International Electronic Journal of Algebra
Volume 37 (2025) 249-272
DOI: 10.24330/ieja.1531556

CHAIN CONDITIONS ON NON-PARALLEL SUBMODULES

Nasrin Shirali, Mohammad Maschizadeh and Sayed Malek Javdannezhad

Received: 8 February 2024; Revised: 28 June 2024; Accepted: 14 July 2024
Communicated by Abdullah Harmancı

Dedicated to the memory of Professor Syed M. Tariq Rizvi

Abstract. In this paper, we investigate modules with ascending and descending chain conditions on non-parallel submodules. We call these modules np-Noetherian and np-Artinian respectively, and give structure theorems for them. It is proved that any np-Artinian module is either atomic or finitely embedded. Also, we give a sufficient condition for np-Noetherian (resp., np-Artinian) modules to be Noetherian (resp., Artinian). We study ascending (resp., descending) chain condition up to isomorphism on non-parallel submodules as npi-Noetherian (resp., npi-Artinian) modules and characterize these modules. It is shown that any npi-Noetherian module has finite type dimension. Next, we investigate some properties of semiprime right np-Artinian (resp., npi-Artinian) rings. In particular, it is proved that if R semiprime ring such that $J(R)$ is not atomic, then R is right np-Artinian if and only if it is semisimple. Further, it is shown that if R is a semiprime right npi-Artinian ring, then either $Z(R)$ is atomic or R is right non-singular. Finally, we investigate when np-Artinian (resp., np-Noetherian) rings and ne-Artinian (resp., ne-Noetherian) rings coincide.

Mathematics Subject Classification (2020): 16P20, 16P40, 16P60
Keywords: Parallel submodule, type dimension, np-Noetherian module, np-Artinian module, iso-Noetherian module, iso-Artinian module

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary right modules. For an R-module M, we write soc(M), rad(M), $E(M)$, $Z(M)$, I-soc(M) and I-rad(M) for the socle, intersection of all maximal submodules, injective hull, singular submodule, iso-socle and iso-radical of M, respectively (see [6], for the last two notions). In particular case $R = M$, we use $J(R)$ instead of rad(R) and it is called the Jacobson radical of R. Also, for a ring R, the lower nil radical Nil$_*$ and the upper nil radical Nil$^*(R)$ will be used for the intersection of all prime ideals of a ring R and the sum of all right nil ideals of R, respectively. The notations $N \subseteq_e M$, $N \parallel M$ and $N \subseteq_t M$ will denote that N is an essential submodule, a parallel submodule and a type submodule of M, respectively. For any subset A of a ring R we set $\mathbf{r}(A) = \{r \in R \mid Ar = 0\}$, i.e., $\mathbf{r}(A)$ is the right annihilator of A in R. In particular, if $A = \{a\}$, then we write $\mathbf{r}(a)$ instead of $\mathbf{r}(\{a\})$. We say that a module M is ne-Noetherian (resp., ne-Artinian), if M satisfies ACC (resp., DCC) on non-essential submodules, i.e., for every ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of non-essential submodules of M, there exists an index $k \geq 1$ such that $N_k = N_i$ for every $i \geq k$. These modules are extensively studied by Smith and Vedadi [11]. A module M is called iso-Noetherian (resp., iso-Artinian) if M satisfies iso-ACC (resp., iso-DCC) on submodules, i.e., for every ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of submodules of M, there exists an index $k \geq 1$ such that N_k is isomorphic to N_i for every $i \geq k$, see [5]. Also, a ring R is called right iso-Noetherian (resp., iso-Artinian) if the right module R_R is iso-Noetherian (resp., iso-Artinian). Recently, Chaturvedi and Prakash [3] introduced and studied the chain conditions up to isomorphism on essential and non-essential submodules as a generalization of iso-Noetherian (resp., iso-Artinian) modules. An R-module M is called nei-Noetherian (resp., nei-Artinian), if M satisfies iso-ACC (resp., iso-DCC) on essential (resp., non-essential) submodules. The authors of [10], introduced the concept of parallel Krull dimension of a module which is Krull-like dimension extension of the concept of DCC on the poset of submodules parallel to itself. The notion of type dimension, which is a generalization of Goldie dimension, first appeared in [14], where many of its properties were proved. Then, it was subsequently used throughout [13]. By then, its usefulness was apparent. Motivated by these papers, in this research we study modules with chain conditions on non-parallel submodules. We say that a module M is np-Noetherian (resp., npi-Noetherian) if M has ACC (resp., iso-ACC) on its parallel submodules. Dually, we introduce np-Artinian (resp., npi-Artinian) modules as a generalization of ne-Artinian (resp., nei-Artinian) modules. In Section 2, we first recall some known definitions and terminologies about type theory which we need in the sequel. In Section 3, we investigate some properties of np-Artinian (resp., np-Noetherian) modules and rings. First, we give an example of an np-Artinian (resp., np-Noetherian) module which is neither ne-Noetherian nor ne-Artinian. We provide characterization theorems for both np-Artinian and np-Noetherian modules and we show that any np-Artinian (np-Noetherian) module has finite type dimension. But we provide an example of a module which has finite type dimension while it is not np-Artinian nor np-Noetherian. Also, in the other main result of this section, we show that every np-Artinian module is either atomic or finitely embedded. We give a sufficient condition for np-Noetherian (resp., np-Artinian) modules to be Noetherian (resp., Artinian). It is easy to see that np-Noetherian (resp., np-Artinian) modules are closed under submodule, but we give some examples to show that these modules are not closed under factor module, finite direct sum and essential extension. However, we show that if N is a type submodule of an np-Noetherian (resp., np-Artinian) module, then $\frac{M}{N}$ is np-Noetherian (resp., np-Artinian). Among other results, we prove that a module M is Artinian (resp., Noetherian) if and only if M is np-Artinian (resp., np-Noetherian) and M satisfies DCC (resp., ACC) on parallel submodules. In Section 4, we investigate some properties of npi-Artinian (resp., npi-Noetherian) modules and rings. We provide structure theorems for these modules and, as a generalization of a result in the previous section, we show that npi-Noetherian modules have finite type dimension but by an example, we show that the converse of this fact is not true in general. We give some examples that illustrate the differences between npi-Artinian (resp., npi-Noetherian) and iso-Artinian (resp., iso-Noetherian) modules. Also, we provide other examples to show that npi-Noetherian (resp., npi-Artinian) modules need not be nei-Noetherian nor np-Artinian. Further, like similar np-Noetherian modules, we show that npi-Noetherian modules are closed under submodule, but we give some examples to show that these modules are not closed under factor module, finite direct sum and essential extension. We prove that npi-Artinian modules have an essential submodule that is a direct sum of atomic submodules. Next, we investigate some properties of npi-Artinian non-atomic modules. For example, we show that such modules have iso-simple submodules and, in a particular case, iso-socle of these modules is an essential submodule. Among other results, we show that if R is a right Rickart np-Artinian (resp., np-Noetherian) ring, then either R is right iso-Artinian (resp., iso-Noetherian) or it has many parallel right ideals. In Section 5, we focus on semiprime right np-Artinian (resp., npi-Artinian) rings. First, we prove that any right np-Artinian ring R with a non-atomic Jacobson radical is semiprime if and only if $J(R) = 0$. Using this we are able to prove that any right ne-Artinian ring with non-uniform Jacobson radical is semiprime if and only if it is semisimple. In the main theorem of this section, we generalize the latter result for right np-Artinian rings. In fact, we prove that if R is a semiprime ring such that $J(R)$ is not atomic, then R is right np-Artinian if and only if it is semisimple. Next, we study semiprime npi-Artinian rings. For a right npi-Artinian ring R with non-atomic iso-radical, we show that R is semiprime if and only if $\text{I-rad}(R_R) = 0$. Moreover, we show that if R is a semiprime right npi-Artinian ring, then either

$Z(R)$ is atomic or R is right non-singular. Finally, we investigate when ne-Artinian (resp., ne-Noetherian) and np-Artinian (resp., np-Noetherian) rings coincide. In particular, as a classical case, we show that for semiprime right duo rings, any right np-Artinian ring is right np-Noetherian.

2. Preliminaries

In this section we give some preliminary results that are needed in the sequel.

