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Research Article

New Integrated and Differentiated Sequence Spaces $\int b_p^{r,s}$ and $db_p^{r,s}$

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ABSTRACT

In this work, we construct new sequence spaces by combining the integrated and differentiated sequence spaces with the binomial matrix. Firstly, we provide information about basic matters such as sequence spaces and matrix domain. Subsequently we briefly summarize some sequence spaces generated by the binomial matrix. Thereafter, we define the integrated and differentiated sequence spaces and establish the new sequence spaces. Afterwards, we examine some properties and the inclusion relations of these new sequence spaces. We also determine the α, β and γ −duals of the integrated and differentiated sequence spaces. Finally, we characterize some matrix classes associated with the new sequence spaces.

1. Introduction

Let $w = \{x = (x_k) : x \in \mathbb{R}\ (or\ \mathbb{C}), \forall k \in \mathbb{N}\}$ be a set. Under the pointwise addition and scalar multiplication w is a vector space. Each subspace of w is called a sequence space. The sequence space ℓ_p, which is absolutely p-summable sequences, is a frequently used sequence space.

A Banach sequence space is classified as a BK-space if the maps $p_n: X \to \mathbb{C}$, defined as $p_n(x) = x_n$ are continuous for all $n \in \mathbb{N}$ [1]. Therefore, we can say that the sequence space ℓ_p, with their-norm defined as

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \tag{1}$$

is a BK-space, for $1 \leq p < \infty$.

Let, $A = (a_{nk})$ be an infinite matrix of real (or complex) entries. The A-transform of the sequence x is denoted as

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \tag{2}$$

where the series $(Ax)_n$ is required to be convergent for every $n \in \mathbb{N}$.

Moreover, let X and Y be two sequence spaces and consider the set defined as $X_A = \{x = (x_k) \in w: Ax \in X\}$ for a given infinite matrix A. This set is referred to as the matrix domain of A on the sequence space X. Additionally, the class of all matrix transformations from X into Y is denoted by $(X: Y)$ and it is given by [2],

$$(X: Y) = \{A = (a_{nk}): Ax \in Y \text{ for all } x \in X\}. \tag{3}$$

Let us consider the summation matrix $S = (s_{nk})$ defined as

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \tag{4}$$

where $\forall n, k \in \mathbb{N}$.

The matrix domain of S is used to define the sets $bs = (\ell_\infty)_S$ and $cs = c_S$, which denote the sets of all bounded and convergent series, respectively.

If the entries of an infinite matrix $A=(a_{nk})$ satisfies the conditions $a_{nn}\neq 0$ for all $n,k\in\mathbb{N}$ and $a_{nk}=0$ for $k>n$, then this matrix is called a triangular matrix. A triangular matrix has an inverse which is also a triangular matrix.

The integrated and differentiated sequence spaces were initially introduced by Goes and Goes [3]. Recently, Kirişçi has extensively studied these sequence spaces from various perspectives [4-6].

Additionally, Binomial sequence spaces were defined by Bişgin using the matrix domain of the Binomial matrix [7, 8]. Subsequently, various sequence spaces were constructed by several authors using the matrix domain of the Binomial matrix [9, 10].

2. New Sequence Spaces

In this section, we first provide a brief overview of some previous studies. Next, we introduce new sequence spaces obtained by combining the integrated and differentiated sequence spaces with the binomial matrix. Then, we explore their respective properties.

The Binomial matrix $B^{rs}=(b_{nk}^{rs})$ is defined as follows;

$$b_{nk}^{rs}=\begin{cases}\dfrac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k, & 0\le k\le n\\ 0 \quad, & k>n\end{cases} \tag{5}$$

for all $n,k\in\mathbb{N}$, $r,s\in\mathbb{R}$ and $s\cdot r>0$. (Throughout the article, we assume $s\cdot r>0$ unless otherwise stated.)

The binomial sequence spaces were first defined by Bişgin in [7, 8] as follows;

$$b_0^{r,s}=\left\{\begin{matrix}x=(x_k)\in w:\\ \lim\limits_{n\to\infty}\dfrac{1}{(s+r)^n}\sum\limits_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k=0\end{matrix}\right\}, \tag{6}$$

$$b_c^{r,s}=\left\{\begin{matrix}x=(x_k)\in w:\\ \lim\limits_{n\to\infty}\dfrac{1}{(s+r)^n}\sum\limits_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k \text{ exists}\end{matrix}\right\}, \tag{7}$$

$$b_\infty^{r,s}=\left\{\begin{matrix}x=(x_k)\in w:\\ \sup\limits_{n\in\mathbb{N}}\left|\dfrac{1}{(s+r)^n}.\sum\limits_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k\right|<\infty\end{matrix}\right\} \tag{8}$$

and

$$b_p^{r,s}=\left\{\begin{matrix}x=(x_k)\in w:\\ \sum\limits_{n}\left|\dfrac{1}{(s+r)^n}.\sum\limits_{k=0}^{n}\binom{n}{k}s^{n-k}r^kx_k\right|^p<\infty\end{matrix}\right\}, \tag{9}$$

where $1\le p<\infty$. Throughout the article, unless otherwise specified, we assume $1\le p<\infty$.

