Journal of
New Theory
ISSN: 2149-1402

48 (2024) 90-98
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Chen-Type Inequality for Generic Submanifolds of Quaternionic Space Form and Its Application

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Article Info
Received: 03 Aug 2024
Accepted: 24 Jan 2024
Published: 30 Jan 2024
doi:10.53570/jnt.1527579
Research Article

Abstract — In 1993, the theory of Chen invariants started when Chen wrote basic inequalities for submanifolds in space forms. This inequality is called Chen's first inequality. Afterward, many geometers studied many papers dealing with this new inequality. The present paper aims to establish a Chen inequality for quaternionic generic submanifolds in a quaternionic space form and obtain this inequality for real hypersurfaces.

Keywords *Chen-type inequality, generic submanifold, quaternionic space form, real hypersurfaces*

Mathematics Subject Classification (2020) 53B20, 53C40

1. Introduction

One of the most interesting topics in differential geometry is the submanifolds of the almost Hermitian manifold. We note that the Kaehler manifold's submanifolds are determined by its tangent space behavior under the action of a complex structure J. One of the classes of submanifolds of Kaehler manifolds is holomorphic submanifolds and the other is total real submanifolds. In the first case, the tangent bundle of the submanifold is invariant under J where as in the second case, the normal bundle of the submanifold is invariant under J. CR-submanifolds were introduced by Bejancu in [1] as a natural generalization of invariant submanifolds and anti-invariant submanifolds. Chen investigated the first detailed research on this subject in [2]. Moreover, the topology of CR-submanifolds was widely studied [3–7]. The authors defined the quaternion CR-submanifolds in quaternion Kaehler manifolds [8], and were followed by several geometers [9–18]. Generic submanifold was defined as a generalization of the concept of CR-submanifold [19]. These submanifolds are known by relaxing the condition on the complementary distribution of holomorphic distribution. More precisely, if the maximal complex subspaces $D_p = T_pM \cap J(T_pM)$ determine on M a distribution $D : D_p \subseteq T_pM$, the M is called a generic submanifold of $\bar{M}$. Generic submanifolds have been commonly studied [20–26].

The present article is organized as follows: Section 2 recalls basic notions and results of quaternion Kaehler manifolds. New optimal inequalities were introduced in [27] for anti-holomorphic submanifolds in complex space forms. In recent years, this new inequality has been obtained by distinct researchers for different classes of submanifolds in different ambient manifolds [28, 29]. Section 3 establishes new inequality for quaternionic generic submanifolds in a quaternionic space form and gives some results. The last part of this paper obtains this inequality for real hypersurfaces.

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2. Preliminaries

In this section, some basic concepts will be given from [1,30] for the following sections. A Riemannian manifold $(\bar{M}, \tilde{g})$ of dimension $4m$, for $m \geq 1$, is called quaternion Kaehler manifold with 3-dimensional vector bundle σ of local basis of almost Hermitian structures J_1, J_2, and J_3 if the following conditions are satisfied

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3$$

and

$$\bar{\nabla}_X J_m = \sum_{b=1}^{3} A_{ml}(X) J_l, \quad m \in \{1,2,3\}, \quad \forall X \in \left(T\bar{M}\right)$$

where A_{ml} are certain local 1-forms on $\bar{M}$ such that $A_{ml} + A_{lm} = 0$. For a Riemann submanifold $N \subset \bar{M}$ of a Riemannian manifold $\bar{M}$, Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_W Z = \nabla_W Z + h(W, Z)$$

and

$$\bar{\nabla}_W V = -A_V W + \nabla^{\perp}_W V \tag{2.1}$$

for all $W, Z \in TN$ and $V \in TN^{\perp}$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of N and $\bar{M}$, respectively. Moreover, h and $\nabla^{\perp}_X$ denote the second fundamental form of N, and the normal connection on the normal bundle, respectively. From (2.1), A_ξ the second fundamental tensor and h the second fundamental form are related by

