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A New Differentiable Sphere Theorem and Its Applications

Vladimir Rovenski* and **Sergey Stepanov**

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Abstract

In this paper, we use the Lichnerowicz Laplacian to prove new results: the sphere theorem and the integral inequality for Einstein's infinitesimal deformations, which allow us to characterize spherical space forms. Our version of the sphere theorem states that a closed connected Riemannian manifold (M, g) of even dimension $n > 3$ is diffeomorphic to a Euclidean sphere or a real projective space if the inequality $Ric_{\max}(x) < n\, K_{\min}(x)\, g$ is true at each point $x \in M$, where $Ric_{\max}(x)$ is the maximum of the Ricci curvature, and $K_{\min}(x)$ is the minimum of the sectional curvature of (M, g) at x. Since this inequality implies positive sectional curvature; therefore, our result partially answers Hopf's old open question.

Keywords: *Lichnerowicz Laplacian, differentiable sphere theorem, Einstein's infinitesimal deformation, spherical space form, curvature operator of the second kind.*

AMS Subject Classification (2020): *Primary: 53C20; Secondary: 53C24.*

1. Introduction and main results

The Lichnerowicz Laplacian Δ_L is a fundamental differential operator of order two, perhaps it is more natural than the rough Laplacian, although both operators are related by the Weitzenböck decomposition formula. Examples of this naturalness are the appearance of Δ_L in the linearized Ricci flow equation (e.g., [8]), in Differentiable Sphere Theorems, in the stability analysis of Einstein metrics and Kaluza-Klein theories (e.g., [1] and [2, Chapter 12]), etc. Namely, Δ_L, acting on the vector bundle $S^2_0 M$ of symmetric traceless 2-tensors, can be considered as Einstein's infinitesimal deformations of the metric g.

By the Differentiable Sphere Theorem, see [5], any closed connected Riemannian manifold (M^n, g) with strictly pointwise $1/4$-pinched sectional curvature is diffeomorphic to the spherical space form $\mathbb{S}^n/\Gamma$, where Γ is a finite group of isometries acting freely. Riemannian manifolds isometric to the quotient manifold $\mathbb{S}^n/\Gamma$, where Γ is a finite group of isometries acting freely on the unit n-sphere, are fully classified in [15] and are called spherical space forms.

The kth Betti number $\mathfrak{b}_k(M)$ (called after Enrico Betti) represents the rank of the kth homology group of M. If a Riemannian manifold (M, g) is diffeomorphic to the spherical space form $\mathbb{S}^n/\Gamma$, then it has zero Betti numbers $\mathfrak{b}_k(M)$ for $k = 1, \ldots, n-1$; thus, $\chi(M) > 0$ for even n. Recall the expression of the Euler-Poincaré characteristic of a manifold M in terms of Betti numbers:

$$\chi(M) = \sum\nolimits_{k>0} (-1)^k\, \mathfrak{b}_k(M).$$

For example, a closed locally conformally flat Riemannian manifold of dimension $n \geq 4$ with positive sectional curvature admits a metric of positive constant sectional curvature, hence it is diffeomorphic to a spherical space form, see [11], and $\chi(M) > 0$ for even $n \geq 4$.

The Hopf's conjecture (posed in 1931) on controlling topology through curvature is a popular open question in Riemannian geometry, which can be considered in relation to Differentiable Sphere Theorems, see [5, 4]. The

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** Corresponding author*

modern formulation of Hopf's hypothesis is as follows, see [8, p. 81]: *a closed, even-dimensional Riemannian manifold with positive sectional curvature has positive Euler-Poincaré characteristic*. Note that any closed, odd-dimensional manifold has zero Euler-Poincaré characteristic. This conjecture holds in dimensions 2 and 4, see [3, 7]. Despite the relevance of Hopf's conjecture, it remains open in dimensions 6 and higher.