Subsequently, the sequence space $b_p^{r,s}(\mathrm{G})$, obtained from the composition of the binomial matrix with the double band matrix defined by Bişgin in [9] as follows;

$$b_p^{r,s}(G)=\left\{\begin{matrix}x=(x_k)\in w:\\ \sum\limits_{n}\left|\dfrac{1}{(s+r)^n}\sum\limits_{k=0}^{n}\binom{n}{k}s^{n-k}r^k(ux_k+vx_{k-1})\right|^p<\infty\end{matrix}\right\}, \tag{10}$$

where double band matrix $G=(g_{nk})$ is defined by

$$g_{nk}=\begin{cases}u, & k=n\\ v, & k=n-1\\ 0, & \text{otherwise}\end{cases} \tag{11}$$

for all $n,k\in\mathbb{N}$ and $u,v\in\mathbb{R}\setminus\{0\}$.

Then, the sequence space $b_p^{r,s}(\mathrm{D})$, obtained from the combination of the binomial and triple band matrix, defined by Sönmez in [10] as follows;

$$b_p^{r,s}(D) = \left\{ x = (x_k) \in w: \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (t x_k + u x_{k-1} + v x_{k-2}) \right|^p < \infty \right\}, \quad (12)$$

where triple band matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} t, & k = n \\ u, & k = n-1 \\ v, & k = n-2 \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

for all $n, k \in \mathbb{N}$ and $t, u, v \in \mathbb{R} \setminus \{0\}$.

Lastly, the sequence space $b_p^{r,s}(\mathrm{Q})$ defined by Topal combining the binomial matrix and quadruple band matrix as follows;

$$b_p^{r,s}(Q) = \left\{ x = (x_k) \in w: \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (o x_k + t x_{k-1} + u x_{k-2} + v x_{k-3}) \right|^p < \infty \right\}, \quad (14)$$

where quadruple band matrix $Q = (q_{nk}(o,t,u,v))$ is defined as follows;

$$q_{nk}(o,t,u,v) = \begin{cases} o, & k = n \\ t, & k = n-1 \\ u, & k = n-2 \\ v, & k = n-3 \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

for all $n, k \in \mathbb{N}$ and $o, t, u, v \in \mathbb{R} \setminus \{0\}$.

Now, let us define the matrix $((k+1)\mathrm{I})$ such that;

$$(k+1)\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (16)$$

where $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$. Let, X be a sequence space. Accordingly, the integrated and differentiated sequence spaces are defined by Goes and Goes [3] as follows;

$$\int X = \{x = (x_k) \in w: ((k+1)x_k) \in X\} = X_{(k+1)\mathrm{I}} \quad (17)$$

and

$$dX = \left\{ x = (x_k) \in w: \left(\left(\frac{1}{k+1} \right) x_k \right) \in X \right\} = X_{\left(\frac{1}{k+1}\right)\mathrm{I}}. \quad (18)$$

Here, if we take $k = 0$ we obtain $\int X = X$ and $dX = X$.

Now, we establish the new sequence spaces by combining the binomial matrix and the integrated and differentiated sequence spaces as follows;

$$\int b_p^{r,s} = \left(b_p^{r,s}\right)_{(k+1)\mathrm{I}} = \left\{ x = (x_k) \in w: \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k \cdot (k+1) x_k \right|^p < \infty \right\} = \left[(\ell_p)_{B^{r,s}}\right]_{(k+1)\mathrm{I}} \quad (19)$$

and

$$d b_p^{r,s} = \left(b_p^{r,s}\right)_{\left(\frac{1}{k+1}\right)\mathrm{I}} = \left\{ x = (x_k) \in w: \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k \left(\frac{1}{k+1}\right) x_k \right|^p < \infty \right\} = \left[(\ell_p)_{B^{r,s}}\right]_{\left(\frac{1}{k+1}\right)\mathrm{I}}. \quad (20)$$

Furthermore, by constructing the matrix $T^{r,s} = (t_{nk}^{r,s}) = B^{r,s}(k+1)\mathrm{I}$ so that;

$$t_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k (k+1); & 0 \le k \le n \\ 0 & ; \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. New integrated sequence spaces can be redefined by matrix $T^{r,s} = (t_{nk}^{r,s}) = B^{r,s}(k+1)\mathrm{I}$ as follows;

$$\int b_p^{r,s} = (\ell_p)_{T^{r,s}}. \tag{21}$$

So, for given $x = (x_k) \in w$, the $T^{r,s}$-transform of x is defined as follows,

$$y_k = (T^{r,s}x)_k = \frac{1}{(s+r)^k}\sum_{i=0}^{k}\binom{k}{i}s^{k-i}r^i(i+1)x_i \tag{22}$$

for all $k \in \mathbb{N}$.