Definition 2.1. [4, Definition 4.1.1] Two modules N and P are orthogonal, written as $N \perp P$, if they do not have non-zero isomorphic submodules. Modules N_1 and N_2 are called parallel, denoted as $N_1 \parallel N_2$, if there does not exist a $0 \neq V_2 \subseteq N_2$ with $N_1 \perp V_2$, and also there does not exist a $0 \neq V_1 \subseteq N_1$ such that $N_2 \perp V_1$.

Now, we consider a particular case of the previous definition. Let M be a module and N be a submodule of M. If $N \parallel M$ (resp., $N \nparallel M$), then we say that N is a parallel (resp., non-parallel) submodule of M. A non-zero module M is called atomic, if every submodule of M is a parallel submodule. It is not hard to verify that a non-zero module M is atomic if and only if M does not have non-zero orthogonal submodules if and only if all non-zero submodules of M are parallel to each other. The following characterization of parallel submodules is easy to prove.

Lemma 2.2. *Let M be a module and N be any non-zero submodule of M. The following statements are equivalent.*

1. *N is a parallel submodule of M.*
2. *For any non-zero submodule M' of M we have $M' \not\perp N$.*
3. *For any non-zero submodule M' of M, there exist non-zero submodules M' of M and N' of N such that $M'' \cong N'$.*
4. *For any non-zero submodule M' of M, there exist cyclic submodules aR of M' and bR of N such that $aR \cong bR$.*

Remark 2.3. Let M be a module. It is clear that N is an essential submodule of M if and only if for any non-zero submodule K of M, N and K have a non-zero equal submodule. By part (3) of the above lemma, N is a parallel submodule of M if and only if for any non-zero submodule K of M, N and K have a non-zero isomorphic submodule. Hence, if K is an essential submodule of M, then K is a parallel submodule of M. In particular, any uniform module is atomic. However, the converse of these facts is not true in the general case, for instance, see Example 3.3(3).

In the following lemma, we list some basic properties of parallel submodules, see [10, Lemma 2.2].

Lemma 2.4. *Let N, K and L are submodules of M as an R-module. Then the following facts hold.*

(1) *If $N \cong K$, then $N \parallel K$.*
(2) *If $N \parallel K$, then $N \not\perp K$.*
(3) *If $N \parallel K$ and $K \parallel L$, then $N \parallel L$.*
(4) *If $N \parallel K$ and $L \perp N$, then $L \perp K$.*
(5) *Suppose that $L \subseteq K \subseteq N$. Then $L \parallel N$ if and only if $L \parallel K$ and $K \parallel N$.*

Recall that a non-zero module M is called iso-simple if every non-zero submodule of M is isomorphic to M. By part (1) of the above lemma, we have the following chart of basic implications for modules;

$$\text{simple} \Rightarrow \text{iso-simple} \Rightarrow \text{uniform} \Rightarrow \text{atomic}.$$

It is not hard to verify that all implications are irreversible.

Definition 2.5. [4, Definition 4.1.2] A submodule P of a module M is called a type submodule, denoted as $P \subseteq_t M$, if the following equivalent conditions hold:

(1) If $P \subseteq Y \subseteq M$ with $P \parallel Y$, then $P = Y$.
(2) If $P \subseteq Y \subseteq M$, then $P \perp X$ for some $0 \neq X \subseteq Y$.
(3) P is a complement submodule of M such $P \oplus D \subseteq_e M$ and $P \perp D$ for some $D \subseteq M$.

The following lemma is similar to [14, Proposition 2]. But whose proof is given for the sake of completeness.

Lemma 2.6. *Let M be a module and N be any submodule of M. Then there exists a type submodule P of M such that it is maximal with respect to $N \subseteq P$ and $N \parallel P$.*

Proof. Let N be a submodule of M. Define $\mathcal{A} = \{K \subseteq M \mid N \subseteq K,\ N \parallel K\}$. Then it is clear that $\mathcal{A} \neq \emptyset$. Let $\mathcal{C} = \{L_i\}_{i \in I}$ be a chain of elements of $\mathcal{A}$. Put $L = \bigcup_{i \in I} L_i$. Clearly, $N \subseteq L_i \subseteq L$ and $N \parallel L_i$ for all $i \in I$. We claim that $N \parallel L$. If not, then $L' \perp N$ for some non-zero submodule L' of L. Certainly, there is $j \in I$ such that $L' \subseteq L_j$. Since $N \parallel L_j$, for any non-zero submodule L'_j of L_j we have $N \not\perp L'_j$ which is a contradiction. Therefore, $L \in \mathcal{A}$ and hence $\mathcal{A}$ has a maximal element P, by Zorn's lemma. Now, we show that P is a type submodule of M. Let $N \subseteq P \subseteq P' \subseteq M$ and $P \parallel P'$. Since $P \parallel N$, by Lemma 2.4(3), we have $N \parallel P'$. Hence, the maximality of P implies that $P = P'$. □

Lemma 2.7. [4, Lemma 4.1.7] *Suppose that $A_1, \ldots, A_n$ are pairwise orthogonal atomic submodules of a module M with $A_1 \oplus \cdots \oplus A_n \subseteq_e M$. If $B_1, \ldots, B_n$ are non-zero pairwise orthogonal submodules of M, then $m \leq n$.*

Definition 2.8. A module M has finite type dimension n, denoted by $t.\dim M = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. If such n does not exist, we say that the type dimension of M is infinite, and write $t.\dim M = \infty$. If $M = 0$, then $t.\dim M = 0$.

The following lemma follows directly from the previous definition.

Lemma 2.9. *Let M be a module.*

(1) *M is atomic if and only if $t.\dim M = 1$.*
(2) *$t.\dim M = \infty$ if and only if there exist an infinite number of pairwise orthogonal non-zero submodules of M.*

Proposition 2.10. [4, Lemma 4.1.10] *Let M be a module and N be a non-zero submodule of M. Then the following facts hold.*

(1) *If $M = M_1 \oplus \cdots \oplus M_n$, then $t.\dim M \leq t.\dim M_1 + \cdots + t.\dim M_n$.*
(2) *Let $M = M_1 \oplus \cdots \oplus M_n$. If $M_i \perp M_j$ for all $i \neq j$, then $t.\dim M = t.\dim M_1 + \cdots + t.\dim M_n$. The converse holds if $t.\dim M < \infty$.*
(3) *If $N \parallel M$, then $t.\dim M = t.\dim N$.*
(4) *If $t.\dim M < \infty$, then $t.\dim M = t.\dim N$ implies that $N \parallel M$.*

3. Chain conditions on non-parallel submodules

In this section, we study some properties of modules which satisfy chain conditions on non-parallel submodules. We begin with the following definition.

Definition 3.1. A module M is said to be np-Noetherian (resp., np-Artinian) if M satisfies ACC (resp., DCC) on non-parallel submodules. A ring R is right np-Noetherian (resp., np-Artinian) if R_R is np-Noetherian (resp., np-Artinian).

Remark 3.2. By Lemma 2.4(5), it is easy to see that every submodule of an np-Noetherian (resp., np-Artinian) module is np-Noetherian (resp., np-Artinian). But in Example 3.15 we show that every factor module of an np-Noetherian (resp., np-Artinian) module need not be an np-Noetherian (resp., np-Artinian) module in general.

Example 3.3.

(1) Evidently each atomic module is both np-Noetherian and np-Artinian. But it is easy to see that the converse is not true in general. As an example, consider the Abelian group $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$. Then $t.\dim M = 2$, by [4, Example 4.1.9(1)]. It follows that M is not atomic, see Lemma 2.9. But M is both np-Noetherian and np-Artinian.

(2) It is well-known that $\mathbb{Q}$ is a uniform $\mathbb{Z}$-module and hence it is atomic, by Remark 2.3. Therefore, $\mathbb{Q}_{\mathbb{Z}}$ is both np-Noetherian and np-Artinian. Although we note that $\mathbb{Q}_{\mathbb{Z}}$ is neither Noetherian nor Artinian.

(3) Every ne-Noetherian (resp., ne-Artinian) module is np-Noetherian (resp., np-Artinian), see Remark 2.3. However, the converse is not true in the general case. For example, consider $M = \bigoplus_{i>0} \mathbb{Z}_{p^i}$ as a $\mathbb{Z}$-module. By [10, Example 2.3], M is atomic, so it is np-Noetherian and np-Artinian. But it is clear that M has infinite Goldie dimension. It follows that there exists an infinite strictly ascending (resp., descending) chain of essentially closed submodules in M, by [9, Propositions 6.30 and 6.32]. Note that any closed submodule is non-essential, hence M is neither ne-Noetherian nor ne-Artinian.

