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Hyper-Dual Leonardo Quaternions

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Abstract — In this paper, hyper-dual Leonardo quaternions are defined and studied. Some basic properties of the hyper-dual Leonardo quaternions, including their relationships with the hyper-dual Fibonacci quaternions and hyper-dual Lucas quaternions, are analyzed. In addition, some formulae and identities, such as the recurrence relations, Binet's formula, generating functions, Vajda's identity, certain sum formulae, and some binomial-sum formulae, are investigated for hyper-dual Leonardo quaternions.

Keywords *Dual numbers, hyper-dual numbers, quaternions, Fibonacci numbers, Leonardo numbers*

Mathematics Subject Classification (2020) 11B39, 11R52

1. Introduction

Dual numbers invented in 1873 by Clifford [1] are an extension of real numbers. Hyper-dual numbers are an extension of dual numbers. Fike and Alonso [2] introduced hyper-dual numbers to demonstrate the advantages of hyper-dual numbers in second-order numerical differentiation. Dual and hyper-dual numbers have become a useful tool in mathematics and engineering. For further information about the applications of dual and hyper-dual numbers, see [3–13]. Quaternions discovered by Hamilton [14] are a 4-dimensional hyper-complex number system. Cohen and Shoham [9] defined hyper-dual quaternions by replacing each real number in a quaternion with the associated hyper-dual number.

Integer sequences are an important field of study in mathematics. The Fibonacci sequence is one of the most well-known examples of special integer sequences. This sequence is widely used in many scientific fields, including mathematics, physics, engineering, and art. Another well-known sequence is the Lucas sequence, closely related to the Fibonacci sequence. Many authors have investigated the Fibonacci and Lucas sequences in [15–17], among others. Another integer sequence studied intensively by researchers in recent years and closely related to the Fibonacci sequence is the Leonardo sequence. Some properties of this sequence have been investigated in [18, 19]. Several authors have investigated the properties of hyper-complex numbers with distinct integer sequences from various perspectives. Some examples of recent studies on quaternions and hyper-dual numbers with the Fibonacci, Lucas, and Leonardo sequences can be found in [20–25].

This paper aims to define the hyper-dual Leonardo quaternions by considering the concepts of hyper-dual numbers, quaternions, and Leonardo numbers and to investigate some of their algebraic and combinatorial properties.

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2. Preliminaries

This section provides some basic notions to provide a background for the next section.

Definition 2.1. [1] Let a and b be arbitrary real numbers. Then, a dual number x has the form

$$x = a + b\varepsilon$$

where ε is the dual unit that satisfies the rules $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

Definition 2.2. [2] Let x_1 and x_2 be any dual numbers and ε be the dual unit. Then, a hyper-dual number z is represented as follows:

$$z = x_1 + x_2\varepsilon$$

Furthermore, it is easy to see that any hyper-dual number z can be characterized by

$$z = a_1 + a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_1\varepsilon_2$$

where, for all $i \in \{1, 2, 3, 4\}$, a_i is a real number and ε_1 and ε_2 are the dual units that satisfy the rules

$${\varepsilon_1}^2 = {\varepsilon_2}^2 = (\varepsilon_1\varepsilon_2)^2 = 0, \quad \varepsilon_1 \neq \varepsilon_2, \quad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1, \quad \varepsilon_1 \neq 0, \quad \varepsilon_2 \neq 0, \quad \text{and} \quad \varepsilon_1\varepsilon_2 \neq 0 \tag{2.1}$$

Let $z_1 = a_1 + a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_1\varepsilon_2$ and $z_2 = b_1 + b_2\varepsilon_1 + b_3\varepsilon_2 + b_4\varepsilon_1\varepsilon_2$ be any two hyper-dual numbers. Then, the addition, scalar multiplication (by a scalar λ), and multiplication of two hyper-dual numbers are defined as follows, respectively:

$$z_1 + z_2 = (a_1 + b_1) + (a_2 + b_2)\varepsilon_1 + (a_3 + b_3)\varepsilon_2 + (a_4 + b_4)\varepsilon_1\varepsilon_2$$

$$\lambda z_1 = \lambda a_1 + \lambda a_2\varepsilon_1 + \lambda a_3\varepsilon_2 + \lambda a_4\varepsilon_1\varepsilon_2$$

and

$$z_1 z_2 = (a_1b_1) + (a_1b_2 + a_2b_1)\varepsilon_1 + (a_1b_3 + a_3b_1)\varepsilon_2 + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)\varepsilon_1\varepsilon_2$$

The set of all the hyper-dual numbers forms a 4-dimensional, with the basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$, commutative, and associative algebra over the real numbers. For detailed information about hyper-dual numbers, see [2].