Similarly, by constructing a matrix $U^{r,s} = (u_{nk}^{r,s}) = B^{r,s}\left(\frac{1}{k+1}\right)$I so that;

$$u_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\left(\frac{1}{k+1}\right); & 0 \leq k \leq n \\ 0 & ; \quad k > n \end{cases}$$

for all $k \in \mathbb{N}$. The new differentiated sequence spaces can be redefined by the matrix $U^{r,s} = (u_{nk}^{r,s})$ as follows;

$$db_p^{r,s} = (\ell_p)_{U^{r,s}}. \tag{23}$$

Thus, for given $x = (x_k) \in w$, the $U^{r,s}$-transform of x is defined as follows;

$$y_k = (U^{r,s}x)_k = \frac{1}{(s+r)^k}\sum_{i=0}^{k}\binom{k}{i}s^{k-i}r^i\left(\frac{1}{i+1}\right)x_i \tag{24}$$

for all $k \in \mathbb{N}$.

Theorem 2.1. The sequence space $\int b_p^{r,s}$ with its norm defined as follows;

$$\|x\|_{\int b_p^{r,s}} = \|T^{r,s}x\|_p = \left(\sum_{k=0}^{\infty}|(T^{r,s}x)_k|^p\right)^{\frac{1}{p}} \tag{25}$$

is a BK- space.

Proof: $T^{r,s} = (t_{nk}^{r,s})$ is a triangular matrix and the equation (21) holds. Additionally, since the space ℓ_p with p-norm is a BK-space, according to Theorem 4.3.12 of Wilansky [2], we conclude that the sequence space $\int b_p^{r,s}$ is also a BK-space. Thus, the proof is complete.

Theorem 2.2. The sequence space $db_p^{r,s}$ with its norm defined as follows;

$$\|x\|_{db_p^{r,s}} = \|U^{r,s}x\|_p = \left(\sum_{k=0}^{\infty}|(U^{r,s}x)_k|^p\right)^{\frac{1}{p}} \tag{26}$$

is a BK- space.

Proof: $T^{r,s} = (t_{nk}^{r,s})$ is a triangular matrix and the equation (23) holds. Therefore, the proof can be demonstrated in a similar way as shown in Theorem 2.1.

Theorem 2.3. The sequence space $\int b_p^{r,s}$ is linearly isomorphic to the sequence space ℓ_p.

Proof: Let F be a transformation defined as $F: \int b_p^{r,s} \to \ell_p$, $F(x) = T^{r,s}x$. It is obvious that F is linear. Also, it is clear that $x = \theta$ whenever $T^{r,s}x = \theta$. Consequently, F is injective.
Now, let us consider a sequence $y = (y_n) \in \ell_p$. We define a sequence $x = (x_n)$ for the given sequence $y = (y_n)$ such that,

$$x_n = r^{-n}\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}(-s)^{n-k}(r+s)^k y_k \tag{27}$$

for all $n \in \mathbb{N}$.

$$\big(((k+1)\mathrm{I})x\big)_k = (k+1)x_k = r^{-k}\sum_{l=0}^{k}\binom{k}{l}(-s)^{k-l}(r+s)^l y_l. \tag{28}$$

Then, we have

$$\|x\|_{\int b_p^{r,s}} = \|T^{r,s}x\|_{\ell_p} = \left(\sum_{n=0}^{\infty}|(T^{r,s}x)_n|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (k+1) x_k \right|^p \right)^{\frac{1}{p}}$$
$$= \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \sum_{l=0}^{k} \binom{k}{l} (-s)^{k-l} (r + s)^l y_l \right|^p \right)^{\frac{1}{p}}$$
$$= \left(\sum_{n=0}^{\infty} |y_n|^p \right)^{\frac{1}{p}}$$
$$= \|y\|_{\ell_p}$$
$$= \|F(x)\|_{\ell_p}$$
$$< \infty. \qquad (29)$$

Hence, F is norm preserving from (25) and surjective. As a result, F is an isomorphism and the proof is complete.

Theorem 2.4. The sequence space $db_p^{r,s}$ is linearly isomorphic to the sequence space ℓ_p.

Proof: Let F be a transformation defined as $F: db_p^{r,s} \to \ell_p$, $F(x) = U^{r,s}x$. Now, let us consider the sequence $x = (x_n)$ as follows;

$$x_n = r^{-n}(n+1) \sum_{k=0}^{n} \binom{n}{k} (-s)^{n-k} (r+s)^k y_k \qquad (30)$$

for all $n \in \mathbb{N}$.