We say that a module M is orthogonal decomposable if $M = N_1 \oplus N_2$ for some submodules N_1, N_2 of M with $N_1 \perp N_2$.

We first give a characterization theorem for np-Artinian modules.

Theorem 3.4. *Let M be an R-module. The following statements are equivalent.*

(1) *M is np-Artinian.*

(2) *Every non-empty family of non-parallel submodules of M has a minimal element.*

(3) *Every non-empty chain of non-parallel submodules of M has a minimal element.*

(4) *Every non-parallel submodule of M is Artinian.*

(5) *Every proper type submodule of M is Artinian.*

(6) *Every orthogonal decomposable submodule of M is Artinian.*

Proof. (1) $\Rightarrow$ (2) On the contrary, assume that there exists a non-empty family of non-parallel submodules $\mathcal{F}$ such that $\mathcal{F}$ does not have minimal element. Since $\mathcal{F}$ is non-empty, there exists $N \in \mathcal{F}$. By our hypothesis, there exists $N_1 \in \mathcal{F}$ such that $N \supsetneq N_1$. Since N_1 is not minimal, there exists $N_2 \in \mathcal{F}$ such that $N_1 \supsetneq N_2$. By continuing this process, we will get an infinite descending chain $N \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots$ of non-parallel submodules of M which is a contradiction.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let N be any non-parallel submodule of M. Note that N is Artinian if and only if every chain of submodules of N has a minimal element. Let $\mathcal{N} = \{N_i\}_{i \in I}$ be a chain of submodules of N. Since N is a non-parallel submodule of M, by Lemma 2.4(5), N_i is a non-parallel submodule of M for all $i \in I$. Now, $\mathcal{N}$ is a chain of non-parallel submodules of M, so by (3), there exists $j \in I$ such that

$N_i = N_k$ for all $i \geq k$; i.e., $\mathcal{N}$ has a minimal element. Thus N is Artinian.
(4) $\Rightarrow$ (5) Because every proper type submodule of M is non-parallel in M.
(5) $\Rightarrow$ (6) Let N be an orthogonal decomposable submodule of M. Then $N = N_1 \oplus N_2$ for some non-zero submodules N_1, N_2 with $N_1 \perp N_2$. By Lemma 2.6, for $i = 1, 2$, there exists a type submodule K_i of M such that $N_i \subseteq K_i$ and $N_i \parallel K_i$. We claim that $K_1 \perp K_2$. If not, there exist non-zero submodules K_1' and K_2' of K_1 and K_2, respectively, with $K_1' \cong K_2'$. Since $N_1 \parallel K_1$, there exists a non-zero submodule N_1' of N_1 such that $K_1' \cong N_1'$. Similarly, there exists a non-zero submodule N_2' of N_2 such that $K_2' \cong N_2'$. Therefore, we get $N_1' \cong N_2'$ which is a contradiction. Now, it is clear that K_1, K_2 are proper type submodules of M. Hence, by (5), they are Artinian, so N is Artinian.
(6 $\Rightarrow$ 1) Let $N_1 \supseteq N_2 \supseteq \cdots$ be any descending chain of non-zero non-parallel submodules of M. Since $N_1 \nparallel M$, there exists a non-zero submodule M' of M such that $M' \perp N_1$. Then $M' \oplus N_1$ is an orthogonal decomposable submodule of M, so it is Artinian, by (4). Therefore, N_1 is Artinian and it follows that there exists an index $k \geq 1$ such that $N_k = N_i$ for every $i \geq k$. Consequently, M is np-Artinian. □

Theorem 3.5. *Every np-Artinian module has finite type dimension.*

Proof. Suppose that M is an np-Artinian module with infinite type dimension. By Lemma 2.9(2), there is a family $\{N_i\}_{i=1}^{\infty}$ of pairwise orthogonal non-zero submodules of M. It is easy to see that we have a direct sum $\bigoplus_{i=1}^{\infty} N_i$ of non-zero submodules of M. Clearly this gives the following infinite descending chain

$$\bigoplus_{i=2}^{\infty} N_i \supsetneq \bigoplus_{i=3}^{\infty} N_i \supsetneq \bigoplus_{i=4}^{\infty} N_i \supsetneq \cdots \tag{1}$$

of submodules of M. We claim that $\bigoplus_{i=j}^{\infty} N_i \nparallel \bigoplus_{i=j-1}^{\infty} N_i$ for all $j \geq 3$. If not, then any non-zero submodule of $\bigoplus_{i=j-1}^{\infty} N_i$ is not orthogonal to $\bigoplus_{i=j}^{\infty} N_i$. In particular, $N_{j-1} \not\perp \bigoplus_{i=j}^{\infty} N_i$. Then there exist non-zero submodules N_{j-1}' of N_j and N' of $\bigoplus_{i=j}^{\infty} N_i$ such that $N_{j-1}' \cong N'$. But $N' \cap N_k \neq (0)$ for some $k \geq j$. It follows that N_j and N_{j-1} have isomorphic submodules, which is a contradiction. Therefore, by Lemma 2.4(5), the chain (1) is a chain of non-parallel submodules of M which is a contradiction. Consequently, M has finite type dimension. □

A module M is finitely embedded (or finitely cogenerated), if its socle is essential and finitely generated. For the next result we need the following characterization of finitely embedded modules, whose proof is given for the sake of completeness.

Proposition 3.6. *A module is finitely embedded if and only if it has an essential Artinian submodule.*

Proof. Let M be a finitely embedded module. Then it is clear that $\mathrm{soc}(M)$ is an essential Artinian submodule of M. Conversely, let N be an essential Artinian submodule of M. Note that $\mathrm{soc}(M) \subseteq N$, so it is Artinian. Therefore, $\mathrm{soc}(M)$ is a semisimple Artinian submodule of M. Hence, it is finitely generated. Now, we show that $\mathrm{soc}(M) \subseteq_e M$. Let K be a non-zero submodule of M. Since $N \cap K$ is Artinian, it has a simple submodule. It follows that $(N \cap K) \cap \mathrm{soc}(M) \neq 0$, so $\mathrm{soc}(M) \cap K \neq 0$. □

We have the following interesting result.

Theorem 3.7. *Every np-Artinian module is either atomic or finitely embedded.*

Proof. Suppose that M is not atomic. By Theorem 3.5, M has finite type dimension. Hence, there exists a positive integer $n \geq 2$ such that M contains an essential submodule of the form $A = A_1 \oplus \cdots \oplus A_n$, where A_i's are pairwise orthogonal atomic submodules of M. It is clear that A_i is a non-parallel submodule of M, so by Theorem 3.4, A_i is Artinian for each $1 \leq i \leq n$. Therefore, A is Artinian. Now, Proposition 3.6 implies that M is finitely embedded and we are done. □

Corollary 3.8. *Let M be an np-Artinian module.*

(1) *Any non-atomic submodule of M contains a simple submodule.*
(2) *Any non-zero submodule of M contains an atomic submodule.*
(3) *The intersection of all maximal type submodules of M is equal to 0.*

Proof. (1) Let N be a non-atomic submodule of M. Then N is np-Artinian, by Remark 3.2. Hence, the previous theorem implies that $\mathrm{soc}(N) \subseteq_e N$. It follows that N contains a simple submodule.
(2) If M is not atomic, then by (1) we are done. If M is atomic, then using Lemma 2.4(3), it is easy to see that every non-zero submodule of M is atomic.
(3) Follows by (2) and [13, Lemma 3.1]. □

A commutative ring R is called locally Noetherian if each localization $R_{\mathfrak{m}}$ is Noetherian for all maximal ideals $\mathfrak{m}$. It is well known that over locally Noetherian rings, finitely embedded modules are Artinian, see [12, Theorem 2].