$$\tilde{g}(h(W, Z), \xi) = \tilde{g}(A_\xi W, Z)$$

If quaternionic sectional curvature of a quaternionic Kaehler manifold is constant, then it is called a quaternionic space form and denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(\mathcal{P}, \mathcal{Q})\mathcal{R} = \frac{c}{4}\left\{\tilde{g}(\mathcal{Q}, \mathcal{R})\mathcal{P} - \tilde{g}(\mathcal{P}, \mathcal{R})\mathcal{Q} + \sum_{a=1}^{3} \tilde{g}(\mathcal{R}, J_a\mathcal{Q})J_a\mathcal{P} - \tilde{g}(\mathcal{R}, J_a\mathcal{P})J_a\mathcal{Q} + 2\tilde{g}(\mathcal{P}, J_a\mathcal{Q})J_a\mathcal{R}\right\}$$

for all $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Gamma(TN)$ [1]. For the second fundamental form h, the covariant derivation $(\nabla_{\mathcal{P}} h)(\mathcal{Q}, \mathcal{R})$ is as follows:

$$(\nabla_{\mathcal{P}} h)(\mathcal{Q}, \mathcal{R}) = \nabla^{\perp}_{\mathcal{P}} h(\mathcal{Q}, \mathcal{R}) - h(\nabla_{\mathcal{P}}\mathcal{Q}, \mathcal{R}) - h(\mathcal{Q}, \nabla_{\mathcal{P}}\mathcal{R})$$

for any $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Gamma(TN)$. For the submanifold N, the Gauss, Codazzi, and Ricci equations of N are provided as follows, respectively:

$$R(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W}) = \bar{R}(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W}) + \tilde{g}(h(\mathcal{P}, \mathcal{W}), h(\mathcal{Q}, \mathcal{R})) - \tilde{g}(h(\mathcal{P}, \mathcal{R}), h(\mathcal{Q}, \mathcal{W})) \tag{2.2}$$

$$\left(\bar{R}(\mathcal{P}, \mathcal{Q})\mathcal{R}\right)^{\perp} = (\nabla_{\mathcal{P}} h)(\mathcal{Q})\mathcal{R}) - (\nabla_{\mathcal{Q}} h)(\mathcal{P}, \mathcal{R})$$

and

$$\bar{R}(\mathcal{P}, \mathcal{Q}, \xi, \eta) = R^{\perp}(\mathcal{P}, \mathcal{Q}, \xi, \eta) + \tilde{g}([A_\xi, A_\eta]\mathcal{P}, \mathcal{Q})$$

for all $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{W} \in \Gamma(TN)$ and $\xi, \eta \in \Gamma(TN)^{\perp}$. The H mean curvature vector of a submanifold N is as follows:

$$H = \left(\frac{1}{p}\right) \operatorname{trace} h, \quad p = \dim N$$

Definition 2.1. A submanifold N of a quaternion Kaehler manifold $\bar{M}$ is called a generic submanifold of $\bar{M}$ if there are two subspace D differentiable distribution and $D^{\perp}$ purely real distribution with constant ranks on N such that $D = J_a TN \cap TN$ and $D^{\perp}$ are complementary orthogonal to D.

Thus, from the definition, it is expressed as follows:

$$TN = D \oplus D^{\perp} \quad \text{and} \quad J_a(D) = D$$

For $X \in \Gamma(D^{\perp})$, $a \in \{1,2,3\}$,

$$J_a X = T_a X + F_a X$$

where $T_a X \in \Gamma\left(D^{\perp}\right)$ and $F_a X \in \Gamma\left(\vartheta^{\perp}\right)$. Moreover, $\vartheta^{\perp}$ and ϑ are complementary orthogonal to each other. Thus, from the definition, it is expressed as follows:

$$TN^{\perp} = \vartheta \oplus \vartheta^{\perp} \quad \text{and} \quad J_a(\vartheta) = \vartheta$$

For $V \in \Gamma\left(\vartheta^{\perp}\right)$,

$$J_a V = t_a V + f_a V$$

where $t_a V \in \Gamma(TN)$ and $f_a V \in \Gamma\left(\vartheta^{\perp}\right)$ [25].