Definition 2.3. [14] A quaternion q is of the form

$$q = q_1 + q_2i + q_3j + q_4k$$

where, for all $i \in \{1, 2, 3, 4\}$, q_i is a real number and i, j, and k are the quaternionic units that satisfy the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad \text{and} \quad ki = j = -ik \tag{2.2}$$

Let $p = p_1 + p_2i + p_3j + p_4k$ and $q = q_1 + q_2i + q_3j + q_4k$ be any two quaternions. Then, the addition, scalar (λ) multiplication, and multiplication of two quaternions are defined as follows, respectively:

$$p + q = (p_1 + q_1) + (p_2 + q_2)i + (p_3 + q_3)j + (p_4 + q_4)k$$

$$\lambda q = \lambda q_1 + \lambda q_2i + \lambda q_3j + \lambda q_4k$$

and

$$\begin{aligned} pq = {} & (p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4) + (p_1q_2 + p_2q_1 + p_3q_4 - p_4q_3)i + (p_1q_3 + p_3q_1 + p_4q_2 - p_2q_4)j \\ & +(p_1q_4 + p_4q_1 + p_2q_3 - p_3q_2)k \end{aligned}$$

The set of all the quaternions forms a 4-dimensional, with the basis $\{1, i, j, k\}$, non-commutative, and associative algebra over the real numbers. For further quaternion information, see [14, 26].

Definition 2.4. [9] A hyper-dual quaternion Q is defined as

$$Q = z_1 + z_2 i + z_3 j + z_4 k$$

where, for all $i \in \{1, 2, 3, 4\}$, z_i is a hyper-dual number and i, j, and k are the quaternionic units defined as in (2.2).

Note that the dual units ε_1 and ε_2 commute with the quaternionic units i, j, and k, e.g., $\varepsilon_1 i = i\varepsilon_1$ [9]. In the rest of this section, we provide some definitions and identities of the sequences of Fibonacci, Lucas, and Leonardo numbers.

Definition 2.5. [15] For $n \geq 2$, the Fibonacci and Lucas numbers are defined by the recurrence relations, respectively:

$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \quad F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2} \quad \text{with} \quad L_0 = 2,\ L_1 = 1$$

Here, F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively.

Definition 2.6. [18] The Leonardo numbers are defined recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

or

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3$$

with the initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Here, Le_n denotes the n-th Leonardo number.

Moreover, Ömür and Koparal [24] defined the hyper-dual generalized Fibonacci and Lucas numbers. In particular cases of the hyper-dual generalized Fibonacci and Lucas numbers, the hyper-dual Fibonacci and Lucas numbers can be derived as:

Definition 2.7. [24] The hyper-dual Fibonacci and hyper-dual Lucas numbers are defined as follows, respectively:

$$HDF_n = F_n + F_{n+1}\varepsilon_1 + F_{n+2}\varepsilon_2 + F_{n+3}\varepsilon_1\varepsilon_2 \tag{2.3}$$

and

$$HDL_n = L_n + L_{n+1}\varepsilon_1 + L_{n+2}\varepsilon_2 + L_{n+3}\varepsilon_1\varepsilon_2 \tag{2.4}$$

where ε_1 and ε_2 are the dual units defined as in (2.1).

Definition 2.8. [25] The hyper-dual Leonardo numbers are

$$HDLe_n = Le_n + Le_{n+1}\varepsilon_1 + Le_{n+2}\varepsilon_2 + Le_{n+3}\varepsilon_1\varepsilon_2 \tag{2.5}$$

where ε_1 and ε_2 are the dual units in (2.1).

Moreover, the recurrence relation of the hyper-dual Leonardo numbers is provided by

$$HDLe_n = HDLe_{n-1} + HDLe_{n-2} + A, \quad n \geq 2 \tag{2.6}$$

or

$$HDLe_n = 2HDLe_{n-1} - HDLe_{n-3}, \quad n \geq 3 \tag{2.7}$$

Here, $A := 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$ [25].