Corollary 3.9. *Let R be a locally Noetherian ring. Then an R-module M is np-Artinian if and only if M is either atomic or Artinian.*

Proof. If M is not atomic, then Theorem 3.7 implies that M is finitely embedded, so by the above comment, M is Artinian. The converse is clear. □

Proposition 3.10. [2, Theorem 4.6] *Let R be a Noetherian ring and assume that the right socle of R is essential as a right ideal or as a left ideal, then R is Artinian.*

Corollary 3.11. *Let R be a Noetherian ring. Then R is right np-Artinian if and only if R is either right atomic or Artinian.*

Proof. If R is not right atomic, then Proposition 3.6 implies that R_R is finitely embedded, so $\mathrm{soc}(R_R) \subseteq_e R_R$. Therefore, R is an Artinian ring, by the previous proposition. The converse is clear. □

Next, we investigate np-Noetherian modules. Let us start with an analogue of Theorem 3.4 which characterizes np-Noetherian modules.

Theorem 3.12. *Let M be an R-module. The following statements are equivalent.*

(1) *M is np-Noetherian.*
(2) *Every non-empty family of non-parallel submodules of M has a maximal element.*
(3) *Every non-empty chain of non-parallel submodules of M has a maximal element.*
(4) *Every non-parallel submodule of M is Noetherian.*
(5) *Every non-parallel submodule of M is finitely generated.*
(6) *Every proper type submodule of M is Noetherian.*
(7) *Every orthogonal decomposable submodule of M is finitely generated.*

In this case, M has finite type dimension.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (6) ⇒ (7) Similar to the proof of Theorem 3.4.
(7) ⇒ (5) Let N be a non-parallel submodule of M. By Lemma 2.2, there exists a non-zero submodule M' of M such that $M' \perp N$. Hence, $N \oplus M'$ is an orthogonal decomposable submodule of M, so by (7), it is clear that $N \cong \frac{M' \oplus N}{M'}$ is finitely generated.
(5) ⇒ (1) First, we show that M has finite type dimension. On the contrary, suppose that M has infinite type dimension. Then there is a family $\{H_i\}_{i=1}^{\infty}$ of pairwise orthogonal non-zero submodules of M, by Lemma 2.9(2). Hence, $H_1 \oplus H_2 \oplus H_3 \oplus \cdots$ be a direct sum of non-zero submodules of M. Let $H = H_2 \oplus H_3 \oplus \cdots$. We have to show that $H_1 \perp H$. Otherwise, there are non-zero submodules H_1' of H_1 and H' of H such that $H_1' \cong H'$. Note that $K = H' \cap H_k \neq (0)$ for some $k \geq 2$. Since K is a submodule of H', from $H_1' \cong H'$ we easily get H_1 and H_k have isomorphic submodule which is a contradiction. Therefore, H is a non-parallel submodule of M, so it is finitely generated, by (3). Hence, $H \subseteq H_2 \oplus \cdots \oplus H_k$ for some integer $k \geq 2$, so $H_{k+1} = 0$ which is a contradiction. It follows that M has finite type dimension.
Now, let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be any ascending chain of non-parallel submodules

of M. It is clear that

$$t.\dim(K_1) \leq t.\dim(K_2) \leq t.\dim(K_3) \leq \cdots \leq t.\dim(M) < \infty.$$

There exists a positive integer t such that $t.\dim(K_t) = t.\dim(K_{t+1})$. Since $K_t \nparallel M$, there exists a non-zero submodule M' of M such that $M' \perp K_t$. Since for each $i \geq t$ we have $t.\dim(K_t) = t.\dim(K_i)$, Proposition 2.10(4) implies that $K_t \parallel K_i$, so $M' \perp K_i$, by Lemma 2.4(4). Therefore, it is not hard to check that $M' \perp \bigcup_{i \geq t} K_i$. Hence, $\bigcup_{i \geq t} K_i$ is a non-parallel submodule of M, so it is finitely generated, by (3). This implies that $K_s = K_{s+1} = \cdots$ for some $s \geq t$. This proves (1). □

By Theorem 3.5 and the previous theorem, np-Artinian (resp., np-Noetherian) modules have finite type dimension. In the following example, we show that the converse of these facts is not true in general.

Example 3.13. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. It is clear that $\mathbb{Z} \perp \mathbb{Z}_{p^\infty}$. Hence, by Proposition 2.10(2) and Lemma 2.9(1), we have

$$t.\dim M = t.\dim \mathbb{Z} + t.\dim \mathbb{Z}_{p^\infty} = 1 + 1 = 2.$$

Now, consider $N = 2\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$. Then it is clear that N is an orthogonal decomposable submodule of M which is not Artinian nor Noetherian. Therefore, Theorems 3.4 and 3.12 imply that M is not np-Artinian nor np-Noetherian.

It is easy to see that Example 3.13 also shows that a direct sum of np-Artinian (resp., np-Noetherian) modules are not np-Artinian (resp., np-Noetherian) in general. Next, we show that npi-Noetherian modules are not closed under essential extension.

Example 3.14. Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. It is easy to see that $E(M) = \mathbb{Z}_{2^\infty} \oplus \mathbb{Z}_{3^\infty}$. Also, note that $\mathbb{Z}_{2^\infty}$ is a non-parallel submodule of $E(M)$ which is not Noetherian. Hence, Theorem 3.12 implies that $E(M)$ is not np-Noetherian.

Example 3.15. By Example 3.3(2), $\mathbb{Q}_\mathbb{Z}$ is np-Noetherian and np-Artinian both. Consider the factor module $\frac{\mathbb{Q}}{\mathbb{Z}}$ of $\mathbb{Q}_\mathbb{Z}$. Then it is well-known that $\frac{\mathbb{Q}}{\mathbb{Z}} \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{p^\infty}$, where $\mathbb{P}$ is the set of all prime numbers. Note that for each distinct prime number p_i and p_j, the order of any subgroup of $\mathbb{Z}_{p_i^\infty}$ and $\mathbb{Z}_{p_j^\infty}$ is p_i^n and p_j^m for some non-negative integer n, m, respectively. It follows that $\{\mathbb{Z}_{2^\infty}, \mathbb{Z}_{3^\infty}, \mathbb{Z}_{5^\infty}, \ldots\}$ is a set of pairwise orthogonal non-zero submodules of $\frac{\mathbb{Q}}{\mathbb{Z}}$. Hence, $\frac{\mathbb{Q}}{\mathbb{Z}}$ has infinite type dimension, by Lemma 2.9(2). Consequently, $\frac{\mathbb{Q}}{\mathbb{Z}}$ is neither np-Artinian nor np-Noetherian, by the previous theorem and Theorem 3.5, respectively.

Example 3.15 shows that every factor module of an np-Noetherian (resp., np-Artinian) module need not be an np-Noetherian (resp., np-Artinian) module in general. In the following proposition we find a class of submodules whose factors satisfy this property.

Proposition 3.16. *Let M be an np-Noetherian (resp., np-Artinian) module and $N \subseteq_t M$. Then $\frac{M}{N}$ is np-Noetherian (resp., np-Artinian).*

Proof. Suppose that N is a type submodule of an np-Noetherian module M. By Theorem 3.12, it suffices to show that every type submodule of $\frac{M}{N}$ is Noetherian. Note that type submodules of $\frac{M}{N}$ are exactly of the form $\frac{K}{N}$, where K is a type submodule of M which contains N, see [4, Lemma 4.3.19 (1)]. By Theorem 3.12, N and K are Noetherian. It follows that $\frac{K}{N}$ is Noetherian and we are done. □

According to [10, Section 3], pk-dim$(M) = 0$ (resp., pn-dim$(M) = 0$) if and only if M satisfies DCC (resp., ACC) on parallel submodules.

Proposition 3.17. *Let M be an R-module.*

(1) *M is Artinian if and only if pk-dim$(M) = 0$ and M is np-Artinian.*
(2) *M is Noetherian if and only if pn-dim$(M) = 0$ and M is np-Noetherian.*

Proof. (1) Let M be an np-Artinian module with pk-dim$(M) = 0$. Suppose that $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules of M. If each N_i, $i = 1, 2, \ldots$, is a parallel submodule of M, then nothing to prove. Hence, assume that there exists $k \geq 1$ such that $N_k \nparallel M$. Then there exists a non-zero submodule M' of M such that $M' \perp N_k$, by Lemma 2.2. Clearly it follows that for each $i \geq k$, $M' \perp N_i$, so by Lemma 2.4(2), we have $N_i \nparallel M$. Since M is np-Artinian, there exists $t \geq k$ such that $N_t = N_{t+1} = N_{t+2} = \cdots$. Therefore, M is Artinian. The converse is clear.
(2) Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. If each N_i, $i = 1, 2, \ldots$, is a non-parallel submodule of M, then nothing to prove. Suppose that there exists $k \geq 1$ such that $N_k \parallel M$. Then, by Lemma 2.4 (5), $N_i \parallel M$ for all $i \geq k$. Now, Since M is p-Noetherian, there exists $t \geq k$ such that $N_t = N_{t+1} = N_{t+2} = \cdots$. Therefore, M is Noetherian. The converse is clear. □

Next, we give some various sufficient conditions for np-Noetherian (resp., np-Artinian) modules to be Noetherian (resp., Artinian).

