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Research Article

Completeness theorems related to BVPs satisfying the Lopatinskii condition for higher order elliptic equations

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To Paolo Emilio Ricci on the occasion of his 80th birthday

ABSTRACT. In this paper, we consider a linear elliptic operator E with real constant coefficients of order $2m$ in two independent variables without lower order terms. For this equation, we consider linear BVPs in which the boundary operators $T_1, \dots, T_m$ are of order m and satisfy the Lopatinskii-Shapiro condition with respect to E. We prove boundary completeness properties for the system $\{(T_1\omega_k, \dots, T_m\omega_k)\}$, where $\{\omega_k\}$ is a system of polynomial solutions of the equation $Eu = 0$.

Keywords: Completeness theorems, Lopatinskii condition, elliptic equations of higher order, partial differential equations with constant coefficients.

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1. INTRODUCTION

The problem of the completeness of particular sequences of solutions of a PDE on the boundary of a domain has a long history. The prototype of such results is the theorem which states that harmonic polynomials are complete in $L^p(\partial\Omega)$ $(1 \leqslant p < \infty)$ or in $C^0(\partial\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \Omega$ is connected. This result has been proved by Fichera [5]. Since then, many other results have been obtained. They are related to different BVPs for several PDEs, including some systems. We refer to [4, Section 2] for an introduction to the subject and a quite updated bibliography. Here we mention that there are numerical methods that are founded on the boundary completeness properties of certain sequences of solutions of a given PDE (see [8, p.36–37]).

In the present paper, we deal with a linear elliptic operator E with real constant coefficients of order $2m$ in two independent variables without lower order terms. For this equation, we consider the BVP in a bounded domain $\Omega \subset \mathbb{R}^2$ in which the boundary conditions are given by m linear differential operators $T_1, \dots, T_m$ of order m. We assume that the operators T_j satisfy the Lopatinskii-Shapiro condition with respect to E and that $\mathbb{R}^2 \setminus \Omega$ is connected.

The aim of this paper is to prove that the system $\{(T_1\omega_k, \dots, T_m\omega_k)\}$, where $\{\omega_k\}$ is a system of polynomial solutions of the equation $Eu = 0$, is complete in the subspace of $[L^p(\partial\Omega)]^m$ constituted by the vectors which satisfy the compatibility conditions of the BVP: $Eu = 0$ in Ω, $T_j u = \psi_j$ on $\partial\Omega$ $(j = 1, \dots, m)$. We also prove a similar and more delicate result in the uniform norm. The BVP $Eu = 0$ in Ω, $T_j u = 0$ on $\partial\Omega$ $(j = 1, \dots, m)$ was considered by Paolo Emilio Ricci in his paper [13]. There Ricci developed a theory of the simple layer potential

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for such BVP, mainly using results from complex analysis related to singular integral systems. Our results hinge on Ricci's paper. Later BVPs with higher order boundary conditions were considered in [10]. The present paper could probably be extended to these more general BVPs, using the results contained in [11, 12]. We plan to investigate this topic in future work.

The present paper is organized as follows. In Section 2, after some preliminaries, we recall some of the results obtained by Ricci. Section 3 is devoted to the completeness of the system $\{(T_1\omega_k,\dots,T_m\omega_k)\}$ in L^p norm. The completeness in C^0 norm is proved in Section 4.

2. Ricci's results

Let us consider an elliptic operator of order $2m$

$$E=\sum_{k=0}^{2m}a_k\frac{\partial^{2m}}{\partial x^{2m-k}\partial y^k}$$

a_k being real coefficients. The ellipticity condition we assume is

$$\sum_{k=0}^{2m}a_k\xi^{2m-k}\eta^k\neq 0,\quad \forall\,(\xi,\eta)\in\mathbb{R}^2\setminus\{(0,0)\}\,.$$

Let Ω be a bounded domain in $\mathbb{R}^2$ and denote its boundary by Σ, which is supposed to be $C^{1,h}$ $(0<h\leqslant 1)$. Let us consider the following BVP

$$\begin{cases}Eu=0 & \text{in }\Omega\\ T_ju=\psi_j & \text{on }\Sigma\quad (j=1,\dots,m),\end{cases}\tag{2.1}$$

where the T_j are m boundary operators of order m. This means that we can write

$$T_j=\sum_{h=0}^{m}b_h^j(z)\frac{\partial^m}{\partial x^{m-h}\partial y^h}+\widetilde{T}_j\,,$$
$$\widetilde{T}_j=\sum_{s=0}^{m-1}\sum_{i=0}^{m-1-s}b^j_{i,m-1-s}(z)\frac{\partial^{m-1-s}}{\partial x^{m-1-s-i}\partial y^i},$$

where $z=x+iy$. We assume that all the functions b^j_h and $b^j_{i,m-1-s}$ belong to $H(\Sigma)$, the space of real valued Hölder continuous functions defined on Σ.

Let us denote by $L(w)$ the characteristic polynomial of E

$$L(w)=\sum_{k=0}^{2m}a_kw^{2m-k}$$

and by $L_j(w,z)$ the characteristic polynomial of the boundary operator T_j, i.e.

$$L_j(w,z)=\sum_{h=0}^{m}b^j_h(z)\,w^{m-h}\quad (j=1,2,\dots,m).$$

Let us consider also the polynomial

$$L^{(-)}(w)=(w-w_1)^{\nu_1}\cdots(w-w_p)^{\nu_p},$$

where $w_1,\dots,w_p$ are the zeros of polynomial L with negative imaginary part ($w_i\neq w_j$ if $i\neq j$, $\nu_1+\dots+\nu_p=m$).