Definition 2.9. [20] The Fibonacci and Lucas quaternions are defined as follows, respectively:

$$QF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$QL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

where i, j, and k are the quaternionic units in (2.2).

Definition 2.10. [23] The Leonardo quaternions are defined by

$$QLe_n = Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k \tag{2.8}$$

where i, j, and k are the quaternionic units in (2.2).

Binet's formula for QLe_n is

$$QLe_n = 2\frac{\alpha^{n+1}\hat{\alpha} - \beta^{n+1}\hat{\beta}}{\alpha - \beta} - q_u \tag{2.9}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\hat{\alpha} := 1 + \alpha i + (1+\alpha)j + (1+2\alpha)k$, $\hat{\beta} := 1 + \beta i + (1+\beta)j + (1+2\beta)k$, and $q_u := 1 + i + j + k$ [23]. Then, the following properties hold [23]:

$$QLe_n = 2QF_{n+1} - q_u \tag{2.10}$$

$$QLe_{n+1} - QLe_n = 2QF_n \tag{2.11}$$

$$QLe_{n+2} = QLe_{n+1} + QLe_n + q_u \tag{2.12}$$

$$\sum_{k=1}^{n} QLe_k = QLe_{n+2} - QLe_2 - nq_u \tag{2.13}$$

$$\sum_{k=1}^{n} QLe_{2k-1} = QLe_{2n} - QLe_0 - nq_u \tag{2.14}$$

and

$$\sum_{k=1}^{n} QLe_{2k} = QLe_{2n+1} - QLe_1 - nq_u \tag{2.15}$$

Here, QF_n is the n-th Fibonacci quaternion and QLe_n is the n-th Leonardo quaternion.

Ait-Amrane et al. [27] defined the hyper-dual Horadam quaternions from two perspectives. In the particular case of the hyper-dual Horadam quaternions, the hyper-dual Fibonacci and Lucas quaternions can be derived as follows:

Definition 2.11. [27] The hyper-dual Fibonacci and Lucas quaternions are defined by

$$QHDF_n = HDF_n + HDF_{n+1}i + HDF_{n+2}j + HDF_{n+3}k$$

and

$$QHDL_n = HDL_n + HDL_{n+1}i + HDL_{n+2}j + HDL_{n+3}k$$

respectively, where HDF_n is the n-th hyper-dual Fibonacci number, HDL_n is the n-th hyper-dual Lucas number, and i, j, and k are the quaternionic units in (2.2).

In addition, the hyper-dual Fibonacci and Lucas quaternions can be defined as:

Definition 2.12. [27] The hyper-dual Fibonacci and Lucas quaternions are defined by

$$QHDF_n = QF_n + QF_{n+1}\varepsilon_1 + QF_{n+2}\varepsilon_2 + QF_{n+3}\varepsilon_1\varepsilon_2$$

and

$$QHDL_n = QL_n + QL_{n+1}\varepsilon_1 + QL_{n+2}\varepsilon_2 + QL_{n+3}\varepsilon_1\varepsilon_2$$

respectively, where ε_1 and ε_2 are the dual units in (2.1).

3. Main Results

This section begins with defining the general term of the hyper-dual Leonardo quaternions.

Definition 3.1. For $n \geq 0$, the n-th hyper-dual Leonardo quaternion is

$$QHDLe_n = HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \tag{3.1}$$

where $HDLe_n$ is the n-th hyper-dual Leonardo number and i, j, and k are the quaternionic units in (2.2).