Since the unit sphere in T_xM at an arbitrary point $x \in M$ is a compact set, then there exist the 2-plane $\sigma(x) \subset T_xM$ and the unit vector $X \in T_xM$ such that exist real

$$K_{\min}(x) := \inf{}_{\sigma(x)\subset T_xM} K(\sigma(x)), \quad Ric_{\max}(x) := \sup{}_{X\in T_xM} Ric(X),$$

where $K(\sigma(x))$ is the sectional curvature of $(M.g)$ with respect to the plane $\sigma(x)$ at $x \in M$, and $Ric(X)$ is the Ricci curvature in the direction of any unit vector $X \in T_xM$ at $x \in M$.

Now we can complete the following differentiable sphere theorem for Riemannian manifolds with pinched curvature in the pointwise sense, see [14]: Let (M, g) be an n-dimensional, $n \geq 3$, compact Riemannian manifold with $Ric_{\min}(x) > (n-1)\,\tau_n K_{\max}(x)$ at each point $x \in M$, where $\tau_n = 1 - \frac{6}{5(n-1)}$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{S}^n$. Namely, the following new differentiable sphere theorem for Riemannian manifolds with pinched curvature in the pointwise sense is true.

Theorem 1.1. *An n-dimensional ($n \geq 3$) connected closed Riemannian manifold (M, g), whose Ricci tensor satisfies the inequality*

$$Ric_{\max}(x) < n\,K_{\min}(x), \tag{1.1}$$

is diffeomorphic to the spherical space form $\mathbb{S}^n/\Gamma$. Moreover, if (M, g) is simply connected, then it is diffeomorphic to $\mathbb{S}^n$.

Remark 1.1. In the article, we consider the following condition at each point $x \in M$:

$$Ric(X) < n\,K_{\min}(x). \tag{1.2}$$

Since,

$$Ric(X) \geq (n-1)\,K_{\min}(x)$$

at each point $x \in M$, by (1.2) we conclude that

$$(n-1)\,K_{\min}(x) \leq Ric(X) < n\,K_{\min}(x). \tag{1.3}$$

The double inequality (1.3), and hence (1.2), implies $K_{\min}(x) > 0$ at any point $x \in M$.

Under conditions of Theorem 1.1, (M, g) is a closed Riemannian manifold of positive sectional curvature. Since the spherical space form $\mathbb{S}^n/\Gamma$ has zero Betti numbers $\mathfrak{b}_k(M)$ for $k = 1, \ldots, n-1$, then $\chi(M) > 0$ for even n. Therefore, Theorem 1.1 gives a partial answer to Hopf's old open question.

Remark 1.2. A Riemannian manifold (M, g) is called *strictly pointwise δ-pinched* if

$$0 < \delta K(\sigma_1(x)) < K(\sigma_2(x))$$

for some real $\delta > 0$ and any planes $\sigma_1(x), \sigma_2(x) \subset T_xM$ at any point $x \in M$, see [5, 4]. Since the unit sphere in T_xM is a compact set, there exists a plane $\sigma(x) \subset T_xM$ such that $K_{\min}(x) = K(\sigma(x))$ – the sectional curvature of $\sigma(x)$. If the sectional curvature of (M, g) is weakly pointwise $1/4$-pinched, i.e.,

$$0 < K_{\min}(x) < K(\sigma) < 4\,K_{\min}(x),$$

then the Ricci tensor and $K_{\min}(x)$ satisfy the inequality

$$Ric_{\max}(x) < 4(n-1)\,K_{\min}(x). \tag{1.4}$$

Since (1.4) follows from (1.2), Theorem 1.1 is consistent with the differentiable sphere theorem.

Corollary 1.1. *Let a closed connected Riemannian manifold (M^n, g) of even dimension satisfy (1.1), then (M, g) is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{RP}^n$.*

One can also ask if the scalar curvature $Scal = \mathrm{trace}_g\, Ric$ (a smooth function on a manifold) can control the topology, see [2]. For example, a metric with nonnegative scalar curvature on a torus has to be flat.