Thus, the proof is completed using the method employed in Theorem 2.3.

Theorem 2.5. The sequence space $\int b_p^{r,s}$ is not a Hilbert space under the condition $p \neq 2$.

Proof: Let us assume that $p = 2$. We know from Theorem 2.1. that the sequence space $\int b_2^{r,s}$ is a BK-space with respect to the norm defined by

$$\|x\|_{\int b_2^{r,s}} = \|T^{r,s}x\|_2 = \left(\sum_{k=0}^{\infty} |(T^{r,s}x)_k|^2 \right)^{\frac{1}{2}}. \qquad (31)$$

Therefore, this norm can be constituted in terms of the inner product as follows;

$$\|x\|_{\int b_2^{r,s}} = \langle T^{r,s}x, T^{r,s}x \rangle^{\frac{1}{2}}. \qquad (32)$$

So, $\int b_2^{r,s}$ is a Hilbert space.

Conversely, let us take $p \in [1, \infty) \setminus \{2\}$. We define two sequences $y = (y_k)$ and $z = (z_k)$ as follows;

$$y_k = \left(\frac{1}{k+1} \right) \left(-\frac{s}{r} \right)^k \left(\frac{s - k(r+s)}{s} \right) \qquad (33)$$

and

$$z_k = \left(\frac{1}{k+1} \right) \left(-\frac{s}{r} \right)^k \left(\frac{s + k(r+s)}{s} \right) \qquad (34)$$

for all $k \in \mathbb{N}$. Then we obtain,

$$\|y+z\|^2_{\int b_p^{r,s}} + \|y-z\|^2_{\int b_p^{r,s}} = 8 \neq 2^{\frac{2}{p}+2} = 2\left(\|y\|^2_{\int b_p^{r,s}} + \|z\|^2_{\int b_p^{r,s}} \right). \qquad (35)$$

So, the parallelogram equality is not satisfied by the norm defined in (25). As a result, if $p \neq 2$ then this norm cannot be generated by an inner product. Therefore the sequence space $\int b_p^{r,s}$ cannot be a Hilbert space. Thus, the proof is complete.

Theorem 2.6. The sequence space $db_p^{r,s}$ is not a Hilbert space under the condition $p \neq 2$.

Proof: Let us assume that $p = 2$. In this part of the proof, we utilize the norm defined Theorem 2.2. Thus, as in the previous theorem, it is shown that the sequence $db_2^{r,s}$ is a Hilbert space.

Conversely, let us take $p \in [1, \infty) \setminus \{2\}$. We define two sequence spaces $u = (u_k)$ and $v = (v_k)$ as follows;

$$u_k = (k+1) \left(-\frac{s}{r} \right)^k \left(\frac{s - k(r+s)}{s} \right) \qquad (36)$$

and

$$v_k = (k+1)\left(-\frac{s}{r}\right)^k \left(\frac{s+k(r+s)}{s}\right) \qquad (37)$$

for all $k \in \mathbb{N}$. Thus, by obtaining the same results as in Theorem 2.5. Thus, the proof is complete.

Theorem 2.7. The inclusion $\int \ell_p \subset \int b_p^{r,s}$ strictly holds.

Proof: Let us consider an arbitrary sequence $x = (x_k) \in \int \ell_p$, for $1 < p < \infty$. From the definition of the sequence space $\int \ell_p$, we obtain $\sum_k |(k+1)x_k|^p < \infty$. Therefore, by applying Hölder's inequality we can write;

$$\begin{aligned}|(T^{r,s}x)_k|^p &= \left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j(j+1)x_j\right|^p \\ &\le \left(\frac{1}{|s+r|^k}\right)^p \left[\left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j\right)^{p-1} \cdot \left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j|(j+1)x_j|^p\right)\right] \\ &= \left(\frac{1}{|s+r|^k}\right)^p \left((|s|+|r|)^k\right)^{p-1} \cdot \sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j|(j+1)x_j|^p \\ &= \sum_{j=0}^{k}\binom{k}{j}\left|\frac{s}{s+r}\right|^k \left|\frac{r}{s}\right|^j |(j+1)x_j|^p.\end{aligned}$$

Then we obtain;

$$\begin{aligned}\sum_k |(T^{r,s}x)_k|^p &\le \sum_k \sum_{j=0}^{k}\binom{k}{j}\left|\frac{s}{s+r}\right|^k \left|\frac{r}{s}\right|^j |(j+1)x_j|^p \\ &= \sum_j |(j+1)x_j|^p \sum_{k=j}^{\infty}\binom{k}{j}\left|\frac{s}{s+r}\right|^k\left|\frac{r}{s}\right|^j \\ &= \left|\frac{s}{s+r}\right|\sum_j |(j+1)x_j|^p. \end{aligned} \qquad (38)$$

If we consider the comparison test together with the result we have obtained, we conclude that; $T^{r,s}x \in \ell_p$. So, $x = (x_k) \in \int b_p^{r,s}$. Hence, $\int \ell_p \subset \int b_p^{r,s}$.

