Φ_k [4, 5] of the order k differential of Φ at $v = 0$.

Lemma 3 *Setting $(\Phi)_u = (\Psi)_u - \lambda(\Lambda)_u$. The first derivative of $(\Phi)_u$ at $u = 0$ is demonstrated as*

$$\Phi_1[u]\cdot\xi = \int_0^l (u'' + \lambda u)\xi'' ds. \tag{8}$$

Lemma 4 *The second derivative of $(\Phi)_v$ is*

$$(\Phi_2)_v[u_1,u_2]\cdot\xi = \int_0^l [2v'u_2'u_1''\xi'' + 2v''\xi''u_2'u_1' + 2v'u_1'u_2''\xi'' + 2v''u_1''u_2'\xi' + 2u_2''u_1''\xi'v' + 2v''u_2''u_1'\xi']ds.$$

So, setting $v = 0$, we obtain $\Phi_2[u_1,u_2]\cdot\xi = 0$.

Lemma 5 *We have*

$$\Phi_3[v_a,v_b,v_c]\cdot v_n=\frac{abcn^2\pi^5}{l^7}\sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=n}\left[1-\frac{3n}{2}+abcn\varepsilon_1\varepsilon_2\varepsilon_3\left(\frac{\varepsilon_1}{a}+\frac{\varepsilon_2}{b}+\frac{\varepsilon_3}{c}\right)\right],$$

where $\varepsilon_i=\pm 1$, $i=1,2,3$.

Lemma 6 *When* $\mathcal{F}=\frac{1}{\sqrt{l/2}}[a_0+\sum_{k=1}^{\infty}(a_k\cos(2k\pi s/l)+b_k\sin(2k\pi s/l))]$, *we calculate*

$$\mathcal{F}_0=-\frac{\pi^2}{l^2}\left[\sum_{m:\,odd}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{m^3 a_k}{m^2-4k^2}v_m^*+\sum_{m:\,even}m^2 b_{m/2}v_m^*\right],\quad \mathcal{F}_1[v_a]=0,\quad and$$

$$\mathcal{F}_2[v_a,v_b]=-\frac{ab\pi^3}{l^5}\sum_{k=0}^{\infty}\Bigg(a_k\sum_{m\not\equiv a+b(2)}m\left(\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\frac{(\varepsilon_1 a+\varepsilon_2 b+m)^2}{(\varepsilon_1 a+\varepsilon_2 b+m)^2-4k^2}\right)v_m^*$$
$$+b_k\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1}\varepsilon_3 k(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 k)v^*_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 k}\Bigg).$$

Lyapunov equations

From (8) we find $\lambda^*=(\frac{n\pi}{l})^2$, $v_n=\sqrt{\frac{2}{l}}\sin(\frac{n\pi s}{l})$ is a non-zero function that generates the kernel of the first-order derivative of Φ. We decompose $\Phi(v,\lambda,\alpha,\beta)=0$ which is equivalent to $P\Phi(v,\lambda,\alpha,\beta)=0$, and $Q\Phi(v,\lambda,\alpha,\beta)=0$. Hence, the orthogonal projection of $\mathcal{X}'$ to $\ker L_1$ is $P:\mathcal{X}\longrightarrow\mathcal{X}$ where $v\mapsto\frac{\langle v,v_n\rangle_2}{\langle v_n,v_n\rangle_2}v_n$ and the orthogonal projection $Q:\mathcal{X}\longrightarrow\mathcal{X},\quad v\mapsto v-P(v)$, $\mathcal{X}'$ to the transpose of $\ker L_1$, the orthogonal complement to $\ker L_1$.
Notably, the differential map $(\ker L_1)^\perp\longrightarrow(\ker L_1)^\perp$ is bijective, and its inverse is also continuous, which means isomorphism where $u\mapsto D_uQ\Phi$, at $(0,\lambda^*,0,0)$ is given by $u\mapsto\left[\xi\mapsto\int_0^l(u''+\lambda^*u)\xi''ds\right]$. If it is not isomorphism, we can not apply the implicit function theorem. By the implicit function theorem [11], the rest of the decomposed equation defines a bifurcation function by

$$Q\Phi(xv_n+\mathcal{W}(x,\lambda,\alpha,\beta),\lambda,\alpha,\beta)=0\quad\text{near }(0,\lambda^*,0,0).\tag{9}$$