The main results of our article are the differentiable sphere theorem and the integral inequality for Einstein's infinitesimal deformations of closed Einstein manifolds, which allow to characterize spherical space forms. In the proofs of our theorems (see Section 2) we use the curvature operator of the second kind, acting on the space of symmetric covariant 2-tensors and the Weitzenböck decomposition formula, which relates the Lichnerowicz Laplacian to the rough Laplacian and Weitzenböck operator.

Recall that an n-dimensional $(n \geq 3)$ connected manifold with a Riemannian metric g is said to be an *Einstein manifold* with Einstein's constant $\alpha \in \mathbb{R}$ (related to the scalar curvature by $Scal = n\,\alpha$) if its Ricci tensor satisfies

$$Ric = \alpha\, g,$$

see [2, p. 3]. For example, there exist three kinds of Einstein 4-dimensional manifolds of positive sectional curvature: a sphere $\mathbb{S}^4$ and real projective space $\mathbb{RP}^4$ with positive constant sectional curvature, and the complex projective plane $\mathbb{CP}^2$ with the Fubini-Study metric.

Let (M, g) be a closed Einstein manifold and all Einstein metrics on M close to g be homothetic to g, then the Einstein metric g is said to be rigid. For example, all Einstein metrics, whose sectional curvature is pointwise $\frac{n-2}{n-1}$-pinched, are homothetic to g, see [1]. M. Berger and D. Ebin (see [1] and [2, Chapter 12]) considered generalizations of this result and introduced "Einstein's infinitesimal deformations". The result they gave, roughly speaking, is that the space of all Einstein metrics on a smooth manifold is locally finite-dimensional.

For Einstein manifolds, the inequality (1.2) can be written in the form (1.5). Thus, the next corollary (also proven in [12, Theorem 1]) directly follows from Theorem 1.1.

Corollary 1.2. *Let an n-dimensional $(n \geq 3)$ closed connected Einstein manifold (M, g) with Einstein constant α satisfy the inequality*

$$\alpha < n\, K_{\min}(x), \quad x \in M\,, \tag{1.5}$$

then (M, g) is a spherical space form $\mathbb{S}^n/\Gamma$. In particular, if (M, g) is simply connected, then it is diffeomorphic to $\mathbb{S}^n$.

Remark 1.3. If (M, g) is an n-dimensional closed connected Einstein manifold of nonnegative scalar curvature such that $\alpha < (n+2)\, K_{\min}(x)$ $(x \in M)$, then the Betti numbers $\mathfrak{b}_k(M)$ are zero for $k = 1, \ldots, n-1$, see [12]. Furthermore, if n is even and M is simply-connected, then $\mathfrak{b}_0(M) = \mathfrak{b}_n(M) = 1$; hence, $\chi(M) > 0$.

Theorem 1.2. *Let (M, g) be an n-dimensional $(n \geq 3)$ closed connected Einstein manifold with Einstein constant α. Then for any Einstein's infinitesimal deformation $\varphi \neq 0$, the following integral inequality is true:*

$$\int_M (\alpha - n\, K_{\min}(x))\, \|\varphi\|^2\, dv_g \geq 0\,. \tag{1.6}$$

Moreover, if the equality in (1.6) is achieved for some Einstein's infinitesimal deformation $\varphi \neq 0$, then (M, g) has constant sectional curvature and φ is a parallel tensor.

It is well known that the standard metric of the sphere is rigid, see [2, p. 132].

In the following corollary of Theorem 1.2 we generalize this result.

Corollary 1.3. *Let (M, g) be an n-dimensional $(n \geq 3)$ closed connected Einstein manifold with Einstein constant α. If for any Einstein's infinitesimal deformation $\varphi \neq 0$ of the metric g the following weaker than (1.5) integral inequality holds:*

$$\int_M (\alpha - n\, K_{\min}(x))\, \|\varphi\|^2\, dv_g < 0\,, \tag{1.7}$$

then the metric g is rigid.