Now, we define a sequence $y = (y_k)$ as follows,

$$y_k = \left(-\frac{1}{k+1}\right)^k$$

for all $k \in \mathbb{N}$. From here, it is observed that $(k+1)y = ((-1)^k) \notin \ell_p$ and $T^{r,s}y = \left(\left(\frac{s-r}{s+r}\right)^k\right) \in \ell_p$. So, $y = (y_k) \notin \int \ell_p$ and $y = (y_k) \in \int b_p^{r,s}$. Hence, $\int \ell_p \subset \int b_p^{r,s}$ is strict. Similarly, the case of $p = 1$ can be proven in a similar way. Thus, the proof is complete.

Theorem 2.8. The inclusion $d\ell_p \subset db_p^{r,s}$ strictly holds.

Proof: The proof of this theorem follows a similar method to the one used in the previous theorem. Where, using $U^{r,s}$ instead of $T^{r,s}$.

3. α, β and $\gamma -$ Duals of the Spaces $\int b_p^{r,s}$ and $db_p^{r,s}$

In this part, we determine α, β and $\gamma -$ duals of the differentiated and integrated sequence spaces $\int b_p^{r,s}$ and $db_p^{r,s}$. Given two sequence spaces X and Y, the multiplier space $M(X,Y)$ is defined as follows;

$$X^\alpha = M(X, \ell_1),$$

$$X^\beta = M(X, cs)$$

and

$$X^\gamma = M(X, bs).$$

Lemma 3.1. [11] Let $A = (a_{nk})$ be an infinite matrix; the following conditions hold.

i) $A = (a_{nk}) \in (\ell_1 : \ell_1)$ if and only if $\sup_{k\in\mathbb{N}} \sum_n |a_{nk}| < \infty,$ (39)

ii) $A = (a_{nk}) \in (\ell_1 : \ell_\infty)$ if and only if $\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty,$ (40)

iii) $A = (a_{nk}) \in (\ell_1 : c)$ if and only if (40) holds and $\lim_{n\to\infty} a_{nk} = a_k$ for all $k \in \mathbb{N}$. (41)

Lemma 3.2. [11] Let $A = (a_{nk})$ be an infinite matrix; the following conditions hold.

i) $A=(a_{nk})\in(\ell_p:\ell_1)$ if and only if $\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}a_{nk}\right|^q<\infty$, (42)

ii) $A=(a_{nk})\in(\ell_p:\ell_\infty)$ if and only if $\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|^p<\infty$, (43)

iii) $A=(a_{nk})\in(\ell_p:c)$ if and only if (41) and (43) hold.

Where $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$.

Theorem 3.3. i) The α-dual of the integrated sequence space $\int b_p^{r,s}$ is the set,

$$\xi_1^{r,s}=\left\{\begin{array}{c}a=(a_k)\in w:\\ \sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}\left(\frac{1}{n+1}\right)\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}\cdot(r+s)^k a_n\right|^q<\infty\end{array}\right\} \quad (44)$$

and the α-dual of the integrated sequence space $\int b_1^{r,s}$ is the set,

$$\xi_2^{r,s}=\left\{\begin{array}{c}a=(a_k)\in w:\\ \sup_{k\in\mathbb{N}}\sum_n\left|\left(\frac{1}{n+1}\right)\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k a_n\right|<\infty\end{array}\right\}. \quad (45)$$

ii) The α-dual of the differentiated sequence space $db_p^{r,s}$ is the set,

$$\xi_3^{r,s}=\left\{\begin{array}{c}a=(a_k)\in w:\\ \sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}(n+1)\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k a_n\right|^q<\infty\end{array}\right\} \quad (46)$$

and the α-dual of the differentiated sequence space $db_1^{r,s}$ is the set,

$$\xi_4^{r,s}=\left\{\begin{array}{c}a=(a_k)\in w:\\ \sup_{k\in\mathbb{N}}\sum_n\left|(n+1)\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k a_n\right|<\infty\end{array}\right\}. \quad (47)$$

Proof: i) Consider a sequence $x=(x_n)$ defined as,

$$x_n=\sum_{k=0}^{n}\left[\frac{1}{n+1}\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k\right]y_k \quad (48)$$

for all $n\in\mathbb{N}$. From this, we conclude that for a sequence $a=(a_n)$, we write;

$$\begin{aligned}a_nx_n&=\sum_{k=0}^{n}\left[\frac{1}{n+1}\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k a_n\right]y_k\\ &=\sum_{k=0}^{n}z_{nk}^{r,s}y_k\\ &=(Z^{r,s}y)_n\end{aligned}$$

for all $n\in\mathbb{N}$.