3. Chen-Type Inequality for Generic Submanifolds of Quaternionic Space Form

Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$ with the differentiable distribution D and the purely real distribution $D^{\perp}$ of M. Consider orthonormal frame $\{e_1, e_2, \cdots, e_{2q+p}\}$ on M in such that $\{e_1, e_2, \cdots, e_{2q}\}$ are in D and $\{e_{2q+1}, e_{2q+2}, \cdots, e_{2q+p}\}$ are in $D^{\perp}$.

Chen [31] investigated new types of Riemannian invariants for submanifolds in space forms, now known as the Chen invariants. Chen [32] defined for CR-submanifold N in a Kaehler manifold $\bar{M}$ with τ the scalar curvature of N, and $\tau(D)$ the scalar curvature of the holomorphic distribution D of N as follows:

$$\delta(D)(p) = \tau(p) - \tau(D_p), \quad p \in N$$

Let $\vec{H}_D$ and $\vec{H}_{D^{\perp}}$ be the two partial mean curvature vectors of M, respectively, i.e.,

$$\vec{H}_D = \frac{1}{2q}\sum_{i=1}^{2q} h(e_i, e_i) \quad \text{and} \quad \vec{H}_{D^{\perp}} = \frac{1}{p}\sum_{r=2q+1}^{2q+p} h(e_r, e_r) \tag{3.1}$$

Theorem 3.1. Let M be a quaternionic generic submanifold of quaternionic space form $\bar{M}$ with minimal codimension, i.e., $\dim\vartheta_x = 0$, for $x \in M$, $\dim D_x = 2q$, $\dim D_x^{\perp} = p$, and $\dim TM^{\perp} = 2m - (2q+p)$, then

$$\delta(D) \leq 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^{\perp}}|^2$$

Proof. Let $\{e_1, e_2, \cdots, e_{2q}, e_{2q+1}, \cdots, e_{2q+p}\}$ be orthonormal bases on TM such that $\{e_1, e_2, \cdots, e_{2q}\}$ are in D and $\{e_{2q+1}, e_{2q+2}, \cdots, e_{2q+p}\}$ are in $D^{\perp}$ and let $\{e_{2q+p+1}, e_{2q+p+2}, \cdots, e_{2q+p+2m}\}$ of $TM^{\perp}$. Since

$$\tau = \sum_{1\leq i<j\leq 2q} K(e_i \wedge e_j) + \sum_{2q+1\leq r<s\leq 2q+p} K(e_r \wedge e_s) + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p} K(e_i \wedge e_r)$$

and

$$\tau(D) = \sum_{1\leq i<j\leq 2q} K(e_i \wedge e_j)$$

then

$$\delta(D) = \tau - \tau(D) = \sum_{2q+1\leq r<s\leq 2q+p} K(e_r \wedge e_s) + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p} K(e_i \wedge e_r)$$

From (2.2),

$$\begin{aligned} K(\mathcal{P}\wedge\mathcal{Q}) &= \bar{K}(\mathcal{P}\wedge\mathcal{Q})+\tilde{g}(h(\mathcal{P},\mathcal{P}),h(\mathcal{Q},\mathcal{Q}))-\tilde{g}(h(\mathcal{P},\mathcal{Q}),h(\mathcal{P},\mathcal{Q})) \\ &= \frac{c}{4}\left[1+3\textstyle\sum_{a=1}^{3}\tilde{g}(J_a\mathcal{P},\mathcal{Q})^2\right]+\tilde{g}(h(\mathcal{P},\mathcal{P}),h(\mathcal{Q},\mathcal{Q}))-\tilde{g}(h(\mathcal{P},\mathcal{Q}),h(\mathcal{P},\mathcal{Q})) \end{aligned} \tag{3.2}$$