According to the Lyapunov-Schmidt reduction, the zero of $\Phi(x,\lambda,\alpha,\beta)$ describes the bifurcation of zero, and then we can write the projection of the bifurcation function

$$\Phi(x,\lambda,\alpha,\beta)=Q\Phi(xv_n+\mathcal{W}(x,\lambda,\alpha,\beta),\lambda,\alpha,\beta).\tag{10}$$

The projections of L_3 and $\mathcal{F}_2$ are as follows

$$QL_3[\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha]=-\frac{l^6}{\pi^6}\sum_{a,b,c\neq n}\frac{\mathcal{F}_a\mathcal{F}_b\mathcal{F}_c}{(a^2-n^2)(b^2-n^2)(c^2-n^2)}QL_3[v_a,v_b,v_c],$$

$$QL_3[\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\beta]=-\frac{l^6}{\pi^6}\sum_{c:odd,a,b,c\neq n}\frac{\mathcal{F}_a\mathcal{F}_b(\frac{l^2}{\pi^2})\sqrt{\frac{2}{l}}}{c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}QL_3[v_a,v_b,v_c],$$

$$
\begin{aligned}
QL_3[\tilde{\mathcal{W}}_\alpha, \tilde{\mathcal{W}}_\beta, \tilde{\mathcal{W}}_\beta] &= -\frac{l^6}{\pi^6} \sum_{b,c:odd,a,b,c\neq n} \frac{\mathcal{F}_a(\frac{l^4}{\pi^4})\frac{2}{l}}{b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)} QL_3[v_a, v_b, v_c],\\
QL_3[\tilde{\mathcal{W}}_\beta, \tilde{\mathcal{W}}_\beta, \tilde{\mathcal{W}}_\beta] &= -\frac{l^6}{\pi^6} \sum_{a,b,c:odd,a,b,c\neq n} \frac{(\frac{l^6}{\pi^6})(\frac{2}{l})^{3/2}}{a^2b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)} QL_3[v_a, v_b, v_c],\\
Q\mathcal{F}_2[\tilde{\mathcal{W}}_\alpha, \tilde{\mathcal{W}}_\alpha] &= \frac{l^4}{\pi^4} \sum_{a,b\neq n} \frac{\mathcal{F}_a\mathcal{F}_b}{(a^2-n^2)(b^2-n^2)} Q\mathcal{F}_2[v_a, v_b],\\
Q\mathcal{F}_2[\tilde{\mathcal{W}}_\alpha, \tilde{\mathcal{W}}_\beta] &= \frac{l^6}{\pi^6}\sqrt{\frac{2}{l}} \sum_{a,b\neq n,odd} \frac{\mathcal{F}_a}{b^2(a^2-n^2)(b^2-n^2)} Q\mathcal{F}_2[v_a, v_b],\\
Q\mathcal{F}_2[\tilde{\mathcal{W}}_\beta, \tilde{\mathcal{W}}_\beta] &= \frac{l^8}{\pi^8}\frac{2}{l} \sum_{a,b\neq n,odd} \frac{1}{a^2b^2(a^2-n^2)(b^2-n^2)} Q\mathcal{F}_2[v_a, v_b].
\end{aligned}
$$

7 Numerical analysis

Suppose the bifurcation function [4] is

$$\Phi(x, \lambda, \alpha, \beta) = Q\Phi(xv_n + \mathcal{W}(x, \lambda, \alpha, \beta), \lambda, \alpha, \beta),$$

where

$$
\begin{aligned}
\Phi &= x\tilde{\Phi}_x + \lambda\tilde{\Phi}_\lambda + \alpha\tilde{\Phi}_\alpha + \beta\tilde{\Phi}_\beta + \frac{x^2}{2}\tilde{\Phi}_{xx} + x\lambda\tilde{\Phi}_{x\lambda} + x\alpha\tilde{\Phi}_\alpha + x\beta\tilde{\Phi}_{x\beta} + \frac{\lambda^2}{2}\tilde{\Phi}_{\lambda\lambda} + \cdots + \frac{x^3}{6}\tilde{\Phi}_{xxx}\\
&\quad + \frac{x^2}{2} l(\alpha, \beta) + xQ(\alpha, \beta) + C(\alpha, \beta) + O(4),
\end{aligned}
$$