We recall that the operators T_j satisfy the Lopatinskii condition with respect to E if, for any $z \in \Sigma$, there are no complex constants $c_1, \dots, c_m$ such that the polynomial $L^{(-)}(w)$ divides the polynomial

$$\sum_{j=1}^{m} c_j \, L_j(w,z) \, .$$

From now on, we assume that this condition is satisfied.

Let Γ be a rectifiable Jordan curve in the complex w-plane enclosing the zeros of $L(w)$ in the lower half-plane and oriented in the positive direction. Agmon [1, p.189–190] showed that the function

$$\mathcal{P}(z-\zeta) = \mathbb{R}\mathrm{e} \left\{ \frac{-1}{2\pi^2(2m-2)!} \int_{+\Gamma} \frac{[(x-\xi)w + (y-\eta)]^{2m-2} \log[(x-\xi)w + (y-\eta)]}{L(w)} dw \right\},$$

($z = x + iy$, $\zeta = \xi + i\eta$), where some fixed determination of the logarithm has been chosen, is a fundamental solution of the operator E.

In [7], Fichera gave the concept of simple layer potential for a class of linear elliptic operators of higher order in two independent variables. In the case of the operator E, it is given by

$$u(z) = \sum_{k=0}^{m-1} \int_{\Sigma} \varphi_k(\zeta) \frac{\partial^{m-1}}{\partial \xi^{m-1-k} \partial \eta^k} \mathcal{P}(z-\zeta) ds_\zeta \, ,$$

the functions φ_k being real valued. It is clear that this definition extends the classical one related to the Laplace operator

$$u(z) = \frac{1}{2\pi} \int_{\Sigma} \varphi(\zeta) \, \log|z-\zeta| \, ds_\zeta \, . \tag{2.2}$$

The following jump formula holds (see [7, p. 65–66], [13, p. 7])

$$\begin{aligned} &\lim_{z \to z_0^+} \int_{\Sigma} \varphi(\zeta) \frac{\partial^{2m-1}}{\partial x^{2m-1-l} \partial y^l} \mathcal{P}(z-\zeta) ds_\zeta \\ &= -\, \varphi(z_0) \, \frac{1}{2\pi} \, \mathbb{I}\mathrm{m} \int_{+\Gamma} \frac{w^{2m-1-l}}{L(w) \, (\dot{x}_0 w + \dot{y}_0)} dw \\ &- \frac{1}{2\pi^2} \, \mathbb{R}\mathrm{e} \int_{\Sigma} \varphi(\zeta) ds_\zeta \int_{+\Gamma} \frac{w^{2m-1-l}}{L(w) \, [(x_0-\xi) \, w + (y_0-\eta)]} dw \, , \ (0 \leqslant l \leqslant 2m-1). \end{aligned} \tag{2.3}$$

Here $z_0 \in \Sigma$ and $z \to z_0$ from the interior of Ω and the dot denotes the derivative with respect to the arc length on Σ. In [13] these formulas have been proved for any $z_0 \in \Sigma$ assuming the Hölder continuity of the density φ, but they are still valid a.e. on Σ if $\varphi \in L^1(\Sigma)$ (see, e.g., [3]). Similarly one can prove that

$$\begin{aligned} &\lim_{z \to z_0^-} \int_{\Sigma} \varphi(\zeta) \frac{\partial^{2m-1}}{\partial x^{2m-1-l} \partial y^l} \mathcal{P}(z-\zeta) ds_\zeta \\ &= \varphi(z_0) \, \frac{1}{2\pi} \, \mathbb{I}\mathrm{m} \int_{+\Gamma} \frac{w^{2m-1-l}}{L(w) \, (\dot{x}_0 w + \dot{y}_0)} dw \\ &- \frac{1}{2\pi^2} \, \mathbb{R}\mathrm{e} \int_{\Sigma} \varphi(\zeta) ds_\zeta \int_{+\Gamma} \frac{w^{2m-1-l}}{L(w) \, [(x_0-\xi) \, w + (y_0-\eta)]} dw \, , \ (0 \leqslant l \leqslant 2m-1), \end{aligned} \tag{2.4}$$

where $z \to z_0$ from the exterior of Ω.

Using jump formulas (2.3), Ricci [13] showed that, given the functions $\psi_j \in H(\Sigma)$ $(j = 1,\dots,m)$, there exists a solution of BVP (2.1) in the form of a simple layer potential (2.2) if and only if there exists a solution of the following singular integral system on the boundary

$$
\begin{aligned}
&-\sum_{k=1}^{m}\sum_{h=0}^{m}\frac{1}{2\pi}\,b^j_h(z)\,\varphi_k(z)\,\mathbb{I}\mathrm{m}\int_{+\Gamma}\frac{w^{2m-1-h-k}}{L(w)\,(\dot{x}w+\dot{y})}dw\\
&-\sum_{k=1}^{m}\sum_{h=0}^{m}\frac{1}{2\pi^2}\,b^j_h(z)\,\mathbb{R}\mathrm{e}\int_{\Sigma}\varphi_k(\zeta)\,ds_\zeta\int_{+\Gamma}\frac{w^{2m-1-h-k}}{L(w)\,[(x-\xi)\,w+(y-\eta)]}dw\\
&+\sum_{k=1}^{m}\sum_{s=0}^{m-1}\sum_{i=0}^{m-1-s}b^j_{i,m-1-s}(z)\int_{\Sigma}\varphi_k(\zeta)\,\frac{\partial^{2m-2-s}}{\partial x^{2m-2-s-i-k}\partial y^{i+k}}\mathcal{P}(z-\zeta)ds_\zeta\\
&=(-1)^{m-1}\psi_j(z)\,,\quad j=1,\dots,m\,.
\end{aligned}
\tag{2.5}
$$