From (3.2), for $\mathcal{P}=e_i$, $\mathcal{Q}=e_r$, $i\in\{1,2,\cdots,2q\}$, and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$K(e_i\wedge e_r)=\frac{c}{4}\left[1+3\sum_{a=1}^{3}\tilde{g}(J_ae_i,e_r)^2\right]+\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\tilde{g}(h(e_i,e_r),h(e_i,e_r)) \tag{3.3}$$

Since $J_ae_i\in D$ and $e_r\in D^{\perp}$,

$$\tilde{g}(J_ae_i,e_r)=0$$

By summation in (3.3) over $i\in\{1,2,\cdots,2q\}$ and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}K(e_i\wedge e_r)=2qp\frac{c}{4}+\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\left[\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\tilde{g}(h(e_i,e_r),h(e_i,e_r))\right]$$

From (3.2), for $\mathcal{P}=e_r$, $\mathcal{Q}=e_s$, and $r,s\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$K(e_r\wedge e_s)=\frac{c}{4}\left[1+3\sum_{a=1}^{3}\tilde{g}(J_ae_r,e_s)^2\right]+\tilde{g}(h(e_r,e_r),h(e_s,e_s))-\tilde{g}(h(e_r,e_s),h(e_r,e_s))$$

Since $T_ae_r\in D^{\perp}$ and $e_s\in D^{\perp}$,

$$\tilde{g}(J_ae_r,e_s)\neq 0$$

By summation in (3.3) over $i\in\{1,2,\cdots,2q\}$ and $r\in\{2q+1,2q+2,\cdots,2q+p\}$,

$$\begin{aligned} \sum_{2q+1\le r<s\le 2q+p}K(e_r\wedge e_s) &= \frac{p(p-1)}{2}\frac{c}{4}+\frac{9c}{4}\|T_a\|^2 \\ &\quad+\sum_{2q+1\le r<s\le 2q+p}\left[\tilde{g}(h(e_r,e_r),h(e_s,e_s))-\tilde{g}(h(e_r,e_s),h(e_r,e_s))\right] \end{aligned}$$

where

$$\|T_a\|^2=\sum_{i,j=2q+1}^{2q+p}\tilde{g}(T_ae_r,e_s)^2$$

and

$$\begin{aligned} \delta(D) &= 2qp\frac{c}{4}+\frac{p(p-1)}{2}\frac{c}{4}+\frac{9c}{4}\|T_a\|^2+\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\tilde{g}(h(e_i,e_i),h(e_r,e_r)) \\ &\quad+\sum_{2q+1\le r<s\le 2q+p}\tilde{g}(h(e_r,e_r),h(e_s,e_s))-\sum_{2q+1\le r<s\le 2q+p}\|h(e_r,e_s)\|^2 \\ &\quad-\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2 \end{aligned} \tag{3.4}$$

Moreover,

$$\frac{(2q+p)^2}{2}H^2+p^2|H_{D^{\perp}}|^2-\|h_{D^{\perp}}\|^2 \tag{3.5}$$

where $\|h_{D^{\perp}}\|^2$ is defined by

$$\|h_{D^{\perp}}\|^2=\sum_{2q+1\le r<s\le 2q+p}\|h(e_r,e_s)\|^2 \tag{3.6}$$

Combining (3.4) and (3.5),

$$\delta(D)=2qp\frac{c}{4}+\frac{p(p-1)}{2}\frac{c}{4}+\frac{9c}{4}\|T_a\|^2+\frac{(2q+p)^2}{2}H^2+p^2|H_{D^{\perp}}|^2-\|h_{D^{\perp}}\|^2-\sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2$$

and

$$p^2|H_{D^\perp}|^2 + \frac{(p+2)}{p-1}\left[\frac{(2q+p)^2}{2}H^2 - \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2 - \delta(D)\right] + \frac{c}{4}\left(2qp + \frac{p(p+2)}{2} + 9\|T_a\|^2\right) = \frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2$$

