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Research Article

Viscosity implicit midpoint scheme for enriched nonexpansive mappings

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ABSTRACT. This article proposes and analyses a viscosity scheme for an enriched nonexpansive mapping. The scheme is incorporated with the implicit midpoint rule of stiff differential equations. We deduce some convergence properties of the scheme and establish that a sequence generated therefrom converges strongly to a fixed point of an enriched nonexpansive mapping provided such a point exists. Furthermore, we provide some examples of the implementation of the schemes with respect to certain enriched mappings and show the numerical pattern of the scheme.

Keywords: Enriched nonexpansive mapping, implicit midpoint rule, fixed point, Hilbert space, viscosity iteration.

2020 Mathematics Subject Classification: 47H09, 47H10, 47J25, 47N20, 65J15.

1. INTRODUCTION

The viscosity scheme is among the prominent iterative methods for estimating a fixed point of a nonlinear mapping through strong convergence under certain feasible control conditions. This scheme was introduced by Moudafi in [10] based upon the results of [2]. The scheme was further studied by Xu [24] in the framework of Banach spaces. The scheme uses contraction mapping to induce a nonexpansive mapping to target a particular fixed point having a unique property. For a linear space $\mathcal{H}$ and a mapping $G : \mathcal{H} \to \mathcal{H}$, the viscosity scheme generates a sequence $\{u_n\}$ recursively by

$$u_{n+1} = \beta_n f(u_n) + (1 - \beta_n)G(u_n), \quad \forall n \geq 1,$$

where $\beta_n \in (0, 1)$ and f is a contraction mapping (that is,

$$\|f(u) - f(w)\| \leq \kappa\|u - w\|$$

for some $\kappa \in [0, 1)$). It is evident, based on [10, 24], that, if G is a nonexpansive mapping and $\{\beta_n\}$ satisfies some suitable condition, then the strong convergence of the scheme $\{u_n\}$ to a fixed point of G can be achieved, where the limit point solves the variational inequality problem involving f over the set of fixed points of G. This method is further extended to nonlinear mappings that are more general than nonexpansive mappings and also to nonlinear spaces. For further details on the viscosity scheme and related concepts of fixed points, see, for example, [9, 22] and the references therein. In [5], Berinde introduced an enriched nonexpansive mapping as a generalization of nonexpansive mappings as follows:

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Let $(\mathcal{H}, \|\cdot\|)$ be a normed linear space and a mapping $G : \mathcal{H} \to \mathcal{H}$ is said to be an enriched (α-enriched) nonexpansive if there exists $\alpha \geq 0$ such that

$$\|\alpha(u-w) + Gu - Gw\| \leq (\alpha+1)\|u-w\|, \quad \forall u, w \in \mathcal{H}. \tag{1.1}$$

Later on, Berinde in [7] considered G as an α-enriched nonexpansive mapping and established that a sequence $\{u_n\}$ generated by

$$u_{n+1} = \left(1 - \frac{\delta_n}{\alpha+1}\right)(1-\beta_n)u_n + \frac{\delta_n}{\alpha+1} G\big((1-\beta_n)u_n\big), \quad \forall n \geq 1, \tag{1.2}$$

converges strongly to a fixed point of G, where $\beta_n, \delta_n \in (0,1)$ with some control conditions. The scheme in (1.2) is a modification of the scheme in [27]. For further development concerning enriched nonexpansive mappings and approximation schemes in this direction even beyond linear spaces, see, for example, [6, 14, 16, 11, 8, 15, 18] and the references therein.

On the other hand, most real-life phenomena are addressed in the form of mathematical models that result in differential equations. Some of these differential equations are difficult to solve analytically. In this regard, engineers seek a numerically generated pattern that exhibits the structure of the real solutions. Thus the emphasis is on the need for numerical approaches to solving differential equations. One of these approaches is the implicit midpoint scheme, which is very promising for handling such differential equations. This scheme is appropriate mostly for stiff equations and differential algebra equations [3, 4, 21, 20, 19].

For a differential equation of the form

$$\begin{cases} u' = g(u), \\ u(0) = u_1, \end{cases}$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and smooth and the implicit midpoint scheme generates a sequence $\{u_n\}$ by solving

$$u_{n+1} = u_n + \eta g\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.3}$$

where η is known as step size. This idea was extended in [26] to fixed point theory considering that the state of equilibrium of such differential equation reduces to a fixed point problem. Thereafter, Alghamdi et al. [1] considered a nonexpansive mapping $G : \mathcal{H} \to \mathcal{H}$ and generate $\{u_n\}$ via the implicit midpoint scheme as

$$u_{n+1} = (1-\beta_n)u_n + \beta_n G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.4}$$

where $\beta_n \in (0,1)$ and $u_1 \in \mathcal{H}$. The authors established that, if $\{\beta_n\}$ is such that

$$\liminf_{n\to\infty} \beta_n > 0, \quad \beta_{n+1} \leq \eta\beta_n$$

for some fixed η, then $\{u_n\}$ converges weakly to a fixed point of G. In [12], the scheme (1.4) is modified and analyzed to approximate a fixed point of an α-enriched nonexpansive mapping in the sense that $\{u_n\}$ is updated based on the equation

$$u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right)u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2} G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.5}$$

where $\beta_n \in (0,1)$ for all $n \geq 1$. The authors established the weak convergence using a similar assumption as in [1]. However, the strong convergence result is more desirable in infinite dimensional spaces. In [25], Xu et al. addressed this problem for the case when G is a nonexpansive mapping by applying the viscosity technique to the scheme (1.4) and using different control conditions. The authors' scheme is as follows:

$$u_{n+1} = (1-\beta_n)f(u_n) + \beta_n G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.6}$$

where f is a contraction mapping.

The purpose of this work is to incorporate a contraction mapping in persuading the implicit midpoint scheme for enriched nonexpansive mappings. The proposed scheme is fashioned after (1.4), (1.5) and (1.6). We establish some convergence properties of the proposed scheme and show the strong convergence of the sequence generated therefrom to a fixed point of the mapping that also solves a variational inequality problem. It is worth noting that fixed points of enriched nonexpansive mappings have applications in many practical problems as they incorporate certain Lipschitz mappings with constants greater than 1. Finally, we give some numerical examples of the Lipschitz mappings and use them to show the explicit reduction of the scheme and the numerical implementations.

2. Preliminaries

In the sequel, unless otherwise stated, $\mathcal{E}$ stands for a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Given a mapping $G : \mathcal{E} \to \mathcal{H}$, we call a sequence $\{u_n\}$ an approximate fixed point sequence for G if

$$\|u_n - Gu_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Recall that Hilbert spaces possess Opial's property, that is, for a sequence $\{u_n\} \subset \mathcal{H}$ that converges weakly to u^*,

$$\liminf_{n\to\infty} \|u_n - u^*\| < \liminf_{n\to\infty} \|u_n - y\|, \quad \forall y \in \mathcal{H}\backslash\{u^*\}.$$

Now, we state the demiclosedness principle of an enriched nonexpansive mapping as in [12].