Moreover, considering (2.5) and (2.8), we can obtain

$$\begin{aligned}
QHDLe_n &= HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \\
&= (Le_n + Le_{n+1}\varepsilon_1 + Le_{n+2}\varepsilon_2 + Le_{n+3}\varepsilon_1\varepsilon_2) + (Le_{n+1} + Le_{n+2}\varepsilon_1 + Le_{n+3}\varepsilon_2 + Le_{n+4}\varepsilon_1\varepsilon_2)i \\
&\quad +(Le_{n+2} + Le_{n+3}\varepsilon_1 + Le_{n+4}\varepsilon_2 + Le_{n+5}\varepsilon_1\varepsilon_2)j + (Le_{n+3} + Le_{n+4}\varepsilon_1 + Le_{n+5}\varepsilon_2 + Le_{n+6}\varepsilon_1\varepsilon_2)k \\
&= (Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k) + (Le_{n+1} + Le_{n+2}i + Le_{n+3}j + Le_{n+4}k)\varepsilon_1 \\
&\quad +(Le_{n+2} + Le_{n+3}i + Le_{n+4}j + Le_{n+5}k)\varepsilon_2 + (Le_{n+3} + Le_{n+4}i + Le_{n+5}j + Le_{n+6}k)\varepsilon_1\varepsilon_2 \\
&= QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2
\end{aligned}$$

Therefore, the general term of the hyper-dual Leonardo quaternions can be reidentified in the following.

Definition 3.2. For $n \geq 0$, the n-th hyper-dual Leonardo quaternion is

$$QHDLe_n = QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \tag{3.2}$$

where QLe_n is the n-th Leonardo quaternion and ε_1 and ε_2 are the dual units in (2.1).

The first three hyper-dual Leonardo quaternions are as follows:

$$\begin{aligned}
QHDLe_0 &= (1 + i + 3j + 5k) + (1 + 3i + 5j + 9k)\varepsilon_1 + (3 + 5i + 9j + 15k)\varepsilon_2 \\
&\quad +(5 + 9i + 15j + 25k)\varepsilon_1\varepsilon_2
\end{aligned}$$

$$\begin{aligned}
QHDLe_1 &= (1 + 3i + 5j + 9k) + (3 + 5i + 9j + 15k)\varepsilon_1 + (5 + 9i + 15j + 25k)\varepsilon_2 \\
&\quad +(9 + 15i + 25j + 41k)\varepsilon_1\varepsilon_2
\end{aligned}$$

and

$$\begin{aligned}
QHDLe_2 &= (3 + 5i + 9j + 15k) + (5 + 9i + 15j + 25k)\varepsilon_1 + (9 + 15i + 25j + 41k)\varepsilon_2 \\
&\quad +(15 + 25i + 41j + 67k)\varepsilon_1\varepsilon_2
\end{aligned}$$

Throughout this paper, let $A := 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$, $q_u := 1 + i + j + k$, and $\Delta := Aq_u = q_uA$.

By (2.6) and (3.1), the following recurrence relation of the hyper-dual Leonardo quaternions is obtained:

$$QHDLe_n = QHDLe_{n-1} + QHDLe_{n-2} + \Delta, \quad n \geq 2 \tag{3.3}$$

Moreover, by (2.7) and (3.1), the alternative recurrence relation of the hyper-dual Leonardo quaternions is obtained:

$$QHDLe_n = 2QHDLe_{n-1} - QHDLe_{n-3}, \quad n \geq 3 \tag{3.4}$$

Theorem 3.3. For $n \geq 0$,

i. $QHDLe_n - QHDLe_{n+1}i - QHDLe_{n+2}j - QHDLe_{n+3}k = 3(HDLe_{n+4} + HDLe_{n+2}) + 2A$

ii. $QHDLe_n - QHDLe_{n+1}\varepsilon_1 - QHDLe_{n+2}\varepsilon_2 - QHDLe_{n+3}\varepsilon_1\varepsilon_2 = QLe_n - 2QLe_{n+3}\varepsilon_1\varepsilon_2$

Proof. Let $n \geq 0$.

i. Using (3.1) to the left-hand side (LHS),

$$\begin{aligned} LHS = &HDLe_n + HDLe_{n+1}i + HDLe_{n+2}j + HDLe_{n+3}k \\ &- (HDLe_{n+1} + HDLe_{n+2}i + HDLe_{n+3}j + HDLe_{n+4}k)i \\ &- (HDLe_{n+2} + HDLe_{n+3}i + HDLe_{n+4}j + HDLe_{n+5}k)j \\ &- (HDLe_{n+3} + HDLe_{n+4}i + HDLe_{n+5}j + HDLe_{n+6}k)k \end{aligned}$$