(bar indicates evaluate at $(0, \lambda^*, 0, 0)$). Where

$$
\begin{aligned}
l(\alpha, \beta) &= \tilde{\Phi}_{xx\alpha}\alpha + \tilde{\Phi}_{xx\beta}\beta,\\
Q(\alpha, \beta) &= \frac{1}{2}\left(\tilde{\Phi}_{x\alpha\alpha}\alpha^2 + 2\tilde{\Phi}_{x\alpha\beta}\alpha\beta + \tilde{\Phi}_{x\beta\beta}\beta^2\right),\\
C(\alpha, \beta) &= \frac{1}{6}\left(\tilde{\Phi}_{\alpha\alpha\alpha}\alpha^3 + 3\tilde{\Phi}_{\alpha\alpha\beta}\alpha^2\beta + 3\tilde{\Phi}_{\alpha\beta\beta}\alpha\beta^2 + \tilde{\Phi}_{\beta\beta\beta}\beta^3\right).
\end{aligned}
$$

We now assume the following bifurcation function

$$\Phi(x, \lambda, \alpha, \beta) = Q\Phi(xv_n + \mathcal{W}(x, \lambda, \alpha, \beta), \lambda, \alpha, \beta) = Q(L)_v - \alpha Q(\mathcal{F})_v + \beta Q\delta, \tag{11}$$

where

$$(L)_v = (\Psi)_v - \lambda(\Lambda)_v,\ v = xv_n + \mathcal{W}(x, \lambda, \alpha, \beta).$$

For describing the pitchfork bifurcation we consider [1] the bifurcation set B, hysteresis set H, and the double limit points DL [2] are defined as in the following manner

$$B = \{\alpha : \exists(x, \lambda),\ \Phi(x, \lambda, \alpha, \beta) = 0, \Phi_x(x, \lambda, \alpha, \beta) = \Phi_\lambda(x, \lambda, \alpha, \beta) = 0\}, \tag{12}$$

$$\begin{aligned} H =& \{\alpha : \exists(x,\lambda),\ \Phi(x,\lambda,\alpha,\beta) = 0, \Phi_x(x,\lambda,\alpha,\beta) = \Phi_{xx}(x,\lambda,\alpha,\beta) = 0\}, && (13)\\ DL =& \{\alpha : \exists(x,y,\lambda), \text{with } x \neq y,\ \Phi(x,\lambda,\alpha,\beta) = 0, \Phi(y,\lambda,\alpha,\beta) = 0 && (14)\\ =& \det(d_x\Phi)(y,\lambda,\alpha,\beta)\}. \end{aligned}$$

Here we have,

$$\begin{aligned} B &= \{(\alpha,\beta) : \tilde{\Phi}_\alpha\alpha + \tilde{\Phi}_\beta\beta + C\,(\alpha,\beta) + O(4)\},\\ H &= \{(\alpha,\beta) : \tilde{\Phi}_\alpha\alpha + \tilde{\Phi}_\beta\beta + C(\alpha,\beta) - \frac{2l^{14}}{27n^{12}\pi^{12}} l\,(\alpha,\beta)^3 + O(4)\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\Phi}_\alpha\alpha + \tilde{\Phi}_\beta\beta &= \frac{4\pi n^2}{l^2}\sum_{i=0}^{\infty}\frac{na_i}{n^2-4i^2}\alpha + \left((-1)^{\frac{n-1}{2}}\sqrt{\frac{2}{l}}\right)\beta|_{n=1},\\ C(\alpha,\beta) &= \left(\frac{1}{6}L_3[v,v,v] - \frac{\alpha}{2}\mathcal{F}\alpha[v,v]\right)\cdot v_n|_{v=\tilde{\mathcal{W}}},\\ \tilde{\mathcal{W}} &= -\frac{l^2}{\pi^2}\sum_{m:odd;m\neq n}\frac{1}{m^2-n^2}\left(\frac{4\alpha}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2}v_m * + \frac{l^2}{\pi^2}\frac{\beta}{m^2\sqrt{\frac{l}{2}}}v_m *\right), \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{W}}_\alpha &= -\frac{4l^2}{\pi^3}\sum_{m:odd,\neq n}\frac{m}{m^2-n^2}\sum_{i=0}^{\infty}\frac{a_i}{m^2 4n^2}v_m^*,\\ \tilde{\mathcal{W}}_\beta &= \frac{l^4}{\pi^4}\sum_{m:odd,\neq n}\frac{\sqrt{\frac{2}{l}}}{m^2(m^2-n^2)}v_m^*. \end{aligned}$$