In [13] it is also proved that this singular integral system can be written in the canonical form

$$
\begin{aligned}
&A(z)\Phi(z)+\frac{1}{\pi i}B(z)\int_{+\Sigma}\frac{\Phi(\zeta)}{\zeta-z}d\zeta+\int_{+\Sigma}M(z,\zeta)\Phi(\zeta)d\zeta\\
&=(-1)^{m-1}2\pi\,\Psi(z)\,,\quad z\in\Sigma\,.
\end{aligned}
\tag{2.6}
$$

Here Ψ and Φ are the vectors

$$
\Psi(z)=\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\\ \vdots\\ \psi_m(z)\end{pmatrix},\quad \Phi(z)=\begin{pmatrix}\varphi_0(z)\\ \varphi_1(z)\\ \vdots\\ \varphi_{m-1}(z)\end{pmatrix}
$$

and

$$
A(z)=\{A_{jk}(z)\};\quad B(z)=\{B_{jk}(z)\};\quad M(z,\zeta)=\{M_{jk}(z,\zeta)\},
$$

where

$$
\begin{aligned}
A_{jk}(z)&=-\,\mathbb{I}\mathrm{m}\int_{+\Gamma}\frac{L_j(w,z)\,w^{m-1-k}}{L(w)\,(\dot{x}w+\dot{y})}dw,\\
B_{jk}(z)&=i\,\mathbb{R}\mathrm{e}\int_{+\Gamma}\frac{L_j(w,z)\,w^{m-1-k}}{L(w)\,(\dot{x}w+\dot{y})}dw
\end{aligned}
\tag{2.7}
$$

and $M_{jk}(z,\zeta)$ are weakly singular kernels.

Ricci [13] proved the following result:

Theorem 2.1. *The singular integral system* (2.6) *is regular (i.e.* $\det(A+B)\neq 0;\ \det(A-B)\neq 0$*) if and only if the BVP* (2.1) *satisfies the Lopatinskii condition.*

By (2.7) we get

$$
\begin{aligned}
A+B&=\left\{i\int_{+\Gamma}\frac{L_j(w,z)\,w^{m-1-k}}{L(w)\,(\dot{x}w+\dot{y})}dw\right\},\\
A-B&=\left\{\overline{i\int_{+\Gamma}\frac{L_j(w,z)\,w^{m-1-k}}{L(w)\,(\dot{x}w+\dot{y})}dw}\right\},
\end{aligned}
$$

from which it follows that system (2.6) is of regular type if and only if the function

$$
\delta_0(z)=\det\left\{\int_{+\Gamma}\frac{L_j(w,z)\,w^{m-1-k}}{L(w)\,(\dot{x}w+\dot{y})}dw\right\}
$$

never vanishes on Σ ([13, p.14]).

It is well known (see, e.g., [14]) that if a singular system is of regular type, then the associated homogeneous system has a finite number of eigensolutions and there exists a solution of the system if and only if the data satisfies a finite number of compatibility conditions. More precisely, there exists a complex valued solution of the system (2.6) if and only if the given vector Ψ is such that

$$\int_{+\Sigma} \Psi\, X\, d\zeta = 0$$

for any (complex valued) eigensolution X of the homogeneous adjoint system

$$A'(z)X(z) - \frac{1}{\pi i}B'(z)\int_{+\Sigma}\frac{X(\zeta)}{\zeta - z}d\zeta + \int_{+\Sigma} N(\zeta,z)X(\zeta)d\zeta = 0\,, \quad z \in \Sigma\,. \tag{2.8}$$

Here A' and B' are the transposed matrices of A and B, respectively, and

$$N(\zeta,z) = M'(\zeta,z) - \frac{1}{\pi i}\frac{B'(\zeta) - B'(z)}{\zeta - z}$$

$M'(\zeta,z)$ being the transposed matrix of $M(\zeta,z)$ (see [13, p.12–13]).

We remark that, if we denote by $K\Phi$ and $K'X$ the left hand side of (2.6) and (2.8) respectively, we have

$$\int_{+\Sigma} X\, K\Phi\, d\zeta = \int_{+\Sigma} \Phi\, K'X\, d\zeta \tag{2.9}$$

for any (complex valued) Hölder continuous vector Φ, X (see [13, p.13]).

3. Completeness theorems in L^p norm

From now on, we assume that $\mathbb{R}^2 \setminus \Omega$ is connected.

Let us denote by $\{\omega_k\}$ ($k \in \mathbb{N}$) a complete system of polynomial solutions of the equation $Eu = 0$. This means that any polynomial solution of the equation $Eu = 0$ can be written as a finite linear combination of elements of $\{\omega_k\}$. A method for the explicit construction of the system $\{\omega_k\}$ is given in [2].

Let us denote by $X_1, \dots, X_s$ a base of the eigenspace related to the equation (2.8). It is well known that these vectors are Hölder continuous.

Let $1 \leqslant p < \infty$ and

$$\Lambda^p = \left\{ G = (g_1, \dots, g_m) \in [L^p(\Sigma)]^m \;\middle|\; \int_{+\Sigma} G\, X_h\, d\zeta = 0,\ h = 1, \dots, s \right\}.$$

We remark that $(g_1, \dots, g_m)$ are real valued functions.