Let M be an R-module. It is defined that $T(M) = \bigcap_{i \in I} N_i$, where M_i's are parallel submodules of M, see [10]. Note that the socle of a module M is the intersection of all essential submodules of M. Hence, by Remark 2.3, $T(M) \subseteq \mathrm{soc}(M)$.

Proposition 3.18. *Let M be a module such that $T(M) \nsubseteq \mathrm{rad}(M)$.*

(1) *M is np-Noetherian if and only if M is Noetherian.*

(2) *M is np-Artinian if and only if M is Artinian.*

Proof. (1) Suppose that M is np-Noetherian with $T(M) \nsubseteq \text{rad}(M)$. Then there exists a maximal submodule N of M such that $N \nparallel M$ (otherwise, we have $T(M) \subseteq \text{rad}(M)$ which is a contradiction). Hence, $\frac{M}{N}$ is a simple module, so it is Noetherian. Since $N \nparallel M$, Theorem 3.12 implies that N is a Noetherian submodule of M. Therefore, M is a Noetherian module. The converse is clear.
(2) It is similar to (1). □

Note that for any commutative ring R we have $\text{Nil}(R) = \text{Nil}_*(R)$. Now, the following is immediate.

Corollary 3.19. *Let R be a commutative zero-dimensional ring such that $T(R_R)$ contains a non-nilpotent element. Then R is an np-Noetherian ring if and only if R is Noetherian.*

Lemma 3.20. *Let R be a ring such that either*

(1) *R is not right Artinian but it is np-Artinian, or*

(2) *R is not right Noetherian but it is np-Noetherian.*

If $c \in R$, then cR is a parallel right ideal of R or $\mathbf{r}(c)$ is a parallel right ideal of R.

Proof. Suppose that (1) holds (the proof for (2) is similar). Suppose that cR is a non-parallel right ideal of R. By Theorem 3.4, cR is Artinian. Since $cR \cong \frac{R}{\mathbf{r}(c)}$, it is easy to see that $\mathbf{r}(c)$ is not Artinian. Hence, $\mathbf{r}(c)$ is a parallel right ideal of R. □

Analogous to [11], we shall say that a ring R has many parallel right ideals, if for every element a in R, aR, is a parallel right ideal or $\mathbf{r}(a)$ is a parallel right ideal of R. Now, the following is immediate.

Corollary 3.21. *Let R be an np-Artinian (resp., np-Noetherian) ring. Then either R is right Artinian (resp., Noetherian) or R has many parallel right ideals.*

4. Chain conditions up to isomorphism

In this section, we investigate modules with ascending (resp., descending) chain condition up to isomorphism on non-parallel submodules and generalize some results of the previous section for these modules.

Definition 4.1. A right R-module M is said to be npi-Noetherian (resp., npi-Artinian) if for every ascending (resp., descending) chain $M_1 \subseteq M_2 \subseteq \dots$ (resp., $M_1 \supseteq M_2 \supseteq \dots$) of non-parallel submodules of M, there exists an index n such that M_i is isomorphic to M_n, for every $i \geq n$. We say that a ring R is right npi-Noetherian (resp., npi-Artinian) if R as an R-module is npi-Noetherian (resp., npi-Artinian).

It is clear that any iso-Noetherian (resp., iso-Artinian) module is npi-Noetherian (resp., npi-Artinian). But we note that $\mathbb{Z}_{p^\infty}$ is an npi-Noetherian $\mathbb{Z}$-module, which is not iso-Noetherian. Next, we give an example of an npi-Artinian ring which is not iso-Artinian.

Example 4.2. Let D be any commutative domain which is not field. Set $R = D[x]$. Then R is a domain, so it is npi-Artinian (in fact, R is atomic). Note that any domain is right iso-Artinian if and only if it is a principal right ideal domain, see [5, Section 2]. Since D is not field, R is not a PID. Consequently, R is not iso-Artinian.

Remark 4.3. Any nei-Noetherian module is npi-Noetherian, see Remark 3.2. But the converse is not true in general. For example, if we consider $M = \bigoplus_{i>0} \mathbb{Z}_{p^i}$ as a $\mathbb{Z}$-module. Then M is npi-Noetherian (resp., npi-Artinian), by Example 3.3(3). But M has infinite Goldie dimension, so [3, Proposition 2.12] implies that M is not nei-Noetherian.

Remark 4.4. By Lemma 2.4(5), it is easy to see that every submodule of an npi-Noetherian (resp., npi-Artinian) module is npi-Noetherian (resp., npi-Artinian). But in Remark 4.13, we see that this property does not hold for factors of npi-Noetherian modules.

Next, we give structure theorems for npi-Artinian and npi-Noetherian modules. Before this, let us recall the following definition from [7].

Definition 4.5. Let $\mathcal{F}$ be a non-empty set of submodules of M. Then $N \in \mathcal{F}$ is called an iso-maximal (resp., iso-minimal) element of $\mathcal{F}$ provided that for any $N \subseteq N'$ (resp., $N' \subseteq N$) if $N' \in \mathcal{F}$, then $N \cong N'$. In particular, if $\mathcal{F}$ is the set of all proper (resp., non-zero) submodules of M, then iso-maximal (resp., iso-minimal) elements of M are said to be iso-maximal (resp., iso-minimal or iso-simple) submodules of M.

Theorem 4.6. *Let M be an R-module. The following statements are equivalent.*

(1) *M is npi-Artinian.*
(2) *Every non-empty family of non-parallel submodules of M has an iso-minimal element.*
(3) *Every non-empty chain of non-parallel submodules of M has an iso-minimal element.*
(4) *Every non-parallel submodule of M is iso-Artinian.*

(5) *Every proper type submodule of M is iso-Artinian.*

Proof. (1) $\Rightarrow$ (2) Similar to the proof of (1) $\Rightarrow$ (2) of Theorem 3.4.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (5) Let N be any proper type submodule of M. Then it is clear that N is a non-parallel submodule of M. By the above definition, it is easy to see that N is iso-Artinian if and only if every chain of submodules of N has an iso-minimal element, see [5, Lemma 2.1]. Now, the rest of the proof is similar to the proof of (3) $\Rightarrow$ (4) of Theorem 3.4.
(5) $\Rightarrow$ (4) Let N_1 be a non-parallel submodule of M. If N_1 is a type submodule of M, then we are done. If not, by Lemma 2.6, there exists a type submodule N of M such that $N \parallel N_1$. It is clear that N is a proper type submodule of M (otherwise, N_1 becomes parallel in M). Consider a descending chain $N_2 \supseteq N_3 \supseteq \cdots$ of submodules of N_1. Then we have a descending chain $N \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of submodules of N. Since, by (3), N is iso-Artinian, there exists $k \in \mathbb{N}$ such that $N_i \cong N_k$, for all $i \geq k$. Therefore, N_1 becomes iso-Artinian.
(4) $\Rightarrow$ (1) Let $M_1 \supseteq M_2 \supseteq \cdots$ be a descending chain of non-parallel submodules of M. Since M_1 is non-parallel, it is iso-Artinian. Therefore, there exists $n \in \mathbb{N}$ such that $M_i \cong M_n$, for all $i \geq n$. Thus, M is npi-Artinian. □

Dually, we have the following result for npi-Noetherian modules.

Theorem 4.7. *Let M be an R-module. The following statements are equivalent.*

(1) *M is npi-Noetherian.*
(2) *Every non-empty family of non-parallel submodules of M has an iso-maximal element.*
(3) *Every non-empty chain of non-parallel submodules of M has an iso-maximal element.*
(4) *Every non-parallel submodule of M is iso-Noetherian.*
(5) *Every proper type submodule of M is iso-Noetherian.*

It is clear that np-Artinian (resp., np-Noetherian) modules are npi-Artinian (resp., npi-Noetherian). But in the following example, we show that the converse is not true in general.