Remark 1.4. Since $\alpha \geq (n-1)K_{\min}(x)$ is true at any point of an n-dimensional Einstein manifold (M, g), from (1.7) we get $\int_M K_{\min}(x)\, \|\varphi\|^2 dv_g > 0$, thus $K_{\min}(x)$ is positive somewhere on M. For the unit sphere, (1.7) reduces to $\int_M \|\varphi\|^2\, dv_g > 0$, so it is true.

2. Proof of Theorem 1.1 and its corollaries

Let $S^p M$ be the bundle of symmetric covariant tensor fields over a Riemannian manifold (M, g). For this case, we have the pointwise identity $\dim S^p(T_x M) = \binom{n+p-1}{p}$ at an arbitrary point $x \in M$. The well-known

Lichnerowicz Laplacian $\Delta_L : C^\infty(S^pM) \to C^\infty(S^pM)$ as on the space of covariant symmetric p-tensors. At the same time, the Lichnerowicz Laplacian satisfies the Weitzenböck decomposition formula $\Delta_L = \bar\Delta + \Re_p$, where $\Re_p$ is the Weitzenböck curvature operator (see [1, p. 388]; [2, p. 54]). It is an algebraic operator, representing the restriction of the Weitzenböck curvature operator $\Re$ to symmetric p-tensors. This differential operator, initially introduced by Lichnerowicz in [9, p. 26], is self-adjoint, elliptic and preserves the symmetries of tensor fields. Furthermore, the Weitzenböck curvature operator $\Re_p : S^pM \to S^pM$ satisfies the following identities: $g(\Re(\varphi), \varphi') = g(\varphi, \Re(\varphi'))$ and $trace_g\, \Re(\varphi) = \Re(trace_g\, \varphi)$ for any $\varphi, \varphi' \in S^pM$ (see [9, p. 315]). Next, let (M, g) be covered by a system of coordinate neighborhoods $\{U, x^1, \ldots, x^n\}$, where U denotes a neighborhood and $x^1, \ldots, x^n$ denote local coordinates in U. Then we can define the natural frame $\{X_1 = \partial_1, \ldots, X_n = \partial_n\}$ in an arbitrary coordinate neighborhood $\{U, x^1, \ldots, x^n\}$. In this case, $g_{ij} = g(X_i, X_j)$ are local components of the metric tensor g with the indices $i, j, k, l, \ldots \in \{1, 2, \ldots, n\}$. Next, we denote by $R_{ik} = Ric(\partial_i, \partial_j)$ and $R^i_{kjl}\partial^i = R(\partial_j, \partial_l)\partial_k$ the local components the Ricci Ric and curvature R tensors, respectively, see [14, p. 49]. Then the Lichnerowicz Laplacian $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with respect to local coordinates $x^1, \ldots, x^n$ has the form, e.g., [1, p. 387–388] and [9, p. 316])

$$\Delta_L\varphi_{ij} = \bar\Delta_L\varphi_{ij} + (R_{ik}\varphi^k_j + R_{jk})\varphi^k_i - 2\,R_{ikjl})\varphi^{kl}, \tag{2.1}$$

where $\bar\Delta = \nabla^*\nabla$ is an elliptic operator, called the rough Laplacian, $\varphi_{ij} = \varphi(\partial_i, \partial_j)$ for arbitrary $\varphi \in C^\infty(S^pM)$ and $R_{ikjl} = g_{im}R^m_{kjl}$ for the local components $g_{ij} = g(\partial_i, \partial_j)$ of the metric tensor g, see [14, p. 49]. Next we will consider a smooth section of a subbundle $S^p_0M \subset S^pM$ of covariant symmetric p-tensors which are traceless on any pair of indices, i.e., $\sum_{i=1}^n \varphi(e_i, e_i, X_3, \ldots X_n) = 0$ for an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_xM at an arbitrary point $x \in M$. It is well-known that $\dim S^p_0(T_xM) = \begin{pmatrix} n+p-1 \\ p \end{pmatrix} - \begin{pmatrix} n+p-3 \\ n-1 \end{pmatrix}$.