Let us denote by $K^* : [L^q(\Sigma)]^m \to [L^q(\Sigma)]^m$ ($q = p/(p-1)$) the operator defined by

$$\int_\Sigma G\, K^*F\, ds = \int_\Sigma F\, KG\, ds\,.$$

Recalling (2.9), we have

$$\begin{aligned}\int_\Sigma F\, KG\, ds &= \int_{+\Sigma} F\, KG\, \dot{\bar{\zeta}}\, d\zeta \\ &= \int_{+\Sigma} G\, K'_z(\dot{\bar{z}}F)\, d\zeta = \int_\Sigma G\, K'_z(\dot{\bar{z}}F)\, \dot{\zeta}\, ds\end{aligned}$$

and then $K^*F = \dot{\zeta}\, K'_z(\dot{\bar{z}}F)$. This shows that $\dot{\zeta}X_1, \dots, \dot{\zeta}X_s$ are (complex valued) eigensolutions of the equation $K^*F = 0$. Since the operator K^* maps real vectors to real vectors, we have that $\mathbb{Re}(\dot\zeta X_j)$ and $\mathbb{Im}(\dot\zeta X_j)$ are (not necessarily linearly independent) real eigensolutions of the equation $K^*\Xi = 0$ and that the kernel $\mathrm{Ker}(K^*)$ is spanned by

$$\{\mathbb{Re}(\dot\zeta X_1), \mathbb{Im}(\dot\zeta X_1), \dots, \mathbb{Re}(\dot\zeta X_s), \mathbb{Im}(\dot\zeta X_s)\}.$$

We note that Λ^p is the annihilator of the kernel of K^*:

$$\Lambda^p = {}^{\perp}\mathrm{Ker}(K^*)\,. \tag{3.10}$$

This follows from the remark that if $G \in \Lambda^p$, we have

$$0 = \int_{+\Sigma} G\, X_h\, d\zeta = \int_\Sigma G\, X_h\, \dot\zeta\, ds = \int_\Sigma G\ \mathbb{Re}(\dot\zeta X_h)\, ds + i \int_\Sigma G\ \mathbb{Im}(\dot\zeta X_h)\, ds$$

and then

$$\int_\Sigma G\ \mathbb{Re}(\dot\zeta X_h)\, ds = \int_\Sigma G\ \mathbb{Im}(\dot\zeta X_h)\, ds = 0 \quad (h = 1, \dots, s).$$

We remark that K^* has the following expression

$$\begin{aligned} K^*_k F(z) = &- \sum_{j=1}^m \sum_{h=0}^m \frac{1}{2\pi}\, b^j_h(z)\, F_j(z)\, \mathrm{Im} \int_{+\Gamma} \frac{w^{2m-1-h-k}}{L(w)\,(\dot x w + \dot y)} dw \\ &+ \sum_{j=1}^m \sum_{h=0}^m \frac{1}{2\pi^2}\, \mathrm{Re} \int_\Sigma b^j_h(\zeta)\, F_j(\zeta)\, ds_\zeta \int_{+\Gamma} \frac{w^{2m-1-h-k}}{L(w)\,[(x-\xi)\,w + (y-\eta)]} dw \\ &+ \sum_{j=1}^m \sum_{s=0}^{m-1} \sum_{i=0}^{m-1-s} \int_\Sigma b^j_{i,m-1-s}(\zeta)\, F_j(\zeta)\, \frac{\partial^{2m-2-s}}{\partial\xi^{2m-2-s-i-k}\partial\eta^{i+k}} \mathcal{P}(z-\zeta) ds_\zeta\,. \end{aligned} \tag{3.11}$$

Note that we have also used the property $\mathcal{P}(z-\zeta) = \mathcal{P}(\zeta - z)$ (see [1, p.189]).

Let us consider the system $\{T\omega_k\} = \{(T_1\omega_k, \dots, T_m\omega_k)\}$. It is clear that it is contained in Λ^p. We aim to show that it is complete in Λ^p. Let us begin with a couple of lemmas.

Lemma 3.1. *Let*

$$p_{k,s}(z,\zeta) = p_{k,s}(x,y,\xi,\eta) = \int_{+\Gamma} \frac{(\xi w + \eta)^k (xw + y)^s}{L(w)}\, dw\,, \tag{3.12}$$

where $k \in \mathbb{N}$, $s \in \mathbb{Z}$. The function $p_{k,s}$ is homogeneous of degree $k+s$. For any fixed $z \in \mathbb{C}$, $p_{k,s}$ is a homogeneous polynomial of degree k in ξ, η and is a solution of the equation $E_\zeta p_{k,s} = 0$. For any fixed $\zeta \in \mathbb{C}$, $p_{k,s}$ is a homogeneous function (a homogeneous polynomial, if $s \geqslant 0$) of degree s in x, y and is a solution of the equation $E_z p_{k,s} = 0$.