From (3.1) and (3.6),

$$\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 = \frac{p+2}{p-1}\left[\sum_{s=2q+1}^{2q+p}(h_{ss}^r) + \sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right] - \frac{3}{p-1}\left(\sum_{s=2q+1}^{2q+p}h_{ss}^r\right)^2$$

Moreover, since

$$0 \leq \sum_{i\leq j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n}a_i^2 - 2\sum_{i\leq j}a_ia_j \tag{3.7}$$

then

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n}a_i\right)^2 = \sum_{i=1}^{n}a_i^2 + 2\sum_{i\leq j}a_ia_j \tag{3.8}$$

Thus, from (3.7) and (3.8),

$$\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 = \frac{1}{p-1}\left\{(p+2)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right] - 3\left(\sum_{s=2q+1}^{2q+p}h_{ss}^r\right)^2\right\}$$

$$\begin{aligned}\frac{p+2}{p-1}\|h_{D^\perp}\|^2 - \frac{3p^2}{p-1}|H_{D^\perp}|^2 &= \frac{1}{p-1}\left\{(p-1)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 - 6\sum_{s\leq t}h_{ss}^rh_{tt}^r + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right]\right\}\\ &= \frac{1}{p-1}\left\{2(1-p)\sum_{s=2q+1}^{2q+p}(h_{ss}^r)^2 + 3\sum_{s\leq t}(h_{ss}^r - h_{tt}^r) + (p+2)\left[\sum_{s\neq t}(h_{st}^r)^2 + \sum_{k=2q+p+1}^{2q+p+2m}\sum_{s,t=2q+1}^{2q+p}(h_{st}^k)^2\right]\right\}\\ &\geq 0\end{aligned}$$

Thus,

$$\|h_{D^\perp}\|^2 \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2$$

and

$$2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \delta(D) \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2 + \sum_{i=1}^{2q}\sum_{r=2q+1}^{2q+p}\|h(e_i,e_r)\|^2$$

then

$$2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + p^2|H_{D^\perp}|^2 - \delta(D) \geq \frac{3p^2}{p+2}|H_{D^\perp}|^2$$

Hence,

$$\delta(D) \leq 2qp\frac{c}{4} + \frac{p(p-1)}{2}\frac{c}{4} + \frac{9c}{4}\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

□

Corollary 3.2. If M^n is a quaternionic generic submanifold of the quaternionic Euclidean m space H^m with $c = 0$, then

$$\delta(D) \leq \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

Corollary 3.3. If M^n is a quaternionic generic submanifold of the quaternionic projective m space $HP^m(4c)$ with $c > 0$, then

$$\delta(D) \leq 2qp + \frac{p(p-1)}{2} + 9\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

Corollary 3.4. If M^n is a quaternionic generic submanifold of the quaternionic hyperbolic m space $HH^m(4c)$ with $c < 0$, then

$$\delta(D) \geq 2qp + \frac{p(p-1)}{2} + 9\|T_a\|^2 + \frac{(2q+p)^2}{2}H^2 + \frac{(p-1)p^2}{p+2}|H_{D^\perp}|^2$$

4. An Inequality for Real Hypersurfaces

Indeed, generic submanifolds with $\vartheta = \{0\}$ and $D^\perp = Sp\{J_aN\}$ are real hypersurface of a quaternion Kaehler manifold, where N is the unit normal vector field of the hypersurface. Therefore, Theorem 3.1 lead to the following

Theorem 4.1. Let M be a real hypersurface of quaternionic space form $\bar{M}(4c)$. Then,

$$\delta(D) \leq \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 + (6q+3)c$$

Proof. Let M be a real hypersurface of quaternionic space form $\bar{M}(4c)$. Then, it follows from the definition $\delta(D)$ that

$$\delta(D) = \tau - \tau(D) = \sum_{1\leq a<b\leq 3} K(J_aN \wedge J_bN) + \sum_{i=1}^{2q}\sum_{a=1}^{3} K(e_i \wedge J_aN)$$