The second order derivatives of $\mathcal{W}$, and the projection of the third order derivatives are at the critical point $(0,\lambda^*,0,0)$ as follows

$$\tilde{\mathcal{W}}_{xx} = 0, \tilde{\mathcal{W}}_{x\lambda} = 0, \tilde{\mathcal{W}}_{x\alpha} = 0, \tilde{\mathcal{W}}_{x\beta} = 0, \tilde{\mathcal{W}}_{\lambda\lambda} = 0, \tilde{\mathcal{W}}_{\lambda\alpha} = L^{-1}Q\Lambda_\alpha[\tilde{\mathcal{W}}_\alpha],$$

$$\begin{aligned} Q\Phi_3[\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha] &= \frac{c_0}{l\pi}\left(\frac{4}{\pi}\right)^3, \quad Q\Phi_3[\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\beta] = \frac{lc_1}{\pi^3}\left(\frac{4}{\pi}\right)^2\left(\frac{2}{l}\right)^{1/2},\\ Q\Phi_3[\tilde{\mathcal{W}}_\alpha,\tilde{\mathcal{W}}_\beta,\tilde{\mathcal{W}}_\beta] &= \frac{l^3c_2}{\pi^5}\left(\frac{4}{\pi}\right)\frac{2}{l}, \quad Q\Phi_3[\tilde{\mathcal{W}}_\beta,\tilde{\mathcal{W}}_\beta,\tilde{\mathcal{W}}_\beta] = \frac{l^5c_3}{\pi^7}\left(\frac{2}{l}\right)^{3/2}\left(\frac{2}{l}\right). \end{aligned}$$

[Remark: Here bar indicates the calculation at $(0,\lambda^*,0,0)$.] We get,

$$\begin{aligned} \tilde{\mathcal{W}}_\alpha\alpha + \tilde{\mathcal{W}}_\beta\beta = &-\frac{l^2}{\pi^2}\sum_{m:odd,m\neq n}\frac{1}{m^2-n^2}\left(\frac{4\alpha}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2} + \frac{l^2}{\pi^2}\frac{\beta}{m^2\sqrt{l/2}}\right)v_m^*\\ &-\frac{l^2}{\pi^2}\sum_{l^2}^{\pi^2}\sum_{m:even,m\neq n}\frac{b_{m/2}}{m^2-n^2}v_m*, \end{aligned}$$

and

$$Q\mathcal{F}_2[v_a, v_b] = -\frac{abn\pi^3}{l^5} \sum_{i=0}^{\infty} a_i \sum_{a+b\neq n(2)} \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(\epsilon_1 a + \epsilon_2 b + n)^2}{(\epsilon_1 a + \epsilon_2 b + n)^2 - 4i^2}.$$

We inspect that the bifurcation set is exhibited as a straight line, while the hysteresis set is a curve, and that is going to make a loop when a_0 is present only. In Figure 1-Figure 5, the red line and blue line indicate the bifurcation set and the hysteresis set, respectively. These diagrams depend on the values of $\mathcal{F}$. For this purpose, the length is varied, e.g., $\pi, 2\pi$, and 4π, under different choices of a_0. It is also observed that the bifurcation sets near the origin are almost the same, but the hysteresis sets are changing considerably as a_0 increases.