Proof. Clearly $p_{k,s}(\lambda x, \lambda y, \lambda\xi, \lambda\eta) = \lambda^{k+s} p_{k,s}(x,y,\xi,\eta)$ $(\lambda > 0)$. It is obvious that, for any fixed $z \in \mathbb{C}$, $p_{k,s}$ is a homogeneous polynomial of degree k in ξ, η. Therefore, if $k \leqslant 2m-1$, it satisfies the equation $E_\zeta p_{k,s} = 0$. Let $k \geqslant 2m$. We have

$$\begin{aligned} E_\zeta p_{k,s} &= \sum_{h=0}^{2m} a_h \frac{\partial^{2m}}{\partial\xi^{2m-h}\partial\eta^h} \int_{+\Gamma} \frac{(\xi w + \eta)^k (xw+y)^s}{L(w)}\, dw \\ &= \sum_{h=0}^{2m} a_h\, k(k-1)\cdots(k-2m+1) \int_{+\Gamma} \frac{(\xi w+\eta)^{k-2m}(xw+y)^s w^{2m-h}}{L(w)}\, dw \\ &= k(k-1)\cdots(k-2m+1) \int_{+\Gamma} (\xi w + \eta)^{k-2m} (xw+y)^s\, dw\,. \end{aligned}$$

The holomorphy of $(\xi \cdot + \eta)^{k-2m} (x \cdot + y)^s$ in the interior of Γ gives the result.

A similar argument works for a fixed ζ. □

Lemma 3.2. *There exists $R > 0$ such that, for any $|z| > R$, we have*

$$\mathcal{P}(z-\zeta) = q_0(z,\zeta) + \frac{1}{2\pi^2(2m-2)!}\sum_{h=1}^{\infty}\sum_{j=0}^{2m-2}\binom{2m-2}{j}\frac{(-1)^j}{h}\,\mathbb{R}\mathrm{e}\,p_{j+h,2m-2-j-h}(z,\zeta) \tag{3.13}$$

uniformly for ζ varying on Σ, where q_0 is, for any fixed $z \in \mathbb{C}$, a polynomial of degree at most $2m-2$ in ξ, η (and then satisfies $E_\zeta q_0 = 0$). The series (3.13) can be differentiated with respect to ξ, η term by term and the differentiated series converge uniformly for ζ varying on Σ.

Proof. We first prove that there exists $R > 0$ such that

$$\left|\frac{\xi w + \eta}{xw + y}\right| < 1 \tag{3.14}$$

for any $|z| > R$, $\zeta \in \Sigma$, $w \in \Gamma$. Let us consider the function

$$\psi(u,v,\vartheta) = \sqrt{(u\cos\vartheta + \sin\vartheta)^2 + v^2\cos^2\vartheta}$$

and set

$$m = \min_{\substack{\vartheta\in[0,2\pi]\\ u+iv\in\Gamma}} \psi(u,v,\vartheta), \qquad M = \max_{\substack{\vartheta\in[0,2\pi]\\ u+iv\in\Gamma}} \psi(u,v,\vartheta).$$

It is easy to see that $m > 0$. Then we can write

$$\left|\frac{\xi w + \eta}{xw + y}\right| \leqslant \frac{M}{m}\,\frac{|\zeta|}{|z|}\,.$$

Choosing

$$R \geqslant \frac{M}{m}\max_{\zeta\in\Sigma}|\zeta|\,,$$

we have that (3.14) is satisfied for any $|z| > R$. If $|z| > R$ and $\zeta \in \Sigma$, $w \in \Gamma$, we have

$$\begin{aligned}\log[(x-\xi)w + (y-\eta)] &= \log[(xw+y) - (\xi w + \eta)]\\ &= \log\left[(xw+y)\left(1 - \frac{\xi w+\eta}{xw+y}\right)\right]\\ &= \log(xw+y) + \log\left(1 - \frac{\xi w + \eta}{xw+y}\right),\end{aligned}$$

where we take the principal determination of $\log\left(1 - \frac{\xi w+\eta}{xw+y}\right)$ and the determination of $\log(xw+y)$ is chosen so that this formula holds. Therefore, fixed $|z| > R$,

$$\log[(x-\xi)w + (y-\eta)] = \log(xw+y) - \sum_{h=1}^{\infty}\frac{1}{h}\left(\frac{\xi w+\eta}{xw+y}\right)^h$$

where, thanks to (3.14), the series uniformly converges for $\zeta \in \Sigma$, $w \in \Gamma$. Then

$$\begin{aligned}
&\int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}\log[(x-\xi)w+(y-\eta)]}{L(w)}dw \\
&= \int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}\log(xw+y)}{L(w)}dw \\
&- \sum_{h=1}^{\infty} \frac{1}{h}\int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}}{L(w)}\left(\frac{\xi w+\eta}{xw+y}\right)^{h} dw \\
&= \int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}\log(xw+y)}{L(w)}dw \\
&- \sum_{h=1}^{\infty}\sum_{j=0}^{2m-2} \binom{2m-2}{j}\frac{(-1)^j}{h}\int_{+\Gamma} \frac{(\xi w+\eta)^{j+h}(xw+y)^{2m-2-j-h}}{L(w)}\, dw\,.
\end{aligned}$$

We have then proved (3.13), where

$$q_0(z,\zeta) = -\,\mathbb{R}\mathrm{e}\, \frac{1}{2\pi^2(2m-2)!}\int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}\log(xw+y)}{L(w)}dw\,,$$

which is clearly a polynomial in ξ, η of degree at most $2m-2$.