We recall that the symmetric operator $\mathring{R} : S^2_0M \to S^2_0M$ determined by the equations $\mathring{R}(\varphi)_{ij} = R_{iklj}\varphi^{kl}$ is called as the curvature operator of the second kind (see [6]). We can also say that $\mathring{R} > 0$ if the eigenvalues of $\mathring{R}$ as a bilinear form on S^2_0M are positive definite everywhere on (M, g). In addition, the well-known (see, for example, [6]) that if $\mathring{R} > 0$ (resp., $\mathring{R} \ge 0$), then the sectional curvature is positive (resp., non-negative). Moreover, easy calculations give that (M, g) is Einstein (i.e., $Ric = (s/n)\,g$) if and only if $\mathring{R}$ maps S^2_0M into itself at each point $x \in M$, see [2, p. 49]. In particular, $\mathring{R}$ is identical with $s/(n(n-1) \times (\text{the identity map}))$ on $S^2_0\,\mathbb{S}^n$ for the Euclidean sphere $\mathbb{S}^n$ with standard metric g_0. Therefore, the curvature operator $\mathring{R}$ is positive on the Euclidean sphere $(\mathbb{S}^n, g_0)$. Using the above, we can rewrite (2.1) in the following form:

$$\Delta_L\varphi_{ij} = \bar\Delta_L\varphi_{ij} + \big(R_{ik}\varphi^k_j + R_{jk}\varphi^k_i + 2\mathring{R}(\varphi)_{ij}\big)$$

We showed above that the curvature operator of the second kind arises as a term in the Lichnerovich Laplacian, including the curvature tensor. Let consider the quadratic form $\mathcal{Q}_2(\varphi) : S^2M \otimes S^2M \to \mathbb{R}$ defined by

$$\mathcal{Q}_2(\varphi) = g(\Re_2(\varphi), \varphi) = \Re(\varphi)\varphi_{ij}\varphi^{ij} = 2\big(R_{ij}\varphi^{ik}\varphi^j_k - R_{ijkl}\varphi^{ik}\varphi^{jl}\big). \tag{2.2}$$

In the last case, there exists an orthonormal basis $e_1, \ldots, e_n$ of T_xM at any point $x \in M$ such that $\varphi_x(e_i, e_j) = \epsilon_i\delta_{ij}$ for the Kronecker δ_{ij}. Then the quadratic form $\mathcal{Q}_2(\varphi)$ can be rewritten in the form (see [1, p. 388])

$$g(\Re_2(\varphi_x), \varphi_x) = \sum\nolimits_{i<j} K(e_i, e_j)(\epsilon_i - \epsilon_j)^2,$$

where $K(e_i, e_j)$ is the sectional curvature in the direction of $\sigma(x) = span\{e_i, e_j\} \subset T_xM$ at an arbitrary point $x \in M$ (see [1, pp. 387–388]). In this case we rewrite (2.1) in the form

$$R_{ij}\varphi^{ik}\varphi^j_k - R_{ijkl}\varphi^{ik}\varphi^{jl} = \sum\nolimits_{i<j} K(e_i, e_j)(\epsilon_i - \epsilon_j)^2.$$

Since the unit sphere in T_xM at an arbitrary point $x \in M$ is a closed set, then there exists the number $K(x)$ such that for all 2-planes $\sigma(x) \subset T_xM$, the inequality $K(x) \le K(\sigma(x))$ is satisfied. In other words, $K(x) = K_{\min}(x)$, where $K_{\min}(x)$ is the minimum of the sectional curvature at $x \in M$. Then the following equalities hold:

$$\begin{aligned}&\sum\nolimits_{i<j} K(e_i, e_j)(\epsilon_i - \epsilon_j)^2 \ge K_{\min}(x)\sum\nolimits_{i<j}(\epsilon_i - \epsilon_j)^2 \\ &= K_{\min}(x)\big((n-1)\sum\nolimits_i(\epsilon_i)^2 - 2\sum\nolimits_{i<j}\epsilon_i\epsilon_j\big) = K_{\min}(x)\big(n\sum\nolimits_i(\epsilon_i)^2 - (\sum\nolimits_i\epsilon_i)^2\big) = nK_{\min}(x)\|\varphi\|^2\end{aligned} \tag{2.3}$$

for $\sigma(x) = span\{e_i, e_j\}$ and $(\mathrm{trace}_g \varphi)(x) = \epsilon_1 + \ldots + \epsilon_n = 0$. Then from the above we conclude that at each point $x \in M$ the following inequality is satisfied:

$$R_{ij}\varphi^{ik}\varphi^j_k - R_{ijkl}\varphi^{ik}\varphi^{jl} \geq nK_{\min}(x)\|\varphi\|^2. \tag{2.4}$$

for the local contravariant components $\varphi^{kl} = g^{ki}g^{lj}\varphi_{ij}$ of an arbitrary $\varphi \in S^2_0(T_xM)$ at $x \in M$. Then (2.4) can be rewritten in the form

$$g(\mathring{\Re}(\varphi), \varphi) \geq nK_{\min}(x)\|\varphi\|^2 - R_{ij}\varphi^{ik}\varphi^j_k \tag{2.5}$$

Using the above, we can state that if the inequality $R_{ij}\varphi^{ik}\varphi^j_k < nK_{\min}(x)\|\varphi\|^2$ holds for any $\varphi \in S^2_0(T_xM)$ at each point $x \in M$, then the condition $\mathring{R} > 0$ is satisfied. In conclusion, we note that if the inequality $R_{ij}X^iX^j < nK_{\min}(x)\|X\|^2$ holds for any $X \in T_xM$, then the inequality $R_{ij}\varphi^{ik}\varphi^j_k < nK_{\min}(x)\|\varphi\|^2$ also holds for any nonzero $\varphi \in S^2_0(T_xM)$ at point $x \in M$, see [16, p. 82–83]. Therefore, from (1.6) and the above comments, we conclude that curvature operator of the second kind $\mathring{R}$ is positive everywhere on an n-dimensional $(n \geq 2)$ Riemannian manifold (M, g) if the sectional curvature and Ricci tensor Ric of (M, g) satisfy the inequality $R_{ij}X^iX^j < n\,K_{\min}(x)\|X\|^2$ for any $X \in T_xM$ at any point $x \in M$.

Since the unit sphere in T_xM at an arbitrary point $x \in M$ is a closed set, there exists a real number $Ric_{\max}(x) = \sup_{X\in T_xM} Ric(X)$. Then the above condition: the Ricci curvature Ric of (M, g) satisfies the strict inequality $Ric(X) < nK_{\min}(x)$ for any unit vector $X \in T_xM$ and the minimum $K_{\min}(x)$ of the sectional curvature of (M, g) at an arbitrary its point $x \in M$ can be replaced with the following more exact condition: $Ric_{\max}(x) < nK_{\min}(x)$ at each point $x \in M$. At the same time, we recall that if (M, g) is a closed Riemannian manifold such that $\mathring{R}$ is positive, then M is diffeomorphic to a spherical space form, see [6]. Therefore, Theorem 1.1 holds.

Note that the simplest examples of spherical space forms are the sphere $\mathbb{S}^n$ and the real projective space $\mathbb{RP}^n$. Furthermore, when n is even, these are the only examples, see [5, p. 3]. In particular, for $n = 2$ the "differentiable sphere theorem" can be rewritten as follows: A closed surface of positive Gaussian curvature is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$. Therefore, we can formulate Corollary 1.