By taking into account the equality above, we observe that $ax=(a_nx_n)\in\ell_1$ whenever $x=(x_k)\in\int b_1^{r,s}$ or $x=(x_k)\in\int b_p^{r,s}$ if and only if $Z^{r,s}y\in\ell_1$ whenever $y=(y_k)\in\ell_1$ or $y=(y_k)\in\ell_p$, respectively. Where $1<p<\infty$. So, we that $a=(a_n)\in\{\int b_1^{r,s}\}^\alpha$ or $a=(a_n)\in\left\{\int b_p^{r,s}\right\}^\alpha$ if and only if $Z^{r,s}\in(\ell_1:\ell_1)$ or $Z^{r,s}\in(\ell_p:\ell_1)$ respectively, where $1<p<\infty$. By connecting these results, Lemma 3.1 (i) and Lemma 3.2 (i), we deduce that;

$$a=(a_n)\in\left\{\int b_1^{r,s}\right\}^\alpha\Leftrightarrow \sup_{k\in\mathbb{N}}\sum_n\left|\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k\frac{1}{n+1}a_n\right|<\infty \quad (49)$$

and

$$a=(a_n)\in\left\{\int b_p^{r,s}\right\}^\alpha\Leftrightarrow \sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k\frac{1}{n+1}a_n\right|^q<\infty, \quad (50)$$

where $1 < p < \infty$. These yield us that $\{\int b_1^{r,s}\}^{\alpha} = \xi_2^{r,s}$ and $\{\int b_p^{r,s}\}^{\alpha} = \xi_1^{r,s}$ Where $1 < p < \infty$. Thus, the proof is complete.

ii) The sequence $x = (x_n)$ is defined as;

$$x_n = \sum_{k=0}^{n} \left[(n+1)\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(r+s)^k\right] y_k \tag{51}$$

for all $n \in \mathbb{N}$. The proof is carried out in a similar method in part (i).

Theorem 3.4. i) Consider the sets $\xi_5^{r,s}$, $\xi_6^{r,s}$ and $\xi_7^{r,s}$ defined by

$$\xi_5^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sum_{j=k}^{\infty} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j \\ \text{exists } \forall k \in \mathbb{N} \end{array} \right\}, \tag{52}$$

$$\xi_6^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j \right| < \infty \end{array} \right\} \tag{53}$$

and

$$\xi_7^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j \right|^q < \infty \end{array} \right\}, \tag{54}$$

where $1 < q < \infty$. Then the following statements hold;

I. $\{\int b_1^{r,s}\}^{\beta} = \xi_5^{r,s} \cap \xi_6^{r,s}$,
II. $\{\int b_p^{r,s}\}^{\beta} = \xi_5^{r,s} \cap \xi_7^{r,s}$, $(1 < p < \infty)$
III. $\{\int b_1^{r,s}\}^{\gamma} = \xi_6^{r,s}$,
IV. $\{\int b_p^{r,s}\}^{\gamma} = \xi_7^{r,s}$. $(1 < p < \infty)$

ii) Consider the sets $\xi_8^{r,s}$, $\xi_9^{r,s}$ and $\xi_{10}^{r,s}$ defined by;

$$\xi_8^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sum_{j=k}^{\infty} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k(j+1)a_j \\ \text{exists } \forall k \in \mathbb{N} \end{array} \right\}, \tag{55}$$

$$\xi_9^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k(j+1)a_j \right| < \infty \end{array} \right\} \tag{56}$$

and

$$\xi_{10}^{r,s} = \left\{ \begin{array}{c} a = (a_k) \in w: \\ \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k(j+1)a_j \right|^q < \infty \end{array} \right\}, \tag{57}$$

where $1 < q < \infty$. Then thefollowing statements hold;

I. $\{db_1^{r,s}\}^{\beta} = \xi_8^{r,s} \cap \xi_9^{r,s}$,
II. $\{db_p^{r,s}\}^{\beta} = \xi_8^{r,s} \cap \xi_{10}^{r,s}$, $(1 < p < \infty)$
III. $\{db_1^{r,s}\}^{\gamma} = \xi_9^{r,s}$,
IV. $\{db_p^{r,s}\}^{\gamma} = \xi_{10}^{r,s}$. $(1 < p < \infty)$