Example 4.8. Set $R = \mathbb{Z} \oplus \mathbb{Z}_4$. Then it is easy to verify that

$$\{0,\, 0 \oplus \mathbb{Z}_4,\, 0 \oplus \langle \bar{2} \rangle,\, n\mathbb{Z} \oplus \bar{0}\}$$

for any $n \in \mathbb{N}$, is the set of all non-parallel ideals of the ring R. Clearly, each non-parallel ideal of R is iso-Artinian (resp., iso-Noetherian), so Theorem 4.6 (resp., Theorem 4.7) implies that R is an npi-Artinian (resp., npi-Noetherian) ring. However, we note that $n\mathbb{Z} \oplus \bar{0}$ is not Artinian and hence R is not np-Artinian, by Theorem 3.4.

For the definition of the homogeneous parallel Krull dimension we refer to [10].

Proposition 4.9. *Let M be an npi-Noetherian module and $N \nparallel M$. If N has homogeneous parallel Krull dimension, then N has Krull dimension.*

Proof. Suppose that M is an npi-Noetherian module and N is a non-parallel submodule of M. Then N is iso-Noetherian, by the previous theorem. Now, [7, Proposition 3.7] implies that N has Krull dimension. □

Corollary 4.10. *Let M be an npi-Noetherian module and $N \nparallel M$. If every submodule of N has DCC on its parallel submodules, then N is Artinian.*

Proof. If N has DCC on its parallel submodules, then $pk\text{-dim}(N) = 0$. By the proof of the previous proposition, N has Krull dimension. Now, [10, Theorem 3.11] implies that $k\text{-dim}(N) = 0$. Therefore, N is Artinian. □

Proposition 4.11. [5, Proposition 5.1] *Any iso-Noetherian module has finite uniform dimension.*

Recall that we show that any np-Noetherian module has finite type dimension (see Theorem 3.12). In the next result, we present a generalization of this fact.

Proposition 4.12. *If M is an npi-Noetherian R-module, then M has finite type dimension.*

Proof. On the contrary, suppose that M is an npi-Noetherian module with infinite type dimension. By Lemma 2.9(2), there is a family $\{N_i\}_{i=1}^{\infty}$ of pairwise orthogonal non-zero submodules of M. Hence, we have a direct sum $\bigoplus_{i=1}^{\infty} N_i$ of non-zero submodules of M. But we note that $N = \bigoplus_{i=2}^{\infty} N_i$ is a non-parallel submodule of M (see the proof of the Theorem 3.5). Then N is iso-Noetherian, by Theorem 4.7. Therefore, the previous proposition implies that N has finite uniform dimension, so it has finite type dimension which is a contradiction. □

Remark 4.13.

(1) We have to show that the converse of the previuos proposition is not true in general. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. Then $t.\dim M = 2$, see Example 3.13. But it is easy to see that $\mathbb{Z}_{p^\infty}$ is a non-parallel submodule of M which is not iso-Noetherian. Thus, M is not npi-Noetherian, by Theorem 4.7.

(2) The previous part implies that npi-Noetherian modules are not closed under direct sums.
(3) It is clear that $\mathbb{Q}_{\mathbb{Z}}$ is npi-Noetherian. By Example 3.15, $\frac{\mathbb{Q}}{\mathbb{Z}}$ has infinite type dimension, so by the previous proposition, it is not npi-Noetherian, . Thus, npi-Noetherian modules are not closed under factor module.
(4) By Example 3.14, it is easy to see that npi-Noetherian modules are not closed under essential extension.

We have an analogue of Corollary 3.8 for npi-Artinian modules as follows.

Proposition 4.14. *Let M be an npi-Artinian module. Then*

(1) *Every non-zero submodule of M which is not atomic contains an iso-simple submodule. In particular, if M is non-atomic, then M contains an iso-simple submodule.*
(2) *Every non-zero submodule of M contains an atomic submodule.*
(3) *The intersection of all maximal type submodules of M is equal to 0.*

Proof. (1) Let M be an npi-Artinian module and N be a non-atomic submodule of M. Hence, N has a non-parallel submodule. Suppose that $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ is a descending chain of non-parallel submodules of N. By Remark 4.4, N is npi-Artinian, so there exists $k \in \mathbb{N}$ such that $N_i \cong N_k$, for all $i \geq k$. Since N_k is non-parallel, Theorem 4.6 implies that N_k is iso-Artinian. Note that any iso-Artinian module contains an iso-simple submodule, see [5, Lemma 2.1]. Therefore, N_k contains an iso-simple submodule.
(2) Let N be any non-zero submodule of M. If N is not atomic, then since any iso-simple submodule is atomic, by (1) we are done. If N is atomic, then using Lemma 2.4(3), it is easy to see that every non-zero submodule of M is atomic.
(3) Follows by (2) and [13, Lemma 3.1]. □

Theorem 4.15. *Let M be an npi-Artinian module. Then M has an essential submodule that is a direct sum of atomic submodules.*

Proof. Suppose that M is an npi-Artinian module. By the previous proposition, M contains at least one atomic submodule. Let S denotes the set of all families of independent atomic submodules of M. By Zorn's Lemma, S has maximal member $W = \{V_i \mid i \in I\}$. Put $V = \bigoplus_{i\in I} V_i$. Now, let N be a non-zero submodule of M. Since M is an npi-Artinian module, by Remark 4.4, N is npi-Artinian. Hence, N has an atomic submodule K, by the previous proposition. Thus, we have $K \cap V \neq (0)$ (since otherwise, $W \cup \{K\}$ is a set of independent atomic submodules which contradicts with the maximality of W). It follows that $N \cap V \neq 0$. Hence,

V is an essential submodule of M which is a direct sum of atomic submodules of M. □

Let $\mathcal{U}$ denote the class of all iso-simple R-modules. A module M is called (finitely) generated by $\mathcal{U}$, or (finitely) $\mathcal{U}$-generated, in case there are a (finite) indexed set $(U_\alpha)_{\alpha\in A}$ in $\mathcal{U}$ and an epimorphism $\bigoplus_{\alpha\in A} U_\alpha \to M$. Dually, a module M is (finitely) cogenerated by $\mathcal{U}$, or (finitely) $\mathcal{U}$-cogenerated, in case there is an (finite) indexed set $(U_\alpha)_{\alpha\in A}$ in $\mathcal{U}$ and a monomorphism $M \to \prod_{\alpha\in A} U_\alpha$. Using the notation of [6], for any R-module M we set

$$\text{I-soc}(M) = \text{Tr}_M(\mathcal{U}) = \sum\{\text{Im}(f) \mid f \in \text{Hom}_R(U, M) \text{ for some } U \in \mathcal{U}\},$$
$$\text{I-rad}(M) = \text{Rej}_M(\mathcal{U}) = \bigcap\{\ker(g) \mid g \in \text{Hom}_R(U, M) \text{ for some } U \in \mathcal{U}\}.$$

In [6], it is proved that the (right) iso-radical I-rad(R_R) of a ring R is the intersection of the annihilators of all iso-simple R-modules. In Proposition 5.11, we provide a characterization of semiprime npi-Artinian rings using the notion of iso-radical.

Proposition 4.16. *Let M be an npi-Artinian module. Then I-*soc$(N) \neq 0$ *for each non-atomic submodule N of M. Moreover, if any non-zero submodule of M is non-atomic, then I-*soc(M) *is an essential submodule of M.*

Proof. Let N be a non-atomic submodule of M. By Remark 4.4, N is npi-Artinian, so N has an iso-simple submodule K, by Proposition 4.14. Therefore, I-soc$(N) \neq 0$. Now, suppose that K is a non-zero submodule of M. Since K is non-atomic, by [1, Proposition 8.16], we get

$$0 \neq \text{I-soc}(K) = K \cap \text{I-soc}(K) \subseteq K \cap \text{I-soc}(M) \subseteq \text{I-soc}(M).$$

Thus, I-soc$(M) \subseteq_e M$. □

Recall that a ring R is said to be right semihereditary, if every finitely generated right ideal of R is projective as an R-module. Also, a ring R is called a right Rickart ring, if the right annihilator of any element in R is of the form eR for some idempotent $e \in R$. For the next result, we recall the following characterization of right Rickart rings, which implies that right semihereditary rings are right Rickart.