From the multiplication rules of the quaternionic units in (2.2),

$$LHS = HDLe_n + HDLe_{n+2} + HDLe_{n+4} + HDLe_{n+6}$$

Using (2.6),

$$LHS = 3HDLe_{n+4} + 3HDLe_{n+2} + 2A$$

ii. Using (3.2) to the left-hand side (LHS),

$$\begin{aligned} LHS = &QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \\ &- (QLe_{n+1} + QLe_{n+2}\varepsilon_1 + QLe_{n+3}\varepsilon_2 + QLe_{n+4}\varepsilon_1\varepsilon_2)\varepsilon_1 \\ &- (QLe_{n+2} + QLe_{n+3}\varepsilon_1 + QLe_{n+4}\varepsilon_2 + QLe_{n+5}\varepsilon_1\varepsilon_2)\varepsilon_2 \\ &- (QLe_{n+3} + QLe_{n+4}\varepsilon_1 + QLe_{n+5}\varepsilon_2 + QLe_{n+6}\varepsilon_1\varepsilon_2)\varepsilon_1\varepsilon_2 \end{aligned}$$

Considering the multiplication rules of the dual units in (2.1),

$$LHS = QLe_n - 2QLe_{n+3}\varepsilon_1\varepsilon_2$$

□

Lemma 3.4. For positive integer n, the followings hold:

i. $HDLe_{n-1} + HDLe_{n+1} = 2HDL_{n+1} - 2A$ [25]

ii. $HDLe_n + HDF_n + HDL_n = 2HDLe_n + A$

where $HDLe_n$, HDF_n, and HDL_n are the n-th hyper-dual Leonardo, hyper-dual Fibonacci, and hyper-dual Lucas numbers, respectively.

Proof. *ii.* From (2.3)-(2.5) and the relation $Le_n + F_n + L_n = 2Le_n + 1$ provided in [19], the proof is clear. □

Theorem 3.5. For $n \geq 0$, the followings hold:

i. $QHDLe_{n-1} + QHDLe_{n+1} = 2QHDL_{n+1} - 2\Delta$

ii. $QHDLe_n + QHDF_n + QHDL_n = 2QHDLe_n + \Delta$

iii. $QHDLe_n = 2QHDF_{n+1} - \Delta$

iv. $QHDLe_{n+1} - QHDLe_n = 2QHDF_n$

where $QHDF_n$ and $QHDL_n$ are the n-th hyper-dual Fibonacci and hyper-dual Lucas quaternions, respectively.

PROOF. From (2.10), (2.11), (3.1), and (3.2) and Lemma 3.4, the proofs of *i.*, *ii.*, *iii.*, and *iv.* are obvious. □

Theorem 3.6. For $n \geq 0$, Binet's formula of the hyper-dual Leonardo quaternions is

$$QHDLe_n = 2\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha - \beta}\right) - \Delta \tag{3.5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$,

$$\alpha^* := (1 + \alpha i + (1+\alpha)j + (1+2\alpha)k)(1 + \alpha\varepsilon_1 + (1+\alpha)\varepsilon_2 + (1+2\alpha)\varepsilon_1\varepsilon_2)$$

and

$$\beta^* := (1 + \beta i + (1+\beta)j + (1+2\beta)k)(1 + \beta\varepsilon_1 + (1+\beta)\varepsilon_2 + (1+2\beta)\varepsilon_1\varepsilon_2)$$

PROOF. From (2.9) and (3.2) and the equalities $1+\alpha = \alpha^2$, $1+2\alpha = \alpha^3$, $1+\beta = \beta^2$, and $1+2\beta = \beta^3$,

$$\begin{aligned}
QHDLe_n &= QLe_n + QLe_{n+1}\varepsilon_1 + QLe_{n+2}\varepsilon_2 + QLe_{n+3}\varepsilon_1\varepsilon_2 \\
&= \left(2\frac{\alpha^{n+1}\hat{\alpha} - \beta^{n+1}\hat{\beta}}{\alpha - \beta} - q_u\right) + \left(2\frac{\alpha^{n+2}\hat{\alpha} - \beta^{n+2}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_1 \\
&\quad + \left(2\frac{\alpha^{n+3}\hat{\alpha} - \beta^{n+3}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_2 + \left(2\frac{\alpha^{n+4}\hat{\alpha} - \beta^{n+4}\hat{\beta}}{\alpha - \beta} - q_u\right)\varepsilon_1\varepsilon_2 \\
&= 2\frac{\alpha^{n+1}\hat{\alpha}}{\alpha - \beta}(1 + \alpha\varepsilon_1 + \alpha^2\varepsilon_2 + \alpha^3\varepsilon_1\varepsilon_2) - 2\frac{\beta^{n+1}\hat{\beta}}{\alpha - \beta}(1 + \beta\varepsilon_1 + \beta^2\varepsilon_2 + \beta^3\varepsilon_1\varepsilon_2) \\
&\quad - q_u(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) \\
&= 2\frac{\alpha^*\alpha^{n+1}}{\alpha - \beta} - 2\frac{\beta^*\beta^{n+1}}{\alpha - \beta} - \Delta
\end{aligned}$$