Proof: Since the other parts of the proof can be done similarly, we provide the proof only for case (I) of part (i). Let us consider the sequence $x = (x_n)$ defined in (48) for an arbitrary $a = (a_n) \in w$. Then,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \frac{1}{r^k}\binom{k}{j}(-s)^{k-j}(r+s)^j\left(\frac{1}{k+1}\right)y_j\right]a_k$$
$$= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j\right]y_k$$
$$= (F^{r,s}y)_n \tag{58}$$

for all $n \in \mathbb{N}$. Where the matrix $F^{r,s} = \left(f_{nk}^{r,s}\right)$ is defined by,

$$f_{nk}^{r,s} = \begin{cases} \sum_{j=k}^{n} \dfrac{\frac{1}{r^j}\binom{j}{k}(-s)^{j-k}}{(r+s)^k\left(\frac{1}{j+1}\right)a_j} & , 0 \le k \le n \\ 0 & , k > n \end{cases} \quad (59)$$

for all $n, k \in \mathbb{N}$. So, $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in \int b_1^{r,s}$ if and only if $F^{r,s} y \in c$ whenever $y = (y_k) \in \ell_1$. This outcome makes clear that $a = (a_n) \in \{\int b_1^{r,s}\}^\beta$ if and only if $F^{r,s} \in (\ell_1 : c)$. By combining this result and Lemma 3.1. (iii), we obtain that $a = (a_n) \in \{\int b_1^{r,s}\}^\beta$ if and only if

$$\sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j \right| < \infty$$

and

$$\sum_{j=k}^{\infty} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_j$$

exists for all $k \in \mathbb{N}$.

This result shows us that $\{\int b_1^{r,s}\}^\beta = \xi_5^{r,s} \cap \xi_6^{r,s}$. Thus, the proof is complete.

4. Some Matrix Classes

In this part, we identify certain matrix classes associated with the new sequence spaces.

Now let us prefer the following sequences that we use throughout this section.

$$\rho_{nk}^{r,s} = \sum_{j=k}^{\infty} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k\left(\frac{1}{j+1}\right)a_{nj} \quad (60)$$

and

$$\eta_{nk}^{r,s} = \sum_{j=k}^{\infty} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k(j+1)a_{nj} \quad (61)$$

for all $n, k \in \mathbb{N}$.

Theorem 4.1. Given an infinite matrix $A = (a_{nk})$, the following statements hold.

i) $A = (a_{nk}) \in (\int b_1^{r,s} : \ell_\infty)$ if and only if $\sup_{k,n} |\rho_{nk}^{r,s}| < \infty$, (62)

ii) $A = (a_{nk}) \in (\int b_p^{r,s} : \ell_\infty)$ if and only if $\sup_{n\in\mathbb{N}} \sum_k |\rho_{nk}^{r,s}|^q < \infty$, (63)

$\{a_{nk}\}_{k\in\mathbb{N}} \in \xi_7^{r,s}$ $(1 < p < \infty)$, (64)

iii) $A = (a_{nk}) \in (db_1^{r,s} : \ell_\infty)$ if and only if $\sup_{k,n} |\eta_{nk}^{r,s}| < \infty$, (65)

iv) $A = (a_{nk}) \in (db_p^{r,s} : \ell_\infty)$ if and only if $\sup_{n\in\mathbb{N}} \sum_k |\eta_{nk}^{r,s}|^q < \infty$, (66)

$\{a_{nk}\}_{k\in\mathbb{N}} \in \xi_{10}^{r,s}$ $(1 < p < \infty)$. (67)

Proof: Since the others can be done in a similar method, we only provide the proof for (iv).
Let $1 < p < \infty$. Let us consider an arbitrary sequence $x = (x_k) \in db_p^{r,s}$ that satisfies the conditions (66) and (67). Thus, it is obtained that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{db_p^{r,s}\}^\beta$. This result indicates the existence of the A-transform of x. From the relation (48), we have

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m} \left[\sum_{j=0}^{k} \frac{1}{r^k}\binom{k}{j}(-s)^{k-j}(r+s)^j(k+1)y_j \right] a_{nk}$$
$$= \sum_{k=0}^{m} \left[\sum_{j=k}^{m} \frac{1}{r^j}\binom{j}{k}(-s)^{j-k}(r+s)^k(j+1) \right] a_{nj} y_k. \quad (68)$$

By taking limit (68) side by side as $m \to \infty$, we obtain that

$$\sum_k a_{nk}x_k = \sum_k \eta_{nk}^{r,s} y_k, n \in \mathbb{N}. \quad (69)$$

Then, we derive by taking ℓ_∞-norm (69) side by side and any by applying Hölder's inequality that,

$$\|Ax\|_\infty = \sup_{n\in\mathbb{N}} \left| \sum_k \eta_{nk}^{r,s} y_k \right|$$

$$\leq \sup_{n\in\mathbb{N}}\left(\sum_k\left|\eta_{nk}^{r,s}\right|^q\right)^{\frac{1}{q}}\left(\sum_k|y_k|^p\right)^{\frac{1}{p}} < \infty. \quad (70)$$

Consequently, we conclude that $Ax \in \ell_\infty$. So, $A = (a_{nk}) \in \left(db_p^{r,s} : \ell_\infty\right)$.