Lemma 2.1. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping. Suppose that $\{u_n\}$ is an approximate fixed point sequence for G and also $\{u_n\}$ weakly converges to u^*. Then u^* is a fixed point of G.*

Some identities involving two points in real Hilbert spaces are very crucial in obtaining our main results.

Lemma 2.2. *Let $u, w \in \mathcal{H}$ and $a \in \mathbb{R}$. Then, we have the following:*

(1) $\|u+w\|^2 = \|u\|^2 + \|w\|^2 + 2\langle u, w\rangle$.
(2) $\|u-w\|^2 = \|u\|^2 + \|w\|^2 - 2\langle u, w\rangle$.
(3) $\|au + (1-a)w\|^2 = a\|u\|^2 + (1-a)\|w\|^2 - a(1-a)\|u-w\|^2$.

Lemma 2.3. [23] *Let $\{\ell_n\}$ be a sequence of non-negative real numbers such that*

$$\ell_{n+1} \leq (1-\sigma_n)\ell_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\sigma_n\} \subseteq (0,1)$ and $\{\delta_n\} \subseteq \mathbb{R}$. Suppose that the following conditions are satisfied

$$\text{(C1)} \sum_{n=1}^{\infty} \sigma_n = \infty; \qquad \text{(C2) either} \sum_{n=1}^{\infty} |\delta_n| < \infty \quad \text{or} \quad \limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} \leq 0.$$

Then $\lim_{n\to\infty} \ell_n = 0$.

3. Viscosity Implicit Midpoint Scheme and Its Convergence

Now, we introduce the main algorithm as follows:

Algorithm 3.1. *Initialize $u_1 \in \mathcal{H}$ arbitrary and find u_{n+1} such that*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha(1+\beta_n)+2}u_n + \frac{2\beta_n(\alpha+1)}{\alpha(1+\beta_n)+2}f(u_n) + \frac{2(1-\beta_n)}{\alpha(1+\beta_n)+2}G\left(\frac{u_n+u_{n+1}}{2}\right),$$

where $\beta_n \in (0,1)$ for all $n \geq 1$, $\alpha \geq 0$ and $G : \mathcal{H} \to \mathcal{H}$ is a mapping and f is a contraction mapping with constant κ.

Remark 3.1. *It is worth noting that, for $\alpha = 0$, Algorithm 3.1 reduces to (1.6). The connection is evident since (1.1) implies that every nonexpansive mapping is 0-enriched nonexpansive.*

Remark 3.2. *It is not difficult to obtain from Algorithm 3.1 that u_{n+1} can be rewritten as follows:*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha}G\left(\frac{u_n+u_{n+1}}{2}\right). \tag{3.7}$$

Throughout this manuscript, we denote the fixed point set of a mapping G by $\mathcal{F}(G)$ and the metric projection onto a closed convex set $\mathcal{C}$ by $P_{\mathcal{C}}$.

Lemma 3.4. *Let G be an α-enriched nonexpansive mapping with $\mathcal{F}(G) \neq \emptyset$. Then $\{u_n\}$ generated through Algorithm 3.1 is bounded.*

Proof. Let $u^* \in \mathcal{F}(G)$ and set $w_n = \dfrac{u_n+u_{n+1}}{2}$. Then it follows from (3.7) and triangle inequality that

$$\begin{aligned}
\|u_{n+1}-u^*\| &= \left\|\frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha}G\left(\frac{u_n+u_{n+1}}{2}\right) - u^*\right\| \\
&= \left\|(1-\beta_n)\left(\frac{\alpha}{\alpha+1}w_n + \frac{1}{1+\alpha}G(w_n) - u^*\right) + \beta_n\left(f(u_n)-u^*\right)\right\| \\
&\leq (1-\beta_n)\left\|\frac{\alpha}{\alpha+1}w_n + \frac{1}{1+\alpha}G(w_n) - u^*\right\| + \beta_n\left\|\left(f(u_n)-u^*\right)\right\| \\
&= \frac{1-\beta_n}{\alpha+1}\left\|\alpha(w_n-u^*) + G(w_n) - G(u^*)\right\| + \beta_n\left\|f(u_n)-u^*\right\|.
\end{aligned}$$

Since G is α-enriched nonexpansive mapping, we have

$$\begin{aligned}
\|u_{n+1}-u^*\| &\leq (1-\beta_n)\|w_n-u^*\| + \beta_n\|f(u_n)-u^*\| \\
&= (1-\beta_n)\left\|\frac{1}{2}(u_n-u^*) + \frac{1}{2}(u_{n+1}-u^*)\right\| + \beta_n\|f(u_n)-u^*\| \\
&\leq \frac{1-\beta_n}{2}\|u_n-u^*\| + \frac{1-\beta_n}{2}\|u_{n+1}-u^*\| + \beta_n\|f(u_n)-u^*\|.
\end{aligned}$$

This gives

$$\frac{1+\beta_n}{2}\|u_{n+1}-u^*\| \leq \frac{1-\beta_n}{2}\|u_n-u^*\| + \beta_n\|f(u_n)-u^*\|.$$

From the fact that f is contraction mapping with constant κ, we have

$$\begin{aligned}\frac{1+\beta_n}{2}\left\|u_{n+1}-u^*\right\| &\le \frac{1-\beta_n}{2}\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u_n\right)-f\left(u^*\right)\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\| \\ &\le \frac{1-\beta_n}{2}\left\|u_n-u^*\right\|+\beta_n\kappa\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\| \\ &= \frac{1-\beta_n+2\beta_n\kappa}{2}\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\|.\end{aligned}$$

This implies that

$$\begin{aligned}\left\|u_{n+1}-u^*\right\| &\le \frac{1-\beta_n+2\beta_n\kappa}{1+\beta_n}\left\|u_n-u^*\right\|+\frac{2\beta_n}{1+\beta_n}\left\|f\left(u^*\right)-u^*\right\| \\ &= \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\left\|u_n-u^*\right\|+\frac{2\beta_n(1-\kappa)}{1+\beta_n}\frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa} \\ &\le \max\left\{\left\|u_n-u^*\right\|,\ \frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa}\right\}.\end{aligned}$$

Inductively, we obtain

$$\left\|u_{n+1}-u^*\right\| \le \max\left\{\left\|u_1-u^*\right\|,\ \frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa}\right\}, \quad \forall \ge 1.$$

This completes the proof. □

Lemma 3.5. *Let G be an α-enriched nonexpansive mapping with $\mathcal{F}(G) \neq \emptyset$. Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ satisfying the following conditions:*

(C1) $\beta_n \to 0$ *as* $n \to \infty$ *(C2)* $\sum_{n=1}^{\infty}\beta_n = \infty$ *(C3)* $\sum_{n=1}^{\infty}\left|\beta_{n+1}-\beta_n\right| < \infty.$

Then we have the following:

(P1) $\left\|u_{n+1}-u_n\right\| \to 0$ *as* $n \to \infty$; *(P2)* $\left\|u_n-G\left(u_n\right)\right\| \to 0$ *as* $n \to \infty$.