For $i \in \{1, 2, \cdots, 2q\}$ and $a \in \{1, 2, 3\}$,

$$\begin{aligned}\delta(D) = {} & 6qc + 3c + \sum_{i=1}^{2q}\sum_{a=1}^{3} \tilde{g}(h(e_i, e_i), h(J_aN, J_aN)) \\ & + \sum_{1\leq a<b\leq 3} \tilde{g}(h(J_aN, J_aN), h(J_bN, J_bN)) - \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2 \\ & - \sum_{i=1}^{2q}\sum_{a=1}^{3} \|h(e_i, J_aN)\|^2\end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned}&\sum_{i=1}^{2q}\sum_{a=1}^{3} \tilde{g}(h(e_i, e_i), h(J_aN, J_aN)) + \sum_{1\leq a<b\leq 3} \tilde{g}(h(J_aN, J_aN), h(J_bN, J_bN)) \\ &\quad - \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2 = \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2\end{aligned} \tag{4.2}$$

where $\|h_{D^\perp}\|^2$ is defined by

$$\|h_{D^\perp}\|^2 = \sum_{1\leq a<b\leq 3} \|h(J_aN, J_bN)\|^2$$

Combining (4.1) and (4.2),

$$\begin{aligned}\delta(D) &= 6qc + 3c + \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 - \|h_{D^\perp}\|^2 - \sum_{i=1}^{2q}\sum_{a=1}^{3}\|h(e_i, J_a N)\|^2 \\ &\leq \frac{(2q+3)^2}{2}H^2 + 9|H_{D^\perp}|^2 + (6q+3)c\end{aligned}$$

□

5. Conclusion

This study investigates an inequality for an intrinsic invariant of Chen-type defined on quaternionic generic submanifolds in a quaternionic space form. Its application obtains this inequality for real hypersurfaces. Although results for certain submanifolds have been obtained in previous studies, their generalized state has not been made. Thus, this study will provide new fields for researchers studying generic submanifolds by working in different space forms such as generalized complex space forms, Sasakian space forms, cosymplectic space forms, and locally conformal Kähler space forms.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] A. Bejancu, Geometry of CR-submanifolds, Kluwer, Dortrecht, 1986.

[2] B. Y. Chen, *On CR-submanifolds of Kaehler manifold I*, Journal of Differential Geometry 16 (2) (1981) 305–322.

[3] S. Dragomir, M. H. Shahid, F. Al-Solamy, S. M. Hasan, Geometry of Cauchy–Riemann submanifolds, Springer, Singapore, 2016.

[4] B. Y. Chen, *Cohomology of CR-submanifolds*, Annales de la Faculté des Sciences de Toulouse Mathématiques 3 (2) (1981) 167–172.

[5] B. Şahin, F. Şahin, *Homology of contact CR–warped product submanifolds of an odd–dimensional unit sphere*, Bulletin of the Korean Mathematical Society 52 (1) (2015) 215–222.

[6] F. Şahin, *On the topology of CR–warped product submanifolds*, International Journal of Geometric Methods in Modern Physics 15 (02) (2018) 1850032 7 pages.

[7] F. Şahin, *Homology of submanifolds of six dimensional sphere*, Journal of Geometry and Physics 145 (2019) 103471 6 pages.

[8] M. Barros, F. Urbano, *Quaternion CR–submanifolds of quaternion manifolds*, Kodai Mathematical Journal 4 (3) (1981) 399–417.

[9] E. Abedi, G. Haghighatdoost, M. Ilmakchi, Z. Nazari, *Contact 3–structure QR–warped product submanifold in Sasakian space form*, Turkish Journal of Mathematics 37 (2) (2013) 340–347.

[10] A. Bejancu, H. R. Farran, *On totally umbilical QR–submanifolds of quaternion Kaehlerian manifolds*, Bulletin of the Australian Mathematical Society 62 (1) (2000) 95–103.

[11] R. Güneş, B. Şahin, S. Keleş, *QR–submanifolds and almost contact 3–Structure*, Turkish Journal of Mathematics 24 (3) (2000) 239–250.