Conversely, assume that A= $(a_{nk}) \in \left(db_p^{r,s} : \ell_\infty\right)$. This gives us to $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{db_p^{r,s}\right\}^\beta$ for all $n \in \mathbb{N}$. Then, it is evident that the condition (67) is necessary and that the $\{\eta_{nk}^{r,s}\}_{k,n\in\mathbb{N}}$ exists. Because of $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{db_p^{r,s}\right\}^\beta$, we can see that the condition (69) holds and the sequences $a_n = (a_{nk})_{k\in\mathbb{N}}$ define the continuous linear functionals f_n on $db_p^{r,s}$ by

$$f_n(x) = \sum_k a_{nk}x_k \quad (71)$$

for all $n \in \mathbb{N}$. Additionally, we know from the Theorem 2.4 that the $db_p^{r,s}$ is norm isomorphic to ℓ_p. By connecting this result and the condition (69), we have

$$\|f_n\| = \left\|\left(\eta_{nk}^{r,s}\right)_{k\in\mathbb{N}}\right\|_q, \quad (72)$$

which yield that the functionals f_n are pointwise bounded. Moreover, we derive from the Banach-Steinhaus Theorem that the functionals f_n are uniformly bounded. So there exists a constant $M > 0$ such that;

$$\left(\sum_k\left|\eta_{nk}^{r,s}\right|^q\right)^{\frac{1}{q}} = \|f_n\| \leq M \quad (73)$$

for all $n \in \mathbb{N}$, which shows us that the condition (66) holds. Thus, the proof is completed.

Lemma 4.1. [11] Let $B = (b_{nk})$ be an infinite matrix. Then, $B = (b_{nk}) \in (\ell_1 : \ell_p)$ if and only if

$$\sup_{k\in\mathbb{N}}\sum_n\left|b_{nk}^{r,s}\right|^p < \infty,$$

where $1 < p < \infty$.

Theorem 4.2. Let an infinite matrix $B = (b_{nk})$ be given. Then,

i) $B = (b_{nk}) \in \left(\int b_1^{r,s} : \ell_p\right)$ if and only if $\sup_{k\in\mathbb{N}}\sum_n\left|\rho_{nk}^{r,s}\right|^p < \infty$, (74)

ii) $B = (b_{nk}) \in \left(db_1^{r,s} : \ell_p\right)$ if and only if $\sup_{k\in\mathbb{N}}\sum_n\left|\eta_{nk}^{r,s}\right|^p < \infty$. (75)

Proof: Let a sequence $y = (y_k) \in \int b_1^{r,s}$ be given. Assume that the condition (75) holds. Then, it is clear that $z = (z_k) \in \ell_1$ and $\{b_{nk}\}_{k\in\mathbb{N}} \in \{\int b_1^{r,s}\}^\beta$ for all $n \in \mathbb{N}$. That means B-transform of x exists. As a result of this, the series $\sum_k \rho_{nk}^{r,s} z_k$ are absolutely convergent for all $n \in \mathbb{N}$ and $z = (z_k) \in \ell_1$. Now let us consider the following equality.

$$\sum_k b_{nk}y_k = \sum_k \rho_{nk}^{r,s} z_k, n \in \mathbb{N}. \quad (76)$$

If we apply the Minkowsky inequality to equation (76), we obtain

$$\left(\sum_n|(By)_n|^p\right)^{\frac{1}{p}} \leq \sum_k|z_k|\left(\sum_n\left|\rho_{nk}^{r,s}\right|^p\right)^{\frac{1}{p}}. (77)$$

Thus, it follows that $By \in \ell_p$, namely $B = (b_{nk}) \in \left(\int b_1^{r,s} : \ell_p\right)$.

Conversely, we suppose that $B = (b_{nk}) \in \left(\int b_1^{r,s} : \ell_p\right)$. Namely, $By \in \ell_p$ for all $y = (y_k) \in \int b_1^{r,s}$. So, $\{b_{nk}\}_{k\in\mathbb{N}} \in \{\int b_1^{r,s}\}^\beta$ for all $n \in \mathbb{N}$, which shows us that the relation (76) holds. These results give us that $\left(\rho_{nk}^{r,s}\right) \in (\ell_1 : \ell_p)$. By combining last result and Lemma 4.1, we obtain that the condition (74) holds.

The part (ii) can be proved by using a similar method. Thus, the proof is complete.

5. Conclusion

$T^{r,s} = \left(t_{nk}^{r,s}\right)$ represents the composition of the binomial matrix and the integrated sequence space and $U^{r,s} = \left(u_{nk}^{r,s}\right)$ represents the composition of the binomial matrix and the differentiated sequence space. Since $T^{r,s} = \left(t_{nk}^{r,s}\right)$ and $U^{r,s} = \left(u_{nk}^{r,s}\right)$ is more comprehensive than integrated and differentiated sequence spaces, respectively, our conclusions are more general.

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