In the same manner, we can see the uniform convergence of differentiated series. □

Theorem 3.2. *The system $\{T\omega_k\}$ is complete in Λ^p ($1 \leqslant p < \infty$).*

Proof. We have to show that, if a functional $\Theta \in ([L^p(\Sigma)]^m)^*$ vanishes on $\{T\omega_k\}$, i.e. if

$$\langle \Theta, T\omega_k\rangle = 0, \quad \forall\, k \in \mathbb{N}\,,$$

then it vanishes on Λ^p:

$$\langle \Theta, G\rangle = 0, \quad \forall\, G \in \Lambda^p.$$

Let $1 < p < \infty$ and suppose that $\Theta = (\Theta_1, \dots, \Theta_m) \in [L^q(\Sigma)]^m$ is such that

$$\int_\Sigma \Theta\, T\omega_k\, ds = 0, \quad \forall\, k \in \mathbb{N}. \tag{3.15}$$

Let us denote by $\omega_{k,1}, \dots, \omega_{k,\nu_k}$ a basis of the (real) linear span generated by all the homogeneous real polynomials of degree k satisfying the equation $Eu = 0$ (see [2, p.34–35]). Since for any $z \in \mathbb{C}$, the polynomials (3.12) are homogeneous and satisfy the equation $Eu = 0$, Lemma 3.2 shows that there exist real functions $c_{k,j}(z)$ such that, for any $|z| > R$,

$$\mathcal{P}(z-\zeta) = \sum_{k=0}^{\infty}\sum_{j=1}^{\nu_k} c_{k,j}(z)\,\omega_{k,j}(\zeta)$$

uniformly for $\zeta \in \Sigma$. The same holds for differentiated series. Then, for any $|z| > R$, we can write

$$\int_\Sigma \Theta(\zeta)\, T_\zeta \mathcal{P}(z-\zeta)\, ds_\zeta = \sum_{k=0}^{\infty}\sum_{j=1}^{\nu_k} c_{k,j}(z)\int_\Sigma \Theta(\zeta)\, T\omega_{k,j}(\zeta)\, ds_\zeta\,.$$

In view of (3.15), we have

$$u(z) = 0$$

for any $|z| > R$, where

$$u(z) = \int_\Sigma \Theta(\zeta)\, T_\zeta \mathcal{P}(z-\zeta)\, ds_\zeta\,.$$

The function u being analytic in $\mathbb{C}\setminus\Sigma$, we find

$$u(z)=0,\quad \forall\, z\in\mathbb{C}\setminus\overline{\Omega}.$$

Then we can write

$$\sum_{j=1}^{m}\sum_{h=0}^{m}\int_\Sigma \Theta_j(\zeta)\, b_h^j(\zeta)\,\frac{\partial^m}{\partial\xi^{m-h}\partial\eta^h}\mathcal{P}(z-\zeta)ds_\zeta$$
$$+\sum_{j=1}^{m}\int_\Sigma \Theta_j(\zeta)\,\widetilde{T}_{j,\zeta}\mathcal{P}(z-\zeta)ds_\zeta=0$$

for any $z\in\mathbb{C}\setminus\overline{\Omega}$. This implies

$$\sum_{j=1}^{m}\sum_{h=0}^{m}\int_\Sigma \Theta_j(\zeta)\, b_h^j(\zeta)\,\frac{\partial^{m-1}}{\partial x^{m-1-k}\partial y^k}\frac{\partial^m}{\partial\xi^{m-h}\partial\eta^h}\mathcal{P}(z-\zeta)ds_\zeta$$
$$+\sum_{j=1}^{m}\int_\Sigma \Theta_j(\zeta)\,\widetilde{T}_{j,\zeta}\frac{\partial^{m-1}}{\partial x^{m-1-k}\partial y^k}\mathcal{P}(z-\zeta)ds_\zeta=0,\quad k=0,\dots,m-1$$

for any $z\in\mathbb{C}\setminus\overline{\Omega}$, i.e.

$$(-1)^m\sum_{j=1}^{m}\sum_{h=0}^{m}\int_\Sigma \Theta_j(\zeta)\, b_h^j(\zeta)\,\frac{\partial^{2m-1}}{\partial x^{2m-1-h-k}\partial y^{h+k}}\mathcal{P}(z-\zeta)ds_\zeta$$
$$+\sum_{j=1}^{m}\sum_{s=0}^{m-1}\sum_{i=0}^{m-1-s}\int_\Sigma \Theta_j(\zeta)\, b_{i,m-1-s}^j(\zeta)\frac{\partial^{m-1-s}}{\partial\xi^{m-1-s-i}\partial\eta^i}\frac{\partial^{m-1}}{\partial x^{m-1-k}\partial y^k}\mathcal{P}(z-\zeta)ds_\zeta$$
$$=0,\qquad k=0,\dots,m-1$$

for any $z\in\mathbb{C}\setminus\overline{\Omega}$. Applying (2.4), we get

$$-\sum_{j=1}^{m}\sum_{h=0}^{m}\frac{1}{2\pi}\Theta_j(z)\, b_h^j(z)\,\mathbb{I}\mathrm{m}\int_{+\Gamma}\frac{w^{2m-1-h-k}}{L(w)\,(\dot{x}w+\dot{y})}dw$$
$$+\sum_{j=1}^{m}\sum_{h=0}^{m}\frac{1}{2\pi^2}\,\mathbb{R}\mathrm{e}\int_\Sigma\Theta_j(\zeta)\, b_h^j(\zeta)\,ds_\zeta\int_{+\Gamma}\frac{w^{2m-1-h-k}}{L(w)\,[(x-\xi)\,w+(y-\eta)]}dw$$
$$+\sum_{j=1}^{m}\sum_{s=0}^{m-1}\sum_{i=0}^{m-1-s}\int_\Sigma \Theta_j(\zeta)\, b_{i,m-1-s}^j(\zeta)\frac{\partial^{2m-2-s}}{\partial\xi^{2m-2-s-i-k}\partial\eta^{i+k}}\mathcal{P}(z-\zeta)ds_\zeta=0,$$

$k=0,\dots,m-1$, a.e. on Σ. Comparing this formula with (3.11), we see that this system coincides with $K^*\Theta=0$, i.e. Θ belongs to $\mathrm{Ker}(K^*)$. Recalling (3.10) we have

$$\int_\Sigma \Theta\, G\, ds=0$$

for any $G\in\Lambda^p$. This completes the proof when $1<p<\infty$.