[12] H. S. Kim, J. S. Pak, *Certain QR–submanifolds of maximal QR–dimension in a quaternionic space form*, Balkan Journal of Geometry and its Applications 18 (1) (2013) 31–46.

[13] G. Macsim, A. Mihai, *A δ–invariant for QR–submanifolds in quaternion space forms*, International Electronic Journal of Geometry 11 (2) (2018) 8–17.

[14] G. Macsim, A. Mihai, I. Mihai, *$\delta(2,2)$–invariant for Lagrangian submanifolds in quaternionic space forms*, Mathematics 8 (4) (2020) 480.

[15] Z. Nazari, E. Abedi, *Ricci solitons on QR–hypersurfaces of a quaternionic space form Qn*, Acta Universitatis Sapientiae, Mathematica 8 (2) (2016) 271–281.

[16] B. Şahin, R. Güneş, *On some properties of distributions in QR–submanifolds*, Hacettepe Journal of Mathematics and Statistics 28 (1999) 15–23.

[17] B. Şahin, *On QR–submanifolds of quaternionic space form*, Turkish Journal of Mathematics 25 (3) (2001) 413–425.

[18] B. Şahin, R. Güneş, *QR–submanifolds of a locally conformal quaternion Kaehler manifold*, Publicationes Mathematicae Debrecen 63 (2003) 157–174.

[19] B. Y. Chen, *Differential geometry of real submanifolds in a Kaehler manifold*, Monatshefte für Mathematik 91 (1981) 257–274.

[20] Y. W. Choe, U. H. Ki, R. Takagi, *Compact minimal generic submanifolds with parallel normal section in a complex projective space*, Osaka Journal of Mathematics 37 (2) (2000) 489–499.

[21] U. C. De, A. K. Sengupta, C. Calin, *Generic submanifolds of quasi–Sasakian manifolds*, Demonstratio Mathematica 37 (2) (2004) 429–437.

[22] U. C. De, A. K. Sengupta, *Generic submanifolds of a Lorentzian para–Sasakian manifold*, Soochow Journal of Mathematics 27 (1) (2001) 29–36.

[23] M. Kon, *On minimal generic submanifolds immersed in S^{2m+1}*, Colloquium Mathematicum 90 (2) (2001) 299–304.

[24] G. E. Vilcu, *On generic submanifolds of manifolds endowed with metric mixed 3–structures*, Communications in Contemporary Mathematics 18 (6) (2016) 1550081 21 pages.

[25] A. Yılmaz, D. Poyraz, B. Şahin, *Generic QR–submanifolds of quaternion Kaehler manifolds*, Acta Mathematica Universitatis Comenianae 90 (3) (2021) 341–351.

[26] A. Yılmaz, *Generic submanifolds of quaternion Kaehler manifolds and Wintgen type inequality*, Filomat 38 (12) (2024) 4167–4185.

[27] F. Al–Solamy, B. Y. Chen, S. Deshmukh, *Two optimal inequalities for anti–holomorphic submanifolds and their applications*, Taiwanese Journal of Mathematics 18 (1) (2014) 199–217.

[28] M. D. Siddiqi, A. N. Siddiqui, O. Bahadir, *Generalized Wintgen inequalities for submanifolds of trans–Sasakian space form*, Matimyás Matematika 44 (1) (2021) 1–14.

[29] A. N. Siddiqui, Y. J. Suh, O. Bahadır, *Extremities for statistical submanifolds in Kenmotsu statistical manifolds*, Filomat 35 (2) (2021) 591–603.

[30] A. Bejancu, *QR–submanifolds of quaternion Kaehlerian manifolds*, Chinese Journal of Mathematics 14 (2) (1986) 81–94.

[31] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Archiv der Mathematik 60 (1993) 568–578.

[32] B. Y. Chen, *An optimal inequality for CR–warped products in complex space forms involving CR δ–invariant*, International Journal of Mathematics 23 (3) (2012) 17 pages.