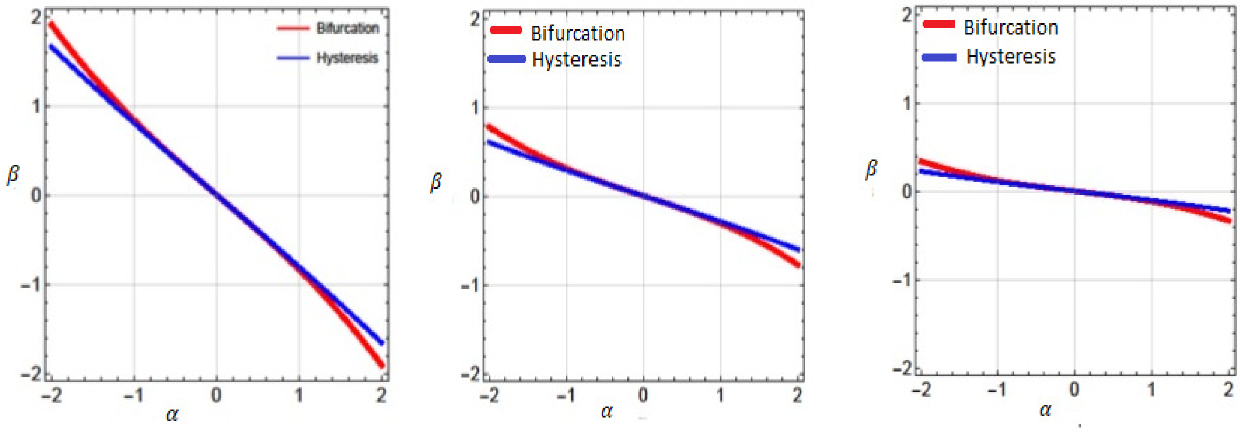

Figure 1. B and H ($a_0 = \frac{1}{2}, a_{\alpha,\beta\geq 1} = 0$) estimates in $\alpha\beta$ plane

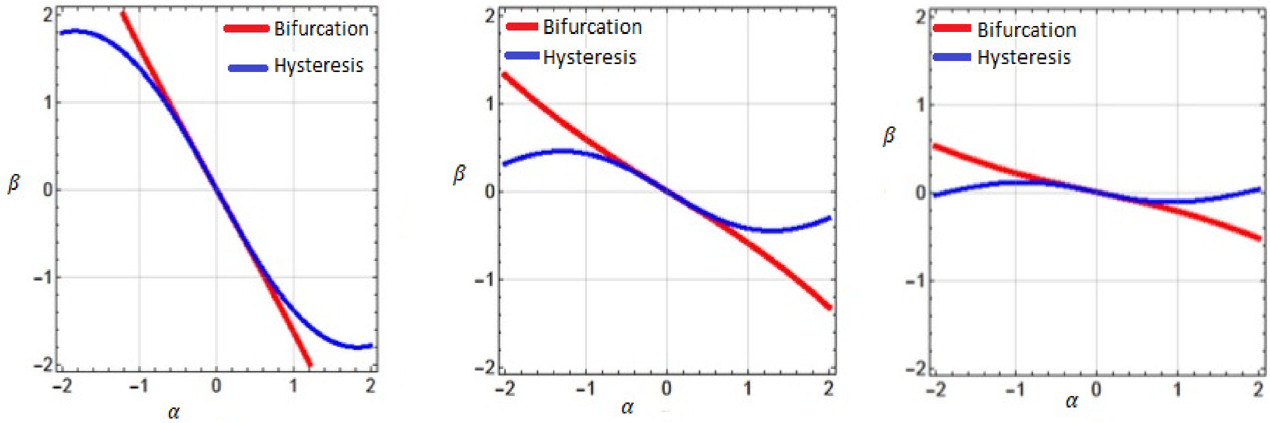

Figure 2. B and H ($a_0 = 1, a_{\alpha,\beta\geq 1} = 0$) estimates in $\alpha\beta$ plane