Proof. Set $w_n = \dfrac{u_n+u_{n+1}}{2}$ and G_α be the mapping defined by

$$G_\alpha(u) = \frac{\alpha}{\alpha+1}u+\frac{1}{\alpha+1}G(u), \quad \forall u \in Dom(G).$$

Then Algorithm 3.1 and (3.7) yield that

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \left\|\frac{\alpha(1-\beta_n)}{\alpha+1}(w_n)+\beta_n f(u_n)+\frac{1-\beta_n}{1+\alpha}G(w_n)-u_n\right\| \\
&= \|\beta_n f(u_n)+(1-\beta_n)G_\alpha(w_n)-u_n\| \\
&= \|\beta_n f(u_n)+(1-\beta_n)G_\alpha(w_n)-\beta_{n-1}f(u_{n-1})-(1-\beta_{n-1})G_\alpha(w_{n-1})\| \\
&= \Big\|(1-\beta_n)(G_\alpha(w_n)-G_\alpha(w_{n-1}))+(\beta_n-\beta_{n-1})(f(u_{n-1})-G_\alpha(w_{n-1})) \\
&+\beta_n(f(u_n)-f(u_{n-1}))\Big\| \\
&\le (1-\beta_n)\|G_\alpha(w_n)-G_\alpha(w_{n-1})\|+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&+\beta_n\|f(u_n)-f(u_{n-1})\| \\
&= \frac{1-\beta_n}{\alpha+1}\|\alpha(w_n-w_{n-1})+G(w_n)-G(w_{n-1})\| \\
&+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\|+\beta_n\|f(u_n)-f(u_{n-1})\|.
\end{aligned}$$

This and the facts that G is an α-enriched nonexpansive mapping and f is a contraction with constant κ yield

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le (1-\beta_n)\|w_n-w_{n-1}\|+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&+\beta_n\kappa\|u_n-u_{n-1}\| \\
&= \frac{1-\beta_n}{2}\|u_{n+1}-u_{n-1}\|+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&+\beta_n\kappa\|u_n-u_{n-1}\| \\
&\le \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n}{2}\|u_n-u_{n-1}\| \\
&+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\|+\beta_n\kappa\|u_n-u_{n-1}\| \\
&= \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\| \\
&+|\beta_n-\beta_{n-1}|\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&\le \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\| \\
&+\eta|\beta_n-\beta_{n-1}|,
\end{aligned}$$

where η is a positive number such that $\eta \ge \sup_{n\ge 1}\|f(u_{n-1})-G_\alpha(w_{n-1})\|$. Consequently, we get

$$\frac{1+\beta_n}{2}\|u_{n+1}-u_n\| \le \frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\|+\eta|\beta_n-\beta_{n-1}|,$$

which resulted to

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \frac{1-\beta_n+2\beta_n\kappa}{1+\beta_n}\|u_n-u_{n-1}\|+\frac{2\eta}{1+\beta_n}|\beta_n-\beta_{n-1}| \\
&= \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\|u_n-u_{n-1}\|+\frac{2\eta}{1+\beta_n}|\beta_n-\beta_{n-1}| \\
&\le \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\|u_n-u_{n-1}\|+2\eta|\beta_n-\beta_{n-1}|.
\end{aligned}$$

Thus Lemma 2.3 and the assumptions on $\{\beta_n\}$ yield the claim (P1). For Claim (P2), we start by obtaining the following inequalities:

$$\begin{aligned}\|u_n - G(u_n)\| &= (\alpha+1)\|u_n - G_\alpha(u_n)\| \\ &\leq (\alpha+1)\Big(\|u_n - u_{n+1}\| + \|u_{n+1} - G_\alpha(w_n)\| + \|G_\alpha(w_n) - G_\alpha(u_n)\| \Big) \\ &= (\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &+ \|\alpha(w_n - u_n) + G(w_n) - G(u_n)\|.\end{aligned}$$

This and the fact that G is an α-enriched nonexpansive mapping yield

$$\begin{aligned}\|u_n - G(u_n)\| &\leq (\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &+ (\alpha+1)\|w_n - u_n\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| \\ &+ (\alpha+1)\|\beta_n f(u_n) + (1-\beta_n)G_\alpha(w_n) - G_\alpha(w_n)\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\beta_n\|f(u_n) - G_\alpha(w_n)\| \\ &\leq \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\eta\beta_n.\end{aligned}$$

As $n \to +\infty$, the last inequality and Claim (P1) yield Claim (P2). This completes the proof. □

Theorem 3.2. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping with a fixed point and $f : \mathcal{E} \to \mathcal{E}$ be a contraction mapping. Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \text{ as } n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ converges strongly to the unique point $u^* \in \mathcal{F}(G)$ with a minimal norm.

Proof. Since f is a contraction mapping, $P_{\mathcal{F}(G)}f$ is also a contraction. Therefore, by the Banach contraction mapping, we have $u^* \in \mathcal{E}$ such that $u^* = P_{\mathcal{F}(G)}f(u^*)$. It is worth noting that the metric projection $P_{\mathcal{F}(G)}$ is well-defined since $\mathcal{F}(G)$ is nonempty closed and convex. By the properties of the metric projection, we have

$$\langle u^* - f(u^*), u^* - p\rangle \leq 0, \quad \forall p \in \mathcal{F}(G).$$

The boundedness of $\{u_n\}$ yields a subsequence $\{u_{n_k}\}$ that weakly converges to a point u^o. By the demiclosedness property of G and $(P2)$ of Lemma 2.2, we have $u^o \in \mathcal{F}(G)$. Moreover, without lost of generality, we have

$$\limsup_{n\to\infty}\langle u^* - f(u^*), u^* - u_n\rangle = \lim_{k\to\infty}\langle u^* - f(u^*), u^* - u_{n_k}\rangle.$$

Consequently, we have

$$\limsup_{n\to\infty}\langle u^* - f(u^*), u^* - u_n\rangle = \langle u^* - f(u^*), u^* - u^o\rangle \leq 0. \tag{3.8}$$