Proposition 4.17. [9, Proposition 7.48] *A ring R is right Rickart if and only if every principal right ideal in R is projective (as an R-module).*

The following result is a generalization of Lemma 3.20 in the case of right Rickart rings, see also [3, Lemma 2.14].

Proposition 4.18. *Let R be right Rickart ring such that R is not right iso-Noetherian (iso-Artinian) but npi-Noetherian (npi-Artinian). Then, for $c \in R$ either cR is a parallel right ideal of R or $\mathbf{r}(c)$ is a parallel right ideal of R.*

Proof. Suppose that $cR \nparallel R$. By Theorem 4.7, cR is iso-Noetherian. Since $cR \cong \frac{R}{\mathbf{r}(c)}$, we may consider the short exact sequence $0 \to \mathbf{r}(c) \to R \to cR \to 0$. Now, Since R is right Rickart, by the previous proposition, cR is projective. Therefore, the short exact sequence splits i.e., $R \cong \mathbf{r}(c) \oplus cR$. If $\mathbf{r}(c)$ is right iso-Noetherian, then R becomes right iso-Noetherian, which is a contradiction. Therefore, $\mathbf{r}(c)$ is not right iso-Noetherian and hence, by Theorem 4.7, $\mathbf{r}(c)$ is a parallel right ideal of R. The proof of the npi-Artinian case is similar. □

Corollary 4.19. *Let R be a right Rickart np-Artinian (resp., np-Noetherian) ring. Then either R is right iso-Artinian (resp., iso-Noetherian) or it has many parallel right ideals.*

5. Semiprime np-Artinian rings

In this section, we give some properties of semiprime right np-Artinian (resp., npi-Artinian) rings. We begin with the following proposition.

Proposition 5.1. *Let R be a right np-Artinian ring such that $J(R)$ is not atomic. Then R is semiprime if and only if $J(R) = 0$.*

Proof. First, suppose that $J(R) = 0$. Since $\mathrm{Nil}_*(R) \subseteq J(R)$, it follows that R is semiprime. Conversely, assume that R is a semiprime right np-Artinian ring. On the contrary, suppose that $J(R) \neq 0$. Then $J(R)$ is right np-Artinian, by Remark 3.2. Since $J(R)$ is not atomic, Corollary 3.8 implies that $J(R)$ contains a minimal right ideal S. Note that $SJ(R) = 0$ and hence, $S^2 \subseteq SJ(R) = 0$. Since R is semiprime, it follows that $S = 0$, which is a contradiction. □

Corollary 5.2. *Let R be a semiprime right np-Artinian ring. Then either $J(R)$ is atomic or $T(R_R) = 0$.*

Proof. It follows from Propositions 3.18 and 5.1. □

By a similar argument, we can prove the following version of Proposition 5.1.

Lemma 5.3. *Let R be a right ne-Artinian ring such that $J(R)$ is not uniform. Then R is semiprime if and only if $J(R) = 0$.*

Now, we present an application of the previous result. First, let us recall a result from [11].

Lemma 5.4. [11, Theorem 2.1] *A semiprime ring R is right ne-Artinian if and only if R is right uniform or right Artinian.*

Proposition 5.5. *Let R be a right ne-Artinian ring such that $J(R)$ is not uniform. Then R is semiprime if and only if it is semisimple.*

Proof. Suppose that R is a semiprime right ne-Artinian ring. Then we have $J(R) = 0$, by Lemma 5.3. It is clear that R is not right uniform. Hence, the previous lemma implies that R is right Artinian. Now, R is a right Artinian ring with zero Jacobson radical, so R is semisimple, see [8, Theorem 4.14]. □

It is well known that for a right Artinian ring R, all three radicals $\mathrm{Nil}_*(R)$, $\mathrm{Nil}^*(R)$ and $J(R)$ coincide, for example see [9, Proposition 10.27].

Proposition 5.6. *Let R be any ring such that $\frac{R}{\mathrm{Nil}_*(R)}$ is a right np-Artinian ring. If $J\left(\frac{R}{\mathrm{Nil}_*(R)}\right)$ is not atomic, then $\mathrm{Nil}_*(R) = \mathrm{Nil}^*(R) = J(R)$.*

Proof. We know that for any ring R, $\mathrm{Nil}_*(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R)$. Thus, it suffices to show that $J(R) \subseteq \mathrm{Nil}_*(R)$. It is clear that $\frac{R}{\mathrm{Nil}_*(R)}$ is a semiprime ring, so by Proposition 5.1, we have $J\left(\frac{R}{\mathrm{Nil}_*(R)}\right) = (0)$. Note that $\mathrm{Nil}_*(R) \subseteq J(R)$. Hence, by [9, Proposition 4.6], we have $\frac{J(R)}{\mathrm{Nil}_*(R)} = (0)$. It follows that $\mathrm{Nil}_*(R) = \mathrm{Nil}^*(R) = J(R)$. □

Now, we give an analogue of Proposition 5.5 for np-Artinian rings as follows.

Theorem 5.7. *Let R be a semiprime ring such that $J(R)$ is not atomic. Then R is right np-Artinian if and only if it is semisimple.*

Proof. The sufficiency is clear. Conversely, suppose that R is a semiprime right np-Artinian ring with non-atomic Jacobson radical. By Proposition 5.1, we have $J(R) = 0$. Note that $J(R)$ is the intersection of all maximal right ideals of R, by Theorem 3.7, R is finitely embedded. It follows that there exist maximal right ideals $\mathfrak{m}_1, \ldots \mathfrak{m}_n$ of R such that $\bigcap_{i=1}^{n} \mathfrak{m}_i = 0$. Now, consider the map $\varphi : R \to \bigoplus_{i=1}^{n} \frac{R}{\mathfrak{m}_i}$ defined by $\varphi(r) = (r + \mathfrak{m}_1, \ldots, r + \mathfrak{m}_n)$. It is easy to see that φ is an R-monomorphism. Since each of $\frac{R}{\mathfrak{m}_i}$ is a simple R-module, we conclude that R is a semisimple ring. □

In view of the Theorem 5.7, Proposition 5.1 and Hopkins-Levitzki theorem (see [8, Theorem 4.15]), the following is immediate.

Corollary 5.8. *Let R be a semiprime right np-Artinian ring. If $J(R)$ is non-atomic, then for any R-module M the following are equivalent.*

(1) *M is Noetherian.*

(2) *M is Artinian.*

(3) *M has a composition series.*

Next, we investigate semiprime npi-Artinian rings. Recall that a module M is called Hopfian, if any surjective R-endomorphism of M is an automorphism. Note that any Noetherian module is Hopfian.

Theorem 5.9. *Let R be a semiprime right npi-Artinian ring. Then either $Z(R)$ is atomic or R is right non-singular.*

Proof. On the contrary, suppose $Z(R) \neq 0$. It is clear that $Z(R)$ is right npi-Artinian. Since $Z(R)$ is not atomic, Proposition 4.14 implies that $Z(R)$ contains a non-zero iso-simple right ideal I. Since R is semiprime, $I^2 \neq 0$. Thus, $aI \neq 0$ for some $a \in I$, so $I \cong aI \cong \frac{I}{\mathbf{r}(a)\cap I}$. Now, consider the surjective R-endomorphism $f_a : I \to aI$ defined by $f_a(x) = ax$. Note that iso-simple modules are Noetherian. Therefore, I is Hopfian and so, $\ker(f_a) = \mathbf{r}(a) \cap I = (0)$, which contradicts the fact that $\mathbf{r}(a)$ is an essential right ideal of R. Consequently, $Z(R) = 0$. □

Following [14], a module M is said to be a TS-module, if every type submodule of M is a summand of M.