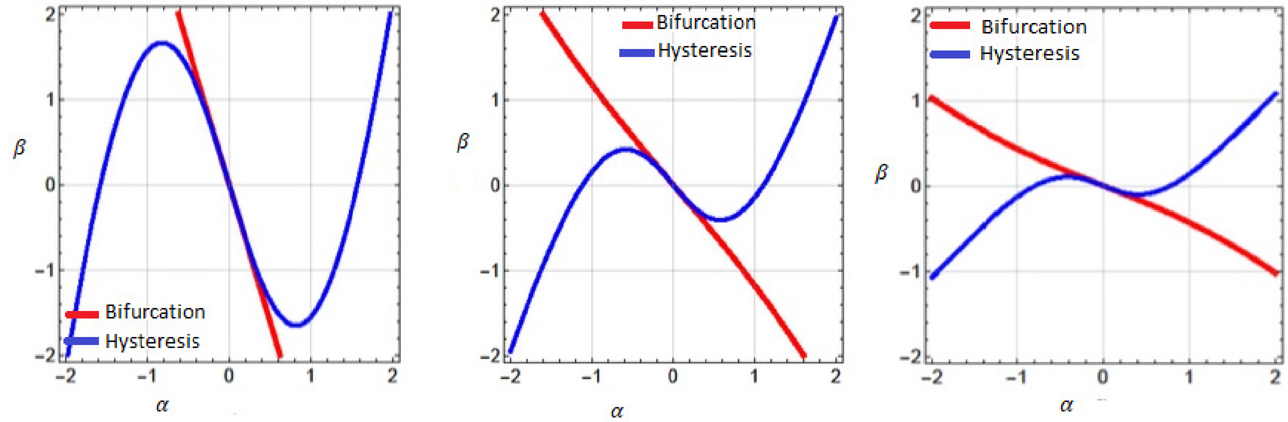

Figure 3. B and H ($a_0 = 2, a_{\alpha,\beta\geq 1} = 0$) estimates in $\alpha\beta$ plane

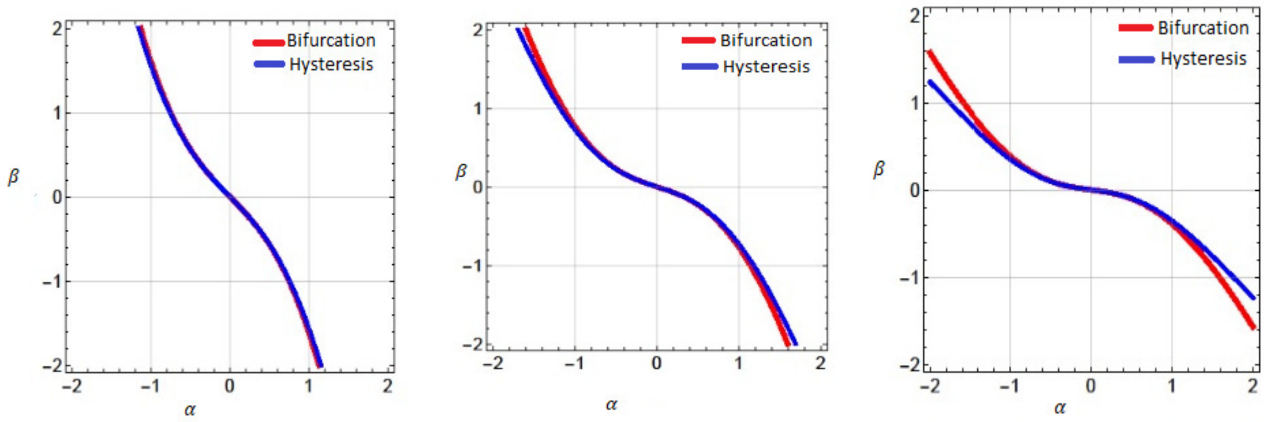

Figure 4. B and H ($a_0 = 1, a_{\alpha,\beta\geq 1} = 1, 2$) estimates in $\alpha\beta$ plane