Let $w_n = \dfrac{u_n + u_{n+1}}{2}$. It follows from (3.7) and Lemma 2.2 (2) that

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \left\| \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha} G\left(\frac{u_n+u_{n+1}}{2}\right) - u^* \right\|^2 \\
&= \left\| (1-\beta_n)\left(\frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^*\right) + \beta_n (f(u_n) - u^*) \right\|^2 \\
&= (1-\beta_n)^2 \left\| \frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^* \right\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n)\left\langle \frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^*, f(u_n) - u^* \right\rangle \\
&= \left(\frac{1-\beta_n}{\alpha+1}\right)^2 \|\alpha(w_n - u^*) + G(w_n) - G(u^*)\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u_n) - u^* \rangle .
\end{aligned}$$

As a consequence of the immediate inequality, the fact that G is α-enriched nonexpansive mapping, f is a contraction with constant κ, and the Cauchy Schwartz inequality yield that

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &\le (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u_n) - u^* \rangle \\
&\le (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u_n) - f(u^*) \rangle \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&\le (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\kappa\beta_n(1-\beta_n)\|G_\alpha(w_n) - G_\alpha(u^*)\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&= (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ \frac{2\kappa\beta_n(1-\beta_n)}{\alpha+1}\|\alpha(w_n - u^*) + G(w_n) - G(u^*)\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&\le (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\kappa\beta_n(1-\beta_n)\|w_n - u^*\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle .
\end{aligned}$$

Now, setting

$$\theta_n = \|u_n - u^*\|$$

and

$$\phi_n = \beta_n^2 \|(f(u_n) - u^*)\|^2 + 2\beta_n(1-\beta_n)\langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle ,$$

we get

$$(1-\beta_n)^2 \|w_n - u^*\|^2 + 2\kappa\beta_n(1-\beta_n)\|w_n - u^*\| \theta_n + \phi_n - \theta_{n+1}^2 \ge 0.$$

Solving this quadratic inequality with respect to $\|w_n - u^*\|$ yields

$$\begin{aligned}\|w_n - u^*\| &\geq \frac{-2\kappa\beta_n(1-\beta_n)\theta_n + \sqrt{4\kappa^2\beta_n^2(1-\beta_n)^2\theta_n^2 - 4\left(1-\beta_n\right)^2\left(\phi_n - \theta_{n+1}^2\right)}}{2\left(1-\beta_n\right)^2} \\ &= \frac{-\kappa\beta_n\theta_n + \sqrt{\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n}}{1-\beta_n}.\end{aligned}$$

This implies that

$$\frac{1}{2}\|u_{n+1} - u^*\| + \frac{1}{2}\|u_n - u^*\| \geq \frac{-\kappa\beta_n\theta_n + \sqrt{\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n}}{1-\beta_n}.$$

Thus it turns out that

$$\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n \leq \left[\frac{1}{2}\left(1-\beta_n\right)\|u_{n+1} - u^*\| + \left(1 + (2\kappa - 1)\beta_n\right)\frac{1}{2}\|u_n - u^*\|\right]^2.$$

Thus, from the fact that $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$, it follows that

$$\begin{aligned}\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n &\leq \frac{1}{4}\left[\left(1-\beta_n\right)^2\|u_{n+1} - u^*\|^2 + \left(1 + (2\kappa-1)\beta_n\right)^2\|u_n - u^*\|^2\right] \\ &\quad + \frac{1}{2}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_{n+1} - u^*\|\,\|u_n - u^*\| \\ &\leq \frac{1}{4}\left[\left(1-\beta_n\right)^2\|u_{n+1} - u^*\|^2 + \left(1 + (2\kappa-1)\beta_n\right)^2\|u_n - u^*\|^2\right] \\ &\quad + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_{n+1} - u^*\|^2 \\ &\quad + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_n - u^*\|^2.\end{aligned}$$

By simple calculations, we can rewrite the last inequality as follows:

(3.9) $$\theta_{n+1}^2 \leq \psi_n\theta_n^2 + \varphi_n,$$

where

$$\psi_n = \frac{\frac{1}{4}\left(1 + (2\kappa-1)\beta_n\right)^2 + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right) - \kappa^2\beta_n^2}{1 - \frac{1}{4}(1-\beta_n)^2 - \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)}$$

and

$$\varphi_n = \frac{\phi_n}{1 - \frac{1}{4}(1-\beta_n)^2 - \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)}.$$

Observe further that

$$\psi_n = \frac{\frac{1}{2}\left(1 + (2\kappa-1)\beta_n\right)\left(1 - (1-\kappa)\beta_n\right) - \kappa^2\beta_n^2}{1 - \frac{1}{2}(1-\beta_n)\left(1 - (1-\kappa)\beta_n\right)}$$

and

$$\varphi_n = \frac{\phi_n}{1 - \frac{1}{2}(1-\beta_n)\left(1 - (1-\kappa)\beta_n\right)}.$$

Now, we complete the proof by showing that $\theta_n \to 0$ as $n \to \infty$. For that, consider a function g defined by

$$g(t) = \frac{2(1-\kappa) - (1-\kappa)^2 t + \kappa^2 t}{1 - \frac{1}{2}(1-t)(1-(1-\kappa)t)}.$$

It can be observed that

$$g(t) = \frac{1}{t}\left[1 - \frac{\frac{1}{2}\left(1+(2\kappa-1)t\right)\left(1-(1-\kappa)t\right) - \kappa^2 t^2}{1-\frac{1}{2}(1-t)\left(1-(1-\kappa)t\right)}\right]$$

and

$$\lim_{t\to 0} g(t) = 4(1-\kappa).$$

This implies that, for $\epsilon = 3(1-\kappa)$, there exists $\delta \in (0,1)$ such that $g(t) > \epsilon$ for all $t \in (0,\delta)$. Thus we have

$$1 - \frac{\frac{1}{2}\left(1+(2\kappa-1)t\right)\left(1-(1-\kappa)t\right) - \kappa^2 t^2}{1-\frac{1}{2}(1-t)\left(1-(1-\kappa)t\right)} > \epsilon t \tag{3.10}$$

for all $t \in (0,\delta)$. By the assumption that $\beta_n \to 0$ as $n \to \infty$, we can have a natural number N^* such that $\beta_n < \delta$ for all $n \geq N^*$. Consequently, it follows from (3.10) that $1 - \psi_n > \epsilon\beta_n$ for all $n \geq N^*$. Thus (3.9) gives

$$\theta_{n+1}^2 \leq (1-\epsilon\beta_n)\theta_n^2 + \varphi_n \quad \forall\, n \geq N^*. \tag{3.11}$$