Corollary 5.10. *Let R be a right npi-Artinian semiprime ring such that $Z(R)$ is not atomic. Then the following are equivalent.*

(1) *R_R has finite type dimension.*

(2) *The maximal right quotient ring of R is a finite direct sum of indecomposable right self-injective regular rings.*

(3) *For every family $\{M_i\}_{i\in I}$ of pairwise orthogonal non-singular modules we have $\bigoplus_{i\in I} E(M_i)$ is injective.*

(4) *Every cyclic (or finitely generated) non-singular module has finite type dimension.*

(5) *Every non-singular TS-module is a direct sum of atomic modules.*

(6) *Every non-singular injective module is a direct sum of atomic modules.*

(7) *Every non-singular module contains a maximal injective type submodule.*

Proof. It follows by the previous theorem and [13, Theorem 2.2]. □

Proposition 5.11. *Let R be a right npi-Artinian ring such that I-rad(R_R) is not right atomic. Then R is semiprime if and only if I-rad$(R_R) = 0$.*

Proof. Let $I =$ I-rad(R_R) be the intersection of the annihilators of all iso-simple right R-modules. Assume by contradiction that R is semiprime and $I \neq 0$. Since I_R is non-atomic npi-Artinian, it contains an iso-simple right ideal J of R, by

Proposition 4.14. Then $J^2 \subseteq JI = 0$, so $J = 0$ which is a contradiction. For the converse, suppose $I = 0$. Let K be an ideal of R such that $K^2 = 0$. If for any iso-simple R-module M we have $MK = 0$, then $K \subseteq I = 0$ and we are done. Now, let M be an iso-simple R-module such that $MK \neq 0$. It is clear that MK is iso-simple and $K \nsubseteq \text{ann}(M)$. Hence, $K \nsubseteq I = (0)$ which is a contradiction. Therefore, R is semiprime. □

At the end of this paper, we investigate when ne-Artinian (resp., ne-Noetherian) and np-Artinian (resp., np-Noetherian) rings coincide. For this, we need the following lemma.

Lemma 5.12. *Let R be any semiprime ring and I, J two ideals of R. Then the following are equivalent.*

(1) $I \perp J$.

(2) $I \cap J = (0)$.

(3) $IJ = (0)$.

Proof. It is easy to see that (1) ⇒ (2) ⇒ (3).
(3) ⇒ (1) By contrary suppose that $I \not\perp J$. Then there exist non-zero ideals I_1 and J_1 of R which are contained in I and J respectively, such that $I_1 \cong J_1$. Let f denotes this isomorphism. Define $\varphi : I_1J_1 \to J_1^2$ by $\varphi(a_1b_1 + \cdots + a_nb_n) = f(a_1)b_1 + \cdots + f(a_n)b_n$ for each $a_i \in I_1$, $b_i \in J_1$ and $n \in \mathbb{N}$. It is easy to see that φ is an R-isomorphism. Since $I_1J_1 \subseteq IJ = (0)$. It follows that $J_1^2 = 0$, so $J_1 = 0$ (since R is semiprime), which is a contradiction. □

Recall that a ring R is called right duo, if any right ideal of R is two sided.

Proposition 5.13. *Let R be a semiprime right duo ring.*

(1) *R is right np-Artinian (resp., np-Noetherian) if and only if R is right ne-Artinian (resp., ne-Noetherian).*

(2) *R is right npi-Artinian (resp., npi-Noetherian) if and only if R is right nei-Artinian (resp., nei-Noetherian).*

Proof. (1) Assume that R is right np-Artinian. We have to show that R is right ne-Artinian. For this goal, by [11, Theorem 1.4], it suffices to show that every non-essential right ideal of R is Artinian. Let I be a non-essential right ideal of R. Since R is right duo, I is an ideal of R which is non-essential as a right ideal. It follows that $I \cap J = (0)$ for some non-zero right ideal J of R. Again, since R is right duo, J is a non-zero ideal of R, so $I \perp J$, by the previous lemma. Now, Lemma 2.2 implies that I is non-parallel as a right ideal. Consequently, since R is np-Artinian, by Theorem 3.4, I is Artinian and we are done. The converse is clear

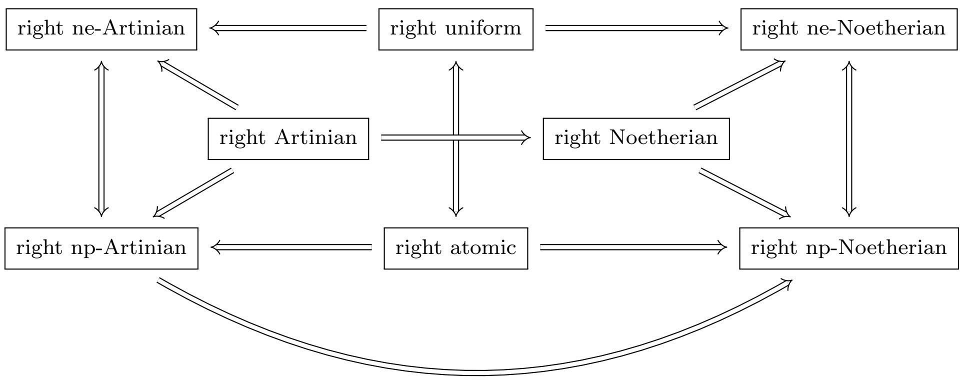
and the proof of np-Noetherian case is similar to the above argument.
(2) Analogous to (1). □

Corollary 5.14. *Let R be a semiprime right duo ring. Then the following statements are equivalent.*

(1) *R is right ne-Noetherian.*
(2) *R is right uniform or right Noetherian.*
(3) *R is right atomic or right Noetherian.*
(4) *R is right np-Noetherian.*

Proof. It follows from the previous proposition and [11, Theorem 2.9]. □

Remark 5.15. In view of the previous corollary, Proposition 5.13 (and its proof), Theorem 5.7 and Hopkins-Levitzki theorem, we have the following diagram for semiprime right duo rings.



Acknowledgement. The authors would like to thank the referees for reading the article carefully and giving useful comments and efforts towards improving the manuscript. The third author is grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (GN: SCU.MM1400.473).
Disclosure statement. The authors report there are no competing interests to declare.

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, second ed., Grad. Texts in Math., 13, Springer-Verlag, New York, 1992.
[2] A. W. Chatters and C. R. Hajarnavis, Rings with Chain Conditions, Res. Notes in Math., 44, Pitman, Boston, 1980.

[3] A. K. Chaturvedi and S. Prakash, *Some variants of ascending and descending chain conditions*, Comm. Algebra, 49(10) (2021), 4324-4333.

[4] J. Dauns and Y. Zhou, Classes of Modules, Pure Appl. Math., Chapman and Hall/CRC, Boca Raton, 2006.

[5] A. Facchini and Z. Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra, 453 (2016), 578-601.

[6] A. Facchini and Z. Nazemian, *Artinian dimension and isoradical of modules*, J. Algebra, 484 (2017), 66-87.

[7] S. M. Javdannezhad, M. Maschizadehand and N. Shirali, *On iso-DICC modules*, to appear in Comm. Algebra, DOI: https://doi.org/10.1080/00927872.2024.2372374.

[8] T. Y. Lam, A First Course in Noncommutative Rings, Grad. Texts in Math., 131, Springer-Verlag, New York, 1991.

[9] T. Y. Lam, Lectures on Modules and Rings, Grad. Texts in Math., 189, Springer-Verlag, New York, 1999.

[10] M. Shirali and N. Shirali, *On parallel krull dimension of modules*, Comm. Algebra, 50(12) (2022), 5284-5295.

[11] P. F. Smith and M. R. Vedadi, *Modules with chain conditions on non-essential submodules*, Comm. Algebra, 32(5) (2004), 1881-1894.

[12] P. Vamos, *The dual of the notion of "finitely generated"*, J. London Math. Soc., 43 (1968), 643-646.

[13] Y. Zhou, *Nonsingular rings with finite type dimension*, Trends Math., (1997), 323-333.

[14] Y. Zhou, *Decomposing modules into direct sums of submodules with types*, J. Pure Appl. Algebra, 138(1) (1999), 83-97.

Nasrin Shirali (Corresponding Author) and **Mohammad Maschizadeh**
Department of Mathematics
Shahid Chamran University of Ahvaz
Ahvaz, Iran
e-mails: shirali_n@scu.ac.ir (N. Shirali)
m-maschizadeh@stu.scu.ac.ir (M. Maschizadeh)

Sayed Malek Javdannezhad
Department of Mathematics
Faculty of Science
Shahid Rajaee Teacher Training University
Tehran, Iran
e-mail: sm.javdannezhad@gmail.com