Remark 2.1. Ogiue and Tachibana proved in [13] that an n-dimensional closed Riemannian manifold (M, g) with positive curvature operator of the second kind is a rational homology spheres. We reformulate this statement in the following form: An n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) is a real homology sphere if its Ricci and sectional curvatures satisfy the inequality $Ric_{\max}(x) < nK_{\min}(x)$ at each point $x \in M$.

3. Proof of Theorem 1.2

Let g be an Einstein metric on a closed manifold M with Einstein constant $\alpha \in \mathbb{R}$. Recall that any Einstein's infinitesimal deformation φ satisfies the following equations with the Lichnerowitz Laplacian, see [2, p. 347]:

$$\Delta_L\,\varphi = 2\,\alpha\,\varphi; \quad \delta\varphi = 0; \quad \mathrm{trace}_g\,\varphi = 0. \tag{3.1}$$

Suppose the opposite, that there exists an Einstein's infinitesimal deformation $\varphi \neq 0$ on (M, g) satisfying the integral inequality (1.7) inverse to (1.6).

From (2.1) we obtain for any $\varphi \in C^\infty(S^2_0M)$ the following equality, see [10, 11]:

$$\frac{1}{2}\,\Delta\,(\|\varphi\|^2) = -g(\Delta_L\,\varphi, \varphi) + \|\nabla\varphi\|^2 + g(\Re_2(\varphi),\, \varphi), \tag{3.2}$$

where $\Delta = div \circ grad$ is the Laplacian (called the Laplace-Beltrami operator). Using (2.2) and (3.1), we rewrite (3.2) as follows:

$$\frac{1}{2}\,\Delta\,\|\varphi\|^2 = \|\nabla\varphi\|^2 + 2\,g\big(\mathring{R}(\varphi), \varphi\big). \tag{3.3}$$

From (3.3) we obtain the following integral formula:

$$\int_M \big(\|\nabla\varphi\|^2 + 2\,g(\mathring{R}(\varphi), \varphi)\big)\,dv_g = 0\,. \tag{3.4}$$

On the other hand, according to (2.5) and our assumption (1.7) we conclude that the integral of the function $g(\mathring{R}(\varphi), \varphi)$ is positive:

$$\int_M g(\mathring{R}(\varphi), \varphi)\,dv_g \geq \int_M \big(n\,K_{\min}(x)\|\varphi\|^2 - R_{ij}\varphi^{ik}\varphi^j_k\big)\,dv_g = \int_M \big(n\,K_{\min}(x) - \alpha\big)\|\varphi\|^2\,dv_g > 0. \tag{3.5}$$

This contradicts the integral formula (3.4), from which it follows that the integral of the function $g(\overset{\circ}{R}(\varphi), \varphi)$ is non-positive for our $\varphi \neq 0$. We conclude that any Einstein's infinitesimal deformation φ of g satisfies the integral inequality (1.6).

If the equality in (1.6) is achieved for some Einstein's infinitesimal deformation $\varphi \neq 0$, then we get the equality in (2.3) for such φ, thus (M, g) has pointwise constant sectional curvature, and by Schur's Theorem, (M, g) has constant sectional curvature.

By the above considerations, see (3.4) and (3.5), the tensor φ is parallel: $\nabla\varphi = 0$, and the equality $g(\overset{\circ}{R}(\varphi), \varphi) = 0$ is valid on M. □

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Affiliations

VLADIMIR ROVENSKI
ADDRESS: University of Haifa, Department of Mathematics, 3498838, Haifa-Israel.
E-MAIL: vrovenski@univ.haifa.ac.il
ORCID ID: 0000-0003-0591-8307

SERGEY STEPANOV
ADDRESS: 1. Finance University, Department of Mathematics, 49-55, Leningradsky Prospect, 125468, Moscow-Russia. 2. Russian Institute for Scientific and Technical Information of the Russian Academy of Sciences, Department of Mathematics, Usievicha street 20, 125190 Moscow, Russia.
E-MAIL: s.e.stepanov@mail.ru
ORCID ID: 0000-0003-1734-8874