Moreover, we have

$$\begin{aligned}
\frac{\phi_n}{\beta_n} &= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\left\langle G_\alpha(w_n) - u^*, f(u^*) - u^*\right\rangle \\
&= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\left\langle G_\alpha(w_n) - u_{n+1}, f(u^*) - u^*\right\rangle \\
&+ \left\langle u_{n+1} - u^*, f(u^*) - u^*\right\rangle \\
&= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\beta_n\left\langle G_\alpha(w_n) - f(u_n), f(u^*) - u^*\right\rangle \\
&+ \left\langle u_{n+1} - u^*, f(u^*) - u^*\right\rangle.
\end{aligned}$$

This, (3.8) and the assumption on $\{\beta_n\}$ yield that

$$\limsup_{n\to\infty} \frac{\phi_n}{\beta_n} \leq 0.$$

So, we have

$$\limsup_{n\to\infty} \frac{\psi_n}{\beta_n} \leq 0.$$

Finally, Lemma 2.3 and (3.11) yield that $\lim_{n\to\infty} \theta_n = 0$. This completes the proof. □

Next, we deduce the following corollary which is the main results of [25]:

Corollary 3.1. *Let $G : \mathcal{E} \to \mathcal{E}$ be a nonexpansive mapping with a fixed point and $f : \mathcal{E} \to \mathcal{E}$ be a contraction mapping. Suppose that $\{u_n\}$ is a sequence generated by (1.6) with $\{\beta_n\}$ satisfying the following conditions:*

(C1) $\beta_n \to 0$ *as* $n \to \infty$; *(C2)* $\sum_{n=1}^{\infty} \beta_n = \infty$; *(C3)* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{u_n\}$ *converges strongly to the unique point* $u^* \in \mathcal{F}(G)$ *with a minimal norm.*

Proof. When $\alpha = 0$, then Algorithm 3.1 reduces to (1.6). Consequently, Theorem 3.2 yields the proof using the fact that a nonexpansive mapping is 0-enriched nonexpansive. □

Recall that a multivalued mapping $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be monotone if, for every $u, w \in \mathcal{H}$, $x \in Mu$ and $y \in Mw$, we have

$$\langle u - w, x - y\rangle \geq 0.$$

Moreover, M is said to be maximal monotone if, for every $(u, x) \in \mathcal{H}$,

$$\langle x - y, u - w\rangle \geq 0$$

for every $(w, y) \in \mathrm{Graph}(M)$ implies $x \in Mu$. It is known that, if M is maximal monotone, then, for any $\xi > 0$, the mapping $(I + \xi M)^{-1}$ is single-valued, nonexpansive and

$$\mathrm{dom}\left((I + \xi M)^{-1}\right) = \mathcal{H}.$$

Furthermore, we have

$$0 \in Mu^* \quad \Longleftrightarrow \quad u \in \mathcal{F}\left((I + \xi M)^{-1}\right).$$

Corollary 3.2. *Let* $M : \mathcal{H} \to 2^{\mathcal{H}}$ *be a maximal monotone. For any* $\xi > 0$ *and* $\eta \geq 1$*, consider* $G : \mathcal{H} \to \mathcal{H}$ *defined by*

$$Gu = \eta(I + \xi M)^{-1}u - (\eta - 1)u, \quad \forall u \in \mathcal{H}.$$

Suppose that $\{u_n\}$ *is a sequence generated by* (1.6) *with* $\alpha = \eta - 1$ *and* $\{\beta_n\}$ *satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \;\text{ as }\; n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ *converges strongly to a zero of* M.

Proof. Using the fact that $(I + \xi M)^{-1}$ is nonexpansive, we can deduce that G is an α-enriched nonexpansive mapping. Indeed, for all $u, w \in \mathcal{H}$, we get

$$\begin{aligned}\|\alpha(u - w) + Gu - Gw\| &= \left\|(\alpha + 1)(I + \xi M)^{-1}u - (\alpha + 1)(I + \xi M)^{-1}w\right\| \\ &= (\alpha + 1)\left\|(I + \xi M)^{-1}u - (I + \xi M)^{-1}w\right\| \\ &\leq (\alpha + 1)\|u - w\|.\end{aligned}$$

Thus Theorem 3.2 guarantees that $\{u_n\}$ converges to a fixed point of G. Let the limit point be u^*. Then we have

$$u^* = Gu^* \quad \Longleftrightarrow \quad u^* = \eta(I + \xi M)^{-1}u^* - (\eta - 1)u^* \quad \Longleftrightarrow \quad u^* = (I + \xi M)^{-1}u^*.$$

Consequently, it follows that $0 \in Mu^*$. This completes the proof. □

A particular case of the immediate corollary is the case when M is equal to the subdifferential of a convex proper and lower semi-continuous function $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. In this regard, we have the next corollary:

Corollary 3.3. *Let* $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ *be a convex proper and lower semi-continuous function. For any* $\xi > 0$ *and* $\eta \geq 1$*, consider* $G : \mathcal{H} \to \mathcal{H}$ *defined by*

$$Gu = \eta(I + \xi \partial f)^{-1}u - (\eta - 1)u, \quad \forall u \in \mathcal{H}.$$

Suppose that $\{u_n\}$ *is a sequence generated by* (1.6) *with* $\alpha = \eta - 1$ *and* $\{\beta_n\}$ *satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \ \text{ as } \ n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ *converges strongly to a minimizer of* f.

Proof. The proof follows from Corollary 3.2 and the fact that

$$0 \in \partial f(u^*) \quad \iff \quad f(u^*) \leq f(u), \quad \forall u \in \mathcal{H}.$$

□

4. Numerical Illustrations

This part contains two numerical problems where the underlined mappings are not nonexpansive but enriched nonexpansive mappings. The purpose is to show the implementation of our method with respect to such mappings and to show the impact of the proposed scheme on handling stiff equations involving enriched nonexpansive mapping.

Example 4.1. *Consider* $\mathcal{H} = \mathbb{R}$ *endowed with the usual norm and take* $\mathcal{E} = \left[\frac{1}{2}, 2\right]$. *Define a mapping* $G : \mathcal{E} \to \mathcal{E}$ *by* $Gu = \frac{1}{u}$, *for all* $u \in \mathcal{E}$. *Then* G *is* $\frac{3}{2}$*-enriched nonexpansive mapping with* 1 *as fixed point but not nonexpansive (see* [5]*). For this example, we set* $f : u \mapsto \frac{u+1}{2}$. *Consequently, Algorithm 3.1 gives*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n + u_{n+1}}{2}\right) + \beta_n \frac{u_n + 1}{2} + \frac{1-\beta_n}{1+\alpha}\left(\frac{2}{u_n + u_{n+1}}\right).$$

Solving for u_{n+1}, *we get*

$$u_{n+1} = \frac{\tau_n u_n - \beta_n - \sqrt{(\beta_n+2)^2 u_n^2 + 2\beta_n(\beta_n+2)u_n + \beta_n^2 + 16c_n(2-\alpha c_n)}}{2(\alpha c_n - 2)} \tag{4.12}$$

for all $n \geq 1$, *where* $\tau_n = 2 - 2\alpha c_n - \beta_n$ *and* $c_n = \dfrac{1-\beta_n}{\alpha+1}$.