If $p=1$, we observe that if $\Theta\in[L^\infty(\Sigma)]^m$, then $\Theta\in[L^s(\Sigma)]^m$ for any $s>1$. Then we can repeat the proof. $\square$

4. Completeness theorems in C^0 norm

In this section, we prove that the completeness property obtained in the previous section is also valid in the uniform norm. Namely, we want to prove the completeness in the space

$$\Lambda^0 = \left\{ G = (g_1, \dots, g_m) \in [C^0(\Sigma)]^m \;\Big|\; \int_{+\Sigma} G\, X_h\, d\zeta = 0,\ h = 1, \dots, s \right\}.$$

Theorem 4.3. *The system $\{T\omega_k\}$ is complete in Λ^0.*

Proof. We have to show that, if a functional in $\Theta \in ([C^0(\Sigma)]^m)^*$ vanishes on $\{T\omega_k\}$, i.e. if

$$\langle \Theta, T\omega_k \rangle = 0, \quad \forall\, k \in \mathbb{N}, \tag{4.16}$$

then it vanishes on Λ^0:

$$\langle \Theta, G \rangle = 0, \quad \forall\, G \in \Lambda^0.$$

It is well known that a functional $\Theta \in ([C^0(\Sigma)]^m)^*$ can be represented as $\Theta = (\mu^1, \dots, \mu^m)$, where μ^j are Borel measures defined on Σ. Therefore conditions (4.16) can be written as

$$\sum_{j=1}^{m} \int_\Sigma T_j \omega_k\, d\mu^j = 0, \quad \forall\, k \in \mathbb{N}. \tag{4.17}$$

The same arguments used in the first part of the proof of Theorem 3.2 lead to

$$\sum_{j=1}^{m} \int_\Sigma T_{j,\zeta} \mathcal{P}(z-\zeta)\, d\mu^j_\zeta = 0$$

for any $z \in \mathbb{C} \setminus \overline{\Omega}$. This implies

$$\sum_{j=1}^{m} \sum_{h=0}^{m} \int_\Sigma b^j_h(\zeta) \frac{\partial^{m-1}}{\partial x^{m-1-k} \partial y^k} \frac{\partial^m}{\partial \xi^{m-h} \partial \eta^h} \mathcal{P}(z-\zeta)\, d\mu^j_\zeta + \sum_{j=1}^{m} \int_\Sigma \widetilde{T}_{j,\zeta} \frac{\partial^{m-1}}{\partial x^{m-1-k} \partial y^k} \mathcal{P}(z-\zeta)\, d\mu^j_\zeta = 0, \quad k = 0, \dots, m-1$$

for any $z \in \mathbb{C} \setminus \overline{\Omega}$, i.e.

$$\begin{aligned}
&(-1)^{m-1} \sum_{j=1}^{m} \sum_{h=0}^{m} \int_\Sigma b^j_h(\zeta) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z-\zeta)\, d\mu^j_\zeta \\
&+ \sum_{j=1}^{m} \sum_{s=0}^{m-1} \sum_{i=0}^{m-1-s} \int_\Sigma b^j_{i,m-1-s}(\zeta) \frac{\partial^{m-1-s}}{\partial \xi^{m-1-s-i} \partial \eta^i} \frac{\partial^{m-1}}{\partial x^{m-1-k} \partial y^k} \mathcal{P}(z-\zeta)\, d\mu^j_\zeta \\
&= 0, \qquad k = 0, \dots, m-1
\end{aligned}$$

for any $z \in \mathbb{C} \setminus \overline{\Omega}$. Let us introduce a family of "parallel curves" Σ_ϱ. Let us denote by $\tau(z)$ a unit vector of class $C^1(\Sigma)$ such that $\tau(z) \cdot \nu(z) \geqslant \beta_0 > 0$, ν being the exterior unit normal to Σ. We can choose $\varrho > 0$ in such a way that the curve Σ_ϱ defined by $z_\varrho = z + \varrho\tau(z)$, $z \in \Sigma$, is the boundary of a domain containing Ω (contained in Ω) if $0 < \varrho \leqslant \varrho_0$ (if $-\varrho_0 \leqslant \varrho < 0$). One can prove that if $\Sigma \in C^1$ such a vector does exist (see [9, p.273–275]). For $0 < \varrho \leqslant \varrho_0$ and for any

$f_k \in H(\Sigma)$ $(k = 0, \dots, m-1)$, we may write

$$\sum_{k=0}^{m-1} \int_{\Sigma_\varrho} f_k(z_\varrho) \Big[\sum_{j=1}^{m} \sum_{h=0}^{m} \int_\Sigma b_h^j(\zeta) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta)\, d\mu_\zeta^j + \sum_{j=1}^{m} \sum_{s=0}^{m-1} \sum_{i=0}^{m-1-s} \int_\Sigma b_{i,m-1-s}^j(\zeta) \frac{\partial^{2m-2-s}}{\partial \xi^{2m-2-s-i-k} \partial \eta^{i+k}} \mathcal{P}(z - \zeta)\, d\mu_\zeta^j \Big] ds_{z_\varrho} = 0. \tag{4.18}$$