□

Theorem 3.7. The ordinary generating function for the hyper-dual Leonardo quaternions is

$$g(x) = \frac{QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2}{1 - 2x + x^3}$$

PROOF. Let

$$g(x) = \sum_{n=0}^{\infty} QHDLe_n x^n$$

be the ordinary generating function for the hyper-dual Leonardo quaternions. Then, from (3.4),

$$\begin{aligned}
g(x) &= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + \sum_{n=3}^{\infty} QHDLe_n x^n \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + \sum_{n=3}^{\infty}(2QHDLe_{n-1} - QHDLe_{n-3})x^n \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 + 2x\sum_{n=3}^{\infty} QHDLe_{n-1}x^{n-1} - x^3\sum_{n=3}^{\infty} QHDLe_{n-3}x^{n-3} \\
&= QHDLe_0 + QHDLe_1 x + QHDLe_2 x^2 - 2x(QHDLe_0 + QHDLe_1 x) + 2x\sum_{n=0}^{\infty} QHDLe_n x^n \\
&\quad - x^3\sum_{n=0}^{\infty} QHDLe_n x^n \\
&= QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2 + 2xg(x) - x^3 g(x)
\end{aligned}$$

Hence,

$$g(x)(1-2x+x^3) = QHDLe_0 + (QHDLe_1 - 2QHDLe_0)x + (QHDLe_2 - 2QHDLe_1)x^2$$

□

Theorem 3.8. The exponential generating function for the hyper-dual Leonardo quaternions is

$$eg(x) = \sum_{n=0}^{\infty} QHDLe_n \frac{x^n}{n!} = 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{\alpha x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{\beta x} - \Delta e^x$$

where α^* and β^* are defined as in Theorem 3.6.

Proof. From (3.5), we obtain

$$\begin{aligned} eg(x) &= \sum_{n=0}^{\infty} QHDLe_n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \Delta\right)\frac{x^n}{n!} \\ &= 2\frac{\alpha^*\alpha}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} - 2\frac{\beta^*\beta}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} - \Delta\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{\alpha x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{\beta x} - \Delta e^x \end{aligned}$$

□

Corollary 3.9. The Poisson generating function for the hyper-dual Leonardo quaternions is

$$pg(x) = 2\frac{\alpha^*\alpha}{\alpha-\beta}e^{(\alpha-1)x} - 2\frac{\beta^*\beta}{\alpha-\beta}e^{(\beta-1)x} - \Delta$$

Proof. Since $pg(x) = eg(x)e^{-x}$, the proof is straghtforward. □

Theorem 3.10. For $n \geq 1$, the followings hold:

i. $\sum_{k=1}^{n} QHDLe_k = QHDLe_{n+2} - QHDLe_2 - n\Delta$

ii. $\sum_{k=1}^{n} QHDLe_{2k-1} = QHDLe_{2n} - QHDLe_0 - n\Delta$

iii. $\sum_{k=1}^{n} QHDLe_{2k} = QHDLe_{2n+1} - QHDLe_1 - n\Delta$

Proof. *i.* From (2.13) and (3.2),

$$\begin{aligned} \sum_{k=1}^{n} QHDLe_k &= \sum_{k=1}^{n}(QLe_k + QLe_{k+1}\varepsilon_1 + QLe_{k+2}\varepsilon_2 + QLe_{k+3}\varepsilon_1\varepsilon_2) \\ &= \left(\sum_{k=1}^{n} QLe_k\right) + \left(\sum_{k=1}^{n} QLe_{k+1}\right)\varepsilon_1 + \left(\sum_{k=1}^{n} QLe_{k+2}\right)\varepsilon_2 + \left(\sum_{k=1}^{n} QLe_{k+3}\right)\varepsilon_1\varepsilon_2 \\ &= (QLe_{n+2} - QLe_2 - nq_u) + (QLe_{n+2} + QLe_{n+1} - QLe_2 - QLe_1 - nq_u)\varepsilon_1 \\ &\quad + (2QLe_{n+2} + QLe_{n+1} - 2QLe_2 - QLe_1 - nq_u)\varepsilon_2 \\ &\quad + (QLe_{n+3} + 2QLe_{n+2} + QLe_{n+1} - QLe_3 - 2QLe_2 - QLe_1 - nq_u)\varepsilon_1\varepsilon_2 \end{aligned}$$

Then, considering (2.12),

$$\sum_{k=1}^{n} QHDLe_k = (QLe_{n+2} - QLe_2 - nq_u) + (QLe_{n+3} - QLe_3 - nq_u)\,\varepsilon_1 + (QLe_{n+4} - QLe_4 - nq_u)\,\varepsilon_2 + (QLe_{n+5} - QLe_5 - nq_u)\,\varepsilon_1\varepsilon_2$$

Then, it follows that

$$\begin{aligned}\sum_{k=1}^{n} QHDLe_k &= (QLe_{n+2} + QLe_{n+3}\varepsilon_1 + QLe_{n+4}\varepsilon_2 + QLe_{n+5}\varepsilon_1\varepsilon_2) \\ &\quad - (QLe_2 + QLe_3\varepsilon_1 + QLe_4\varepsilon_2 + QLe_5\varepsilon_1\varepsilon_2) - nq_u(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) \\ &= QHDLe_{n+2} - QHDLe_2 - n\Delta\end{aligned}$$

This completes the proof of *i.* In a similar manner, *ii.* and *iii.* can be proved by using (2.14) and (2.15). □

Theorem 3.11. For $n \geq 0$, the followings hold:

i. $QHDLe_{2n} = \sum\limits_{k=0}^{n} \binom{n}{k}(QHDLe_k + \Delta) - \Delta$

ii. $QHDLe_{2n+1} = \sum\limits_{k=0}^{n+1} \binom{n+1}{k}(QHDLe_{k-1} + \Delta) - \Delta$

Proof. *i.* From (3.5),

$$\begin{aligned}QHDLe_{2n} &= 2\left(\frac{\alpha^*\alpha^{2n+1} - \beta^*\beta^{2n+1}}{\alpha - \beta}\right) - \Delta \\ &= 2\left(\frac{\alpha^*\alpha(\alpha^2)^n - \beta^*\beta(\beta^2)^n}{\alpha - \beta}\right) - \Delta \\ &= 2\left(\frac{\alpha^*\alpha(1+\alpha)^n - \beta^*\beta(1+\beta)^n}{\alpha - \beta}\right) - \Delta\end{aligned}$$

Since $(1+\alpha)^n = \sum\limits_{k=0}^{n} \binom{n}{k}\alpha^k$ and $(1+\beta)^n = \sum\limits_{k=0}^{n} \binom{n}{k}\beta^k$, then

$$\begin{aligned}QHDLe_{2n} &= 2\left(\frac{\alpha^*\alpha}{\alpha - \beta}\sum_{k=0}^{n}\binom{n}{k}\alpha^k - \frac{\beta^*\beta}{\alpha - \beta}\sum_{k=0}^{n}\binom{n}{k}\beta^k\right) - \Delta \\ &= 2\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\alpha^*\alpha^{k+1} - \beta^*\beta^{k+1}}{\alpha - \beta}\right) - \Delta \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(2\frac{\alpha^*\alpha^{k+1} - \beta^*\beta^{k+1}}{\alpha - \beta} - \Delta\right) + \sum_{k=0}^{n}\binom{n}{k}\Delta - \Delta \\ &= \sum_{k=0}^{n}\binom{n}{k}(QHDLe_k + \Delta) - \Delta\end{aligned}$$

ii. The proof is similar to the proof of *i.* □

Theorem 3.12. (Vajda's Identity) For non-negative integers n, r, and s,

$$\begin{aligned}QHDLe_{n+r}QHDLe_{n+s} - QHDLe_nQHDLe_{n+r+s} &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^s - \alpha^*\beta^*\beta^s)F_r \\ &\quad + \Delta(QHDLe_n + QHDLe_{n+r+s}) \\ &\quad - \Delta(QHDLe_{n+r} + QHDLe_{n+s})\end{aligned}$$

where F_r is the r-th Fibonacci number.