To show the numerical patterns of the scheme for this example, we set $\beta_n = \frac{1}{n+1}$ *and use* α *as* $3/2$. *The first few generated values when truncated to six decimal places, are shown in Table 1. In the table, 'IMS Alg' stands for our proposed implicit midpoint scheme which reduces to* (4.12) *and 'MKM Alg' stands for the modified Krasnosel'skiĭ-Mann scheme of Berinde* [7] *which is stated in* (1.2)*. We note here that the sequence* $\{\delta_n\}$ *is considered as* $\delta_n = \frac{n}{2n+2}$ *to meet up with the assumption in* [7].

TABLE 1. Few numerical values of $\{u_n\}$

	Case 1		*Case 2*		*Case 3*		*Case 4*	
n	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*
1	2	2	1.85	1.85	0.75	0.75	0.5	0.5
2	1.374738	1	1.313463	0.940608	0.927576	0.604167	0.872325	0.625
3	1.107046	0.777778	1.087951	0.756091	0.982499	0.680109	0.970009	0.681111
4	1.024461	0.752976	1.019912	0.746526	0.996253	0.72764	0.993625	0.727846
5	1.004859	0.771613	1.003945	0.769573	0.999269	0.763835	0.998758	0.763896
6	1.000887	0.79504	1.000719	0.79431	0.999867	0.792278	0.999774	0.792299
7	1.000152	0.8162	1.000124	0.815913	0.999977	0.815116	0.999961	0.815124
8	1.000025	0.834232	1.00002	0.834111	0.999996	0.833776	0.999994	0.833779
9	1.000004	0.849452	1.000003	0.849398	0.999999	0.849249	0.999999	0.849251
10	1.000001	0.862341	1	0.862317	1	0.862248	1	0.862249
11	1	0.873338	1	0.873326	1	0.873293	1	0.873294
12	1	0.882797	1	0.882791	1	0.882775	1	0.882775
13	1	0.891001	1	0.890998	1	0.89099	1	0.89099
14	1	0.898172	1	0.898171	1	0.898167	1	0.898167
15	1	0.904487	1	0.904486	1	0.904484	1	0.904484
16	1	0.910085	1	0.910084	1	0.910083	1	0.910083
17	1	0.915077	1	0.915077	1	0.915076	1	0.915076
18	1	0.919555	1	0.919555	1	0.919555	1	0.919555
19	1	0.923592	1	0.923592	1	0.923592	1	0.923592
20	1	0.92725	1	0.92725	1	0.92725	1	0.92725

Remark 4.3. *Table 1 shows that based on the Example 4.1, the proposed scheme (IMS Alg) converges faster than the modified Krasnosel'skiĭ-Mann scheme. Indeed, IMS Alg reaches the fixed point value* (1) *in less than ten loops.*

Example 4.2. *For any $\xi > 0$, consider the stiff equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -\xi y(t), \quad y(0) = y_1 = \beta, \quad \forall t \geq 0.$$

This represents a model of a lot of physical Phenomena most of which arise through sciences and engineering. This problem has the solution

$$y(t) = \beta e^{-\xi t}, \quad y(t) \to 0 \text{ as } t \to \infty.$$

The aim of numerical methods for solving such initial value problems is primarily to exhibit the structure of the solution. So, in most cases due to the tediousness of establishing an analytical solution of stiff equations, engineers employ numerical methods to describe the solution. Since our proposed algorithm is based on the implicit midpoint rule (which is prominent in handling stiff equations), we investigate *the performance of the proposed scheme in exhibiting the structure of the solution in comparison with the modified Krasnosel'skiĭ-Mann scheme.*

Now, consider G as a mapping such that $u \mapsto -(\xi+1)u$. Then G is not nonexpansive mapping. However G is $\xi/2$-enriched nonexpansive mapping since

$$\begin{aligned}\left\|\frac{\xi}{2}(u-w)+Gu-Gw\right\| &= \left\|\frac{5}{2}(u-w)-(\xi+1)(u-w)\right\| \\ &= \left\|\left(\frac{\xi}{2}-\xi-1\right)(u-w)\right\| \\ &= \frac{\xi+2}{2}\|u-w\| \\ &= \left(\frac{\xi}{2}+1\right)\|u-w\|.\end{aligned}$$

For this example, we take $f : u \mapsto \dfrac{u}{5}$ and so Algorithm 3.1 gives

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right)+\frac{\beta_n}{5}u_n-(\xi+1)\frac{1-\beta_n}{1+\alpha}\left(\frac{u_n+u_{n+1}}{2}\right).$$

Solving for u_{n+1} and substituting $\alpha=\xi/2$, we get

$$u_{n+1} = \frac{7\beta_n-5}{5(3-\beta_n)}u_n.$$

To extract numerically the strsucture of the solution using our proposed scheme and that of (1.2), we maintain the sequence values of $\{\beta_n\}$ for the two algorithms as in the Example 4.1 and set $\beta=1$. The measure of how far the iterate u_n is from the value of the exact solution $\beta e^{-\xi(n-1)}$ at each n (up to $n=20$) is shown in Table 2 and Figure 1-6. In the table, the column VIMS represents in absolute value how far our proposed scheme is from the value of the exact solution. Cases 1-6 similarly show how far is the iterate (1.2) is to the value of the exact solution when δ_n $(n\in\mathbb{N})$ is set as $\frac{1}{2}$, $\frac{n}{2n+2}$, $\frac{n}{n+100}$, $\frac{4}{5}$, $\frac{n}{5n+3}$ and $\frac{2n}{3n+7}$, respectively.

5. Conclusion Remarks

In this work, we analyzed the convergence of a viscosity implicit midpoint scheme to a fixed point of an enriched nonexpansive mapping within the setting of Hilbert spaces. We established that the sequence generated by this scheme converges strongly to a particular fixed point of the underlying mapping. We provided examples where the mappings are not nonexpansive but are instead enriched nonexpansive, and we derived the explicit form of the proposed scheme. The numerical results obtained using this scheme are reported, demonstrating the distance between the iterates of the proposed scheme and those of the exact solution, in comparison to the well-known modified Krasnosel'skiĭ-Mann scheme by Berinde [7]. Despite the computational demands, our numerical data shows that, for the example considered, the proposed scheme achieves a higher degree of numerical stability than the Krasnosel'skiĭ-Mann scheme of Berinde [7]. Given that geodesically connected spaces can be viewed as nonlinear analogs of normed linear spaces [17, 13], it would be an interesting direction for future studies to extend the analyses presented here to such settings.