Due to the weak singularity of the kernel, we have that

$$\lim_{\varrho \to 0^+} \int_{\Sigma_\varrho} f_k(z_\varrho)\, ds_{z_\varrho} \int_\Sigma b_{i,m-1-s}^j(\zeta) \frac{\partial^{2m-2-s}}{\partial \xi^{2m-2-s-i-k} \partial \eta^{i+k}} \mathcal{P}(z - \zeta)\, d\mu_\zeta^j$$
$$= \int_\Sigma b_{i,m-1-s}^j(\zeta)\, d\mu_\zeta^j \int_\Sigma f_k(z) \frac{\partial^{2m-2-s}}{\partial \xi^{2m-2-s-i-k} \partial \eta^{i+k}} \mathcal{P}(z - \zeta)\, ds_z\,.$$

Concerning the first term in (4.18), we may write

$$\int_{\Sigma_\varrho} f_k(z_\varrho)\, ds_{z_\varrho} \int_\Sigma b_h^j(\zeta) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta)\, d\mu_\zeta^j$$
$$= \int_\Sigma b_h^j(\zeta)\, d\mu_\zeta^j \int_{\Sigma_\varrho} f_k(z_\varrho) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta) ds_{z_\varrho}$$

and

$$\int_{\Sigma_\varrho} f_k(z_\varrho) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta)\, ds_{z_\varrho}$$
$$= \int_{\Sigma_\varrho} f_k(z_\varrho) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta)\, ds_{z_\varrho}$$
$$- \int_\Sigma f_k(z) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z - \zeta_{-\varrho})\, ds_z$$
$$+ \int_\Sigma f_k(z) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z - \zeta_{-\varrho})\, ds_z.$$

By means of the results proved in [6] (see also [3, p.58–60]) and recalling (2.3), we see that

$$\lim_{\varrho \to 0^+} \Big(\int_{\Sigma_\varrho} f_k(z_\varrho) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z_\varrho - \zeta) ds_{z_\varrho} - \int_\Sigma f_k(z) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z - \zeta_{-\varrho})\, ds_z \Big) = 0$$

and

$$\lim_{\varrho \to 0^+} \int_\Sigma f_k(z) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(z - \zeta_{-\varrho})\, ds_z$$
$$= \lim_{\varrho \to 0^+} \int_\Sigma f_k(z) \frac{\partial^{2m-1}}{\partial \xi^{2m-1-h-k} \partial \eta^{h+k}} \mathcal{P}(\zeta_{-\varrho} - z)\, ds_z$$
$$= - f_k(\zeta) \frac{1}{2\pi} \mathbb{I}\mathrm{m} \int_{+\Gamma} \frac{w^{2m-1-h-k}}{L(w)\, (\dot{\xi} w + \dot{\eta})} dw$$
$$- \frac{1}{2\pi^2} \mathbb{R}\mathrm{e} \int_\Sigma f_k(z)\, ds_z \int_{+\Gamma} \frac{w^{2m-1-h-k}}{L(w)\, [(\xi - x)\, w + (\eta - y)]} dw$$

uniformly for ζ varying on Σ. So, letting $\varrho \to 0^+$ in (4.18) and keeping in mind (2.5), we get

$$\sum_{j=1}^{m} \int_{\Sigma} K_j f \, d\mu^j = 0 \tag{4.19}$$

for any $f = (f_0, \dots, f_{m-1}) \in [H(\Sigma)]^m$. Thanks to Theorem 2.1 the operator K can be reduced on the left (and on the right). This means that there exists a singular integral operator S of the form

$$Sg(z) = C(z)g(z) + \frac{1}{\pi i} D(z) \int_{+\Sigma} \frac{g(\zeta)}{\zeta - z} d\zeta$$

such that

$$KSg(z) = g(z) + \int_{+\Sigma} R(z,\zeta)\, g(\zeta) d\zeta,$$

where $R(z,\zeta) = \{R_{jk}(z,\zeta)\}$ is a kernel with a weak singularity. Taking $f = Sg$ in (4.19), we find

$$\sum_{j=1}^{m} \int_{\Sigma} K_j \, Sg \, d\mu^j = 0 \,, \quad \forall\, g \in [H(\Sigma)]^m,$$

i.e.,

$$\sum_{j=1}^{m} \int_{\Sigma} g_j \, d\mu^j + \sum_{j,k=1}^{m} \int_{\Sigma} d\mu^j_z \int_{+\Sigma} R_{jk}(z,\zeta)\, g_k(\zeta) d\zeta = 0 \,, \quad \forall\, g \in [H(\Sigma)]^m.$$

By Tonelli and Fubini's theorems, we get

$$\sum_{j=1}^{m} \int_{\Sigma} g_j \, d\mu^j = -\sum_{j,k=1}^{m} \int_{+\Sigma} g_k(\zeta) d\zeta \int_{\Sigma} R_{jk}(z,\zeta)\, d\mu^j_z \,, \quad \forall\, g \in [H(\Sigma)]^m.$$

This shows that μ^j are absolutely continuous measures and their Radon Nykodym derivatives with respect to the one-dimensional Lebesgue measure on Σ belong to $L^r(\Sigma)$ for some $r > 1$. In other words, there exist $\Theta^j \in L^r(\Sigma)$ $(r > 1)$ such that

$$d\mu^j = \Theta^j ds \quad (\Theta^j \in L^r(\Sigma)).$$

Conditions (4.17) become

$$\sum_{j=1}^{m} \int_{\Sigma} T_j \omega_k \, \Theta^j ds = 0, \quad \forall\, k \in \mathbb{N}\,.$$

But this coincides with (3.15) and the result follows from what we have proved in Theorem 3.2. □

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