Proof. Applying (3.5) to the left-hand side (LHS),

$$
\begin{aligned}
LHS &= \left(2\left(\frac{\alpha^*\alpha^{n+r+1}-\beta^*\beta^{n+r+1}}{\alpha-\beta}\right)-\Delta\right)\left(2\left(\frac{\alpha^*\alpha^{n+s+1}-\beta^*\beta^{n+s+1}}{\alpha-\beta}\right)-\Delta\right)\\
&\quad-\left(2\left(\frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}\right)-\Delta\right)\left(2\left(\frac{\alpha^*\alpha^{n+r+s+1}-\beta^*\beta^{n+r+s+1}}{\alpha-\beta}\right)-\Delta\right)\\
&= 4\left(\frac{(\alpha\beta)^{n+1}(\alpha^r-\beta^r)(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)}{(\alpha-\beta)^2}\right)\\
&\quad-\Delta\left(QHDLe_{n+r}+QHDLe_{n+s}-QHDLe_n-QHDLe_{n+r+s}\right)\\
&= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)F_r\\
&\quad+\Delta(QHDLe_n+QHDLe_{n+r+s}-QHDLe_{n+r}-QHDLe_{n+s})
\end{aligned}
$$

Here, $F_r=\frac{\alpha^r-\beta^r}{\alpha-\beta}$ [15]. □

In the particular case of Theorem 3.12, we have the following results:

Corollary 3.13. (Catalan's Identity) For non-negative integers n and s such that $n \geq s$,

$$
\begin{aligned}
QHDLe_{n-s}QHDLe_{n+s}-(QHDLe_n)^2 &= \frac{4}{\sqrt{5}}(-1)^{n+s}(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)F_s\\
&\quad+\Delta(2QHDLe_n-QHDLe_{n-s}-QHDLe_{n+s})
\end{aligned}
$$

Proof. Taking $r \to -s$ in Theorem 3.12 and considering the relation $F_{-r}=(-1)^{r+1}F_r$ [15], the proof is obvious. □

Corollary 3.14. (Cassini's Identity) For positive integer n,

$$
\begin{aligned}
QHDLe_{n-1}QHDLe_{n+1}-(QHDLe_n)^2 &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha-\alpha^*\beta^*\beta)\\
&\quad+\Delta(QHDLe_{n-2}-QHDLe_{n-1})
\end{aligned}
$$

Proof. Taking $r \to -s$ and $s=1$ in Theorem 3.12 and using (3.3), the proof is clear. □

Corollary 3.15. (d'Ocagne's Identity) For positive integers n and m,

$$
\begin{aligned}
QHDLe_{n+1}QHDLe_m-QHDLe_nQHDLe_{m+1} &= \frac{4}{\sqrt{5}}(-1)^{n+1}(\beta^*\alpha^*\alpha^{m-n}-\alpha^*\beta^*\beta^{m-n})\\
&\quad+\Delta(QHDLe_{m-1}-QHDLe_{n-1})
\end{aligned}
$$

Proof. Taking $s \to m-n$ and $r=1$ in Theorem 3.12 and using (3.3), the proof is clear. □

4. Conclusion

In this study, the hyper-dual Leonardo quaternions have been proposed from two different perspectives. At first, the hyper-dual quaternions have been defined using the hyper-dual Leonardo numbers as coefficients in quaternions. Then, as equivalent to this first definition, the hyper-dual Leonardo quaternions have been defined using the Leonardo quaternions as coefficients in hyper-dual numbers. Some of their properties, such as non-homogeneous and homogeneous recurrence relations, Binet's formula, certain sum formulae, and binomial-sum formulae, have been provided. The ordinary, exponential, and Poisson-generating functions, Vajda's identity, and, in particular cases, Catalan's, Cassini's, and d'Ocagne's identities of the hyper-dual Leonardo quaternions have been presented. For future studies, researchers may define hyper-dual split quaternions provided in [10] with the Leonardo number coefficients.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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