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TABLE 2. Few numerical values of $\left\{\left|u_n - e^{-3/2(n-1)}\right|\right\}$

	VIMS	*MKM Alg*					
n		*Case 1*	*Case 2*	*Case 3*	*Case 4*	*Case 5*	*Case 6*
1	*0.950213*	*0.950213*	*0.950213*	*0.950213*	*0.950213*	*0.950213*	*0.950213*
2	*0.131109*	*0.546823*	*0.403966*	*0.266767*	*0.718252*	*0.332538*	*0.375395*
3	*0.021521*	*0.380174*	*0.228247*	*0.08656*	*0.664256*	*0.142341*	*0.20435*
4	*0.006226*	*0.308042*	*0.161237*	*0.036165*	*0.707768*	*0.075549*	*0.144595*
5	*0.001335*	*0.263439*	*0.122915*	*0.015374*	*0.80004*	*0.0418*	*0.112944*
6	*0.000422*	*0.235351*	*0.100118*	*0.007188*	*0.943077*	*0.024625*	*0.096086*
7	*0.000104*	*0.216107*	*0.08491*	*0.003459*	*1.143201*	*0.014917*	*0.086329*
8	*3.31E-05*	*0.202607*	*0.074303*	*0.001744*	*1.41472*	*0.009279*	*0.08094*
9	*8.96E-06*	*0.192958*	*0.06657*	*0.000905*	*1.778503*	*0.005877*	*0.07841*
10	*2.82E-06*	*0.186067*	*0.060769*	*0.000485*	*2.263781*	*0.003778*	*0.077983*
11	*8.09E-07*	*0.181234*	*0.056321*	*0.000266*	*2.910575*	*0.002458*	*0.079242*
12	*2.53E-07*	*0.177998*	*0.052857*	*0.00015*	*3.773353*	*0.001615*	*0.081977*
13	*7.55E-08*	*0.176042*	*0.050131*	*8.59E-05*	*4.926092*	*0.00107*	*0.086104*
14	*2.36E-08*	*0.175143*	*0.047975*	*5.04E-05*	*6.469265*	*0.000714*	*0.091624*
15	*7.20E-09*	*0.175143*	*0.04627*	*3.01E-05*	*8.53943*	*0.000479*	*0.098605*
16	*2.26E-09*	*0.175925*	*0.044927*	*1.83E-05*	*11.32237*	*0.000323*	*0.107172*
17	*7.00E-10*	*0.177404*	*0.043883*	*1.13E-05*	*15.07112*	*0.000219*	*0.117505*
18	*2.20E-10*	*0.179516*	*0.04309*	*7.13E-06*	*20.13071*	*0.000149*	*0.129837*
19	*6.90E-11*	*0.182215*	*0.04251*	*4.56E-06*	*26.97213*	*0.000102*	*0.144465*
20	*2.18E-11*	*0.185469*	*0.042115*	*2.95E-06*	*36.23898*	*6.99E-05*	*0.161749*

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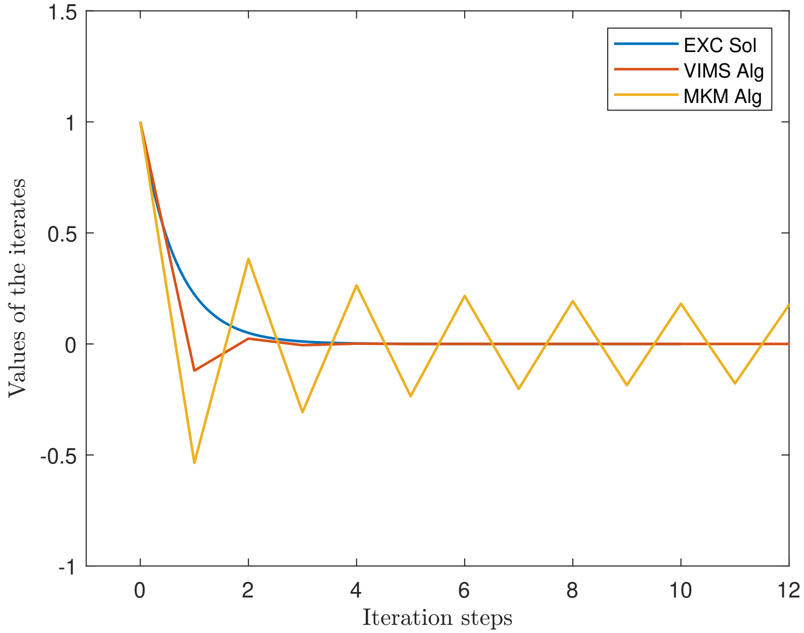