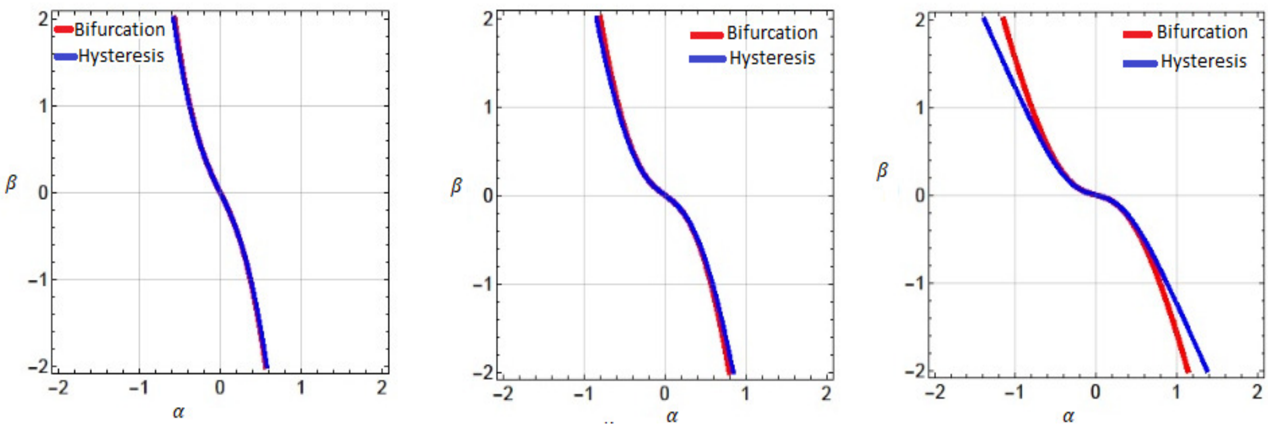

Figure 5. B and H ($a_0 = 2, a_{\alpha,\beta\geq 1} = 1, 2$) estimates in $\alpha\beta$ plane

8 Conclusion

We describe the perturbation or imperfection of the Euler beam problem, confirming the smoothness of the considered problem. We present our model as non-degenerate. The explicit notion of imperfections is unfolding parameters that are essential to indicate an arbitrarily small perturbation of the given problem. Versatility is crucial since it allows us to talk about flaws or potentially disturbed bifurcation diagrams using the versal unfolding parameters. We are also interested in talking about stability analysis and eigenvalues.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

No Data associated with the manuscript.

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

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Author's contributions

A.A.: Conceptualization, Methodology, Software, Writing original draft preparation, Supervision, Writing-reviewing and editing. M-S.H.: Conceptualization, Investigation, and Supervision. S.D.S.: Conceptualization, Writing-reviewing and editing. M.T.: Methodology and Investigation. All authors discussed the results and contributed to the final manuscript

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References

[1] Golubitsky, M. and Schaeffer, D. A theory for imperfect bifurcation via singularity theory. *Communications on Pure and Applied Mathematics*, 32(1), 21-98, (1979).

[2] Golubitsky, M., Stewart, I. and Schaeffer, D.G. Further Examples of Hopf Bifurcation with Symmetry. In *Singularities and Groups in Bifurcation Theory* Vol.2, (pp. 363-411). New York, NY: Springer, (1988). [CrossRef]

[3] Golubitsky, M. and Schaeffer, D. A theory for imperfect bifurcation via singularity theory. *Communications on Pure and Applied Mathematics*, 32(1), 21-98, (1979).

[4] Afroz, A. *Bifurcation Analysis of Euler Buckling Problem from the viewpoint of Singularity Theory*. Ph.D. Thesis, Department of Mathematics, Saitama University, (2019). [https://sucra.repo.nii.ac.jp/records/19055]

[5] Afroz, A. and Fukui, T. Bifurcation of Euler buckling problem, revisited. *Hokkaido Mathematical Journal*, 50(1), 111-150, (2021). [CrossRef]

[6] Ioffe, A.D. *Variational Analysis of Regular Mapping*. Springer: Cham, (2017). [CrossRef]

[7] Ambrosetti, A. and Álvarez, D.A. *An Introduction to Nonlinear Functional Analysis and Eliptic Problems* (Vol. 82). Springer Science & Business Media: Berlin, (2011).

[8] Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer: New York, (2011). [CrossRef]

[9] Sofonea, M., Han, W. and Shillor, M. *Analysis and Approximation of Contact Problems with Adhesion or Damage*. Chapman and Hall/CRC: New York, (2005). [CrossRef]

[10] Hamilton, R.S. The inverse function theorem of Nash and Moser. *Bulletin of the American Mathematical Society*, 7(1), 65-222, (1982).

[11] Abraham, R., Marsden, J.E. and Ratiu, T. *Manifolds, Tensor Analysis, and Applications* (Vol. 75). Springer Science & Business Media: New York, (2012).

[12] Han, S.M., Benaroya, H. and Wei, T. Dynamics of transversely vibrating beams using four engineering theories. *Journal of Sound and Vibration*, 225(5), 935-988, (1999). [CrossRef]

[13] Marsden, J.E. and Hughes, T.J. *Mathematical Foundations of Elasticity*. Dover Publications: New York, (1994).

[14] Kielhöfer, H. *Bifurcation Theory*. Springer: New York, (2012).

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