FIGURE 1. Numerical stability due to Case 1

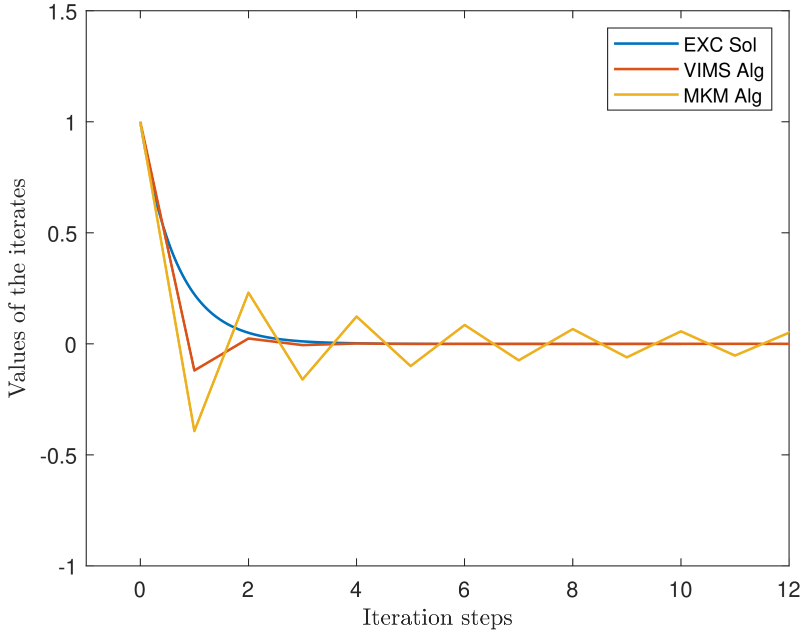

FIGURE 2. Numerical stability due to Case 2

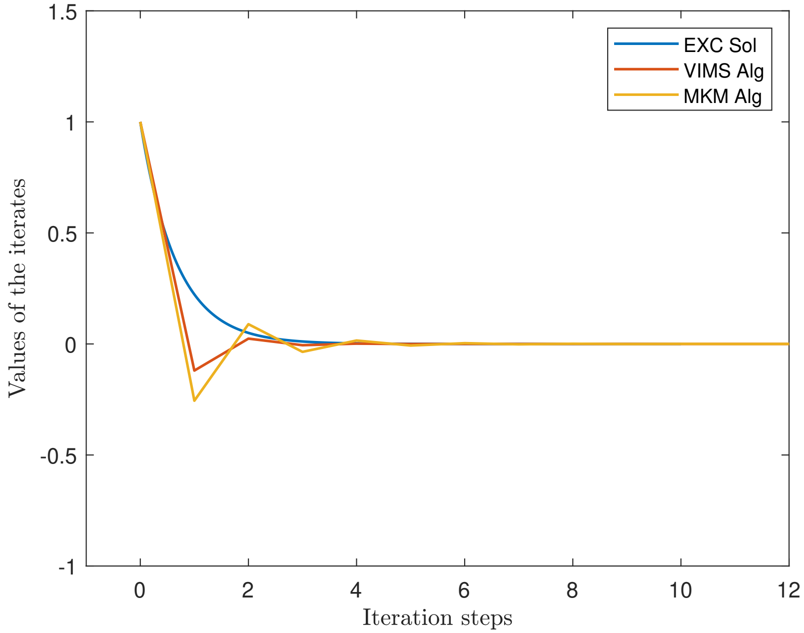

FIGURE 3. Numerical stability due to Case 3

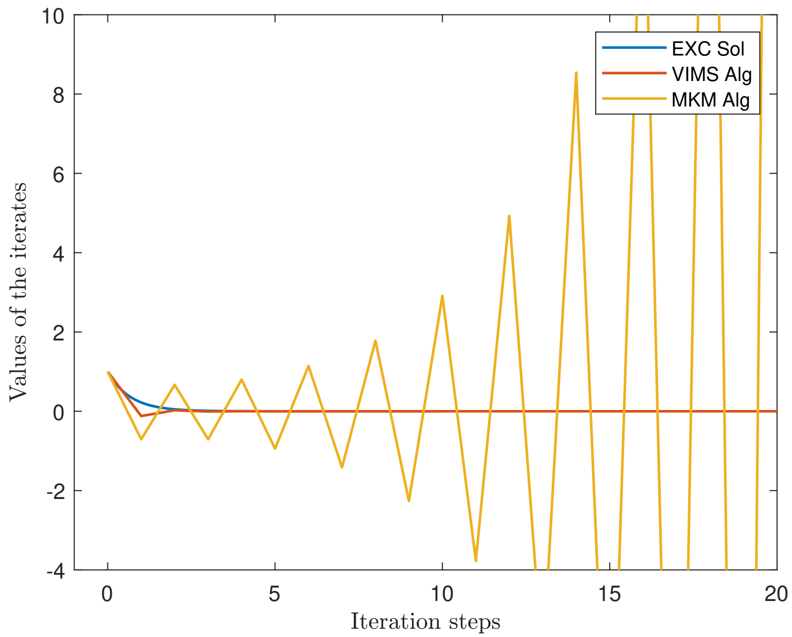

FIGURE 4. Numerical stability due to Case 4

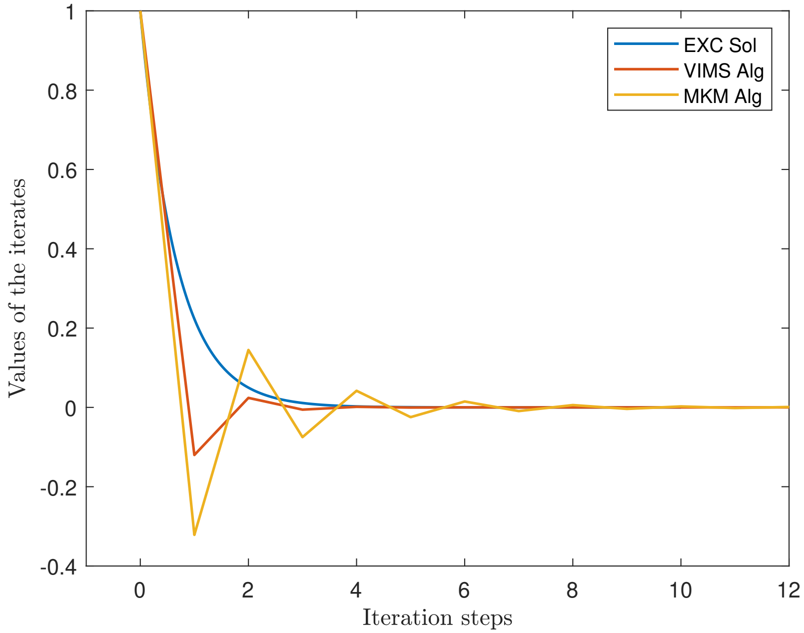

FIGURE 5. Numerical stability due to Case 5

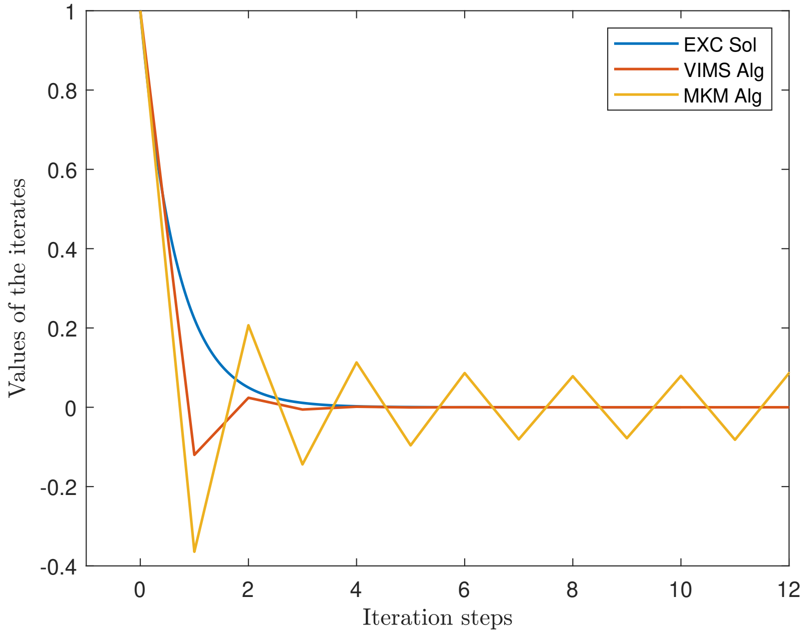

FIGURE 6. Numerical stability due to Case 6

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