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Solitons on Nearly Cosymplectic Manifold Exhibitting Schouten Van Kampen Connection

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Abstract
The following research investigates various types of soliton of NC (Nearly Cosymplectic) manifolds with SVK (Schouten-van Kampen) connections, which are steady, shrinking, or expanding. Further, we investigate the geometric characteristics of Ricci solitons, Yamabe solitons, η-ricci soliton etc. We also study the curvature features of the SVK connection on an NC manifold. In addition, an example is developed to demonstrate the results.

Keywords: Einstein manifold, Nearly cosymplectic manifold, Quasi Einstein manifold, Ricci soliton, Schouten van Kampen connection, Yamabe soliton
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1. Introduction

The Schouten Van Kampen connection is one of the most significant connections having the property of an affine connection [1]-[4]. The contact metric manifolds and solitons on these manifolds with respect to Schouten van Kampen connection are studied by several authors [3]-[7]. Nearly cosymplectic structures were introduced by Blair [8] and first appeared essentially as the hypersurface of nearly Kahler manifolds. A nearly cosymplectic manifold is defined as an almost contact metric manifold with a normality condition having closed 1-form η and 2-form F [9]. Various geometrical properties of a Nearly cosymplectic manifold was investigated by Endo [2].

Hamilton introduced the idea of the Ricci flow to find out the canonical metric over a smooth manifold [10, 11]. By the introduction of Ricci flow it is easy to study manifolds with positive curvature. Perelman proved the Poincare conjecture using Ricci flow [12, 13]. The term Ricci soliton refers to the limit of the solutions of the Ricci flow. In general, an almost ricci soliton is a simplification of an Einstien metric. For a complete vector field Y on a Riemannian manifold M of dimension n, a Riemannian metric g on M is termed a nearly Ricci soliton if it satisfies

$$L_Y g + 2S + 2\alpha g = 0, \tag{1.1}$$

where α is a smooth function, S stands for the Ricci tensor, and L is the Lie derivative. A metric g that satisfies (1.1) is referred to as a Ricci soliton if α is a constant. If $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$, then a Ricci soliton is expanding, steady, or shrinking, respectively. The concept of η-Ricci soliton was introduced by Cho and Kimura [9]. An almost η-Ricci soliton is a Riemannian manifold M with Riemannian metric g if for a smooth vector field Y such that,

$$L_Y g + 2S + 2\alpha g + 2\beta\eta \otimes \eta = 0, \tag{1.2}$$

for both the smooth functions, α and β.

If both α and β are constant, the metric g is referred as a η-Ricci soliton. A Ricci soliton has constant curvature for a compact manifold of dimension two or three [2, 14].

Hamilton [11] proposed the idea of Yamabe flow. A vector field Y that is static on a Riemannian manifold M generates the Yamabe solitons, which are self-similar outcomes of the Yamabe flow and are transformed by a family of diffeomorphisms with one parameter. On a Riemannian manifold (M, g), a triplet (g, Y, γ) is said to be a nearly Yamabe soliton if [10]

$$\frac{1}{2}L_Y g = (r - \gamma) \tag{1.3}$$

where γ is a smooth function and r is the scalar curvature of manifold (M,g). When γ remains constant, the almost Yamabe soliton transforms into a Yamabe soliton. If $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$, respectively, then a Yamabe soliton is expanding, steady, or shrinking. Yamabe and Ricci soliton coincide for the manifold of dimension 2 but they have distinct behaviors for the manifolds of dimensions greater than 2. Furthermore, Nearly Yamabe solitons always represent Einstein manifolds. And, the Riemannian metric g becomes a Yamabe metric if the Riemannian manifold M has constant scalar curvature [4].

In this study, we investigate several forms of Ricci and Yamabe solitons over NC manifold of dimension n with an SVK connection. Section 2 gives a brief description of the NC manifold and SVK connection. Section 3 introduces the SVK connection on the NC manifold and establishes the formulas for curvature tensor, Ricci tensor, Ricci operator, and scalar curvature. In Section 4, we investigate the Ricci solitons for an NC manifold with an SVK connection. The final section investigates Yamabe solitons on an n-dimensional NC manifold with SVK connection.

2. Preliminaries

Consider an (2n+1) dimensional almost contact manifold with structure (M, ϕ, ξ, η, g), where ξ is the vector field, η is a 1-form, g is the Riemannian Metric, and ϕ is a (1,1) tensor field. The following prerequisites are satisfied by this (ϕ, ξ, η, g) structure [9].

$$\phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1,$$

$$\phi^2 X = -X + \eta(X)\xi, \eta(X) = g(X, \xi). \tag{2.1}$$

Let g be compatible i.e.

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

As ϕ is a skew-symmetric operator with g, as per the definition above, η is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and the bilinear form F = g(X, ϕY) defines a 2-form [15].

An almost contact metric manifold with (M, ϕ, ξ, η, g) is said to be a Nearly Cosymplectic manifold if

$$(\triangledown_X \phi)Y + (\triangledown_Y \phi)X = 0,$$

for each vector field X, Y. It is clear that this condition is the same as $(\triangledown_X \phi)X = 0$.

The Reeb vector field ξ is defined for nearly cosymplectic manifolds is killing if it fulfills the requirements $\triangledown_\xi \xi = 0$ and $\triangledown_\xi \eta = 0$.

Moreover, the type (1,1) tensor field H defined by [9]

$$\triangledown_X \xi = HX \tag{2.3}$$

is anti-commutative with ϕ and skew-symmetric. Additionally, H providing

$$H\xi = 0, \eta(HX) = 0,$$

$$TraceH = 0,$$

$$\phi H = -H\phi, g(HX,Y) = g(X,HY).$$

These formulae also hold [15]-[18]

$$g((\triangledown_{X\phi})Y,HZ) = \eta(Y)g(H^2X,\phi Z) - \eta(X)g(H^2Y,\phi Z),$$

$$(\triangledown_X H)Y = g(H^2X,Y)\xi - \eta(Y)H^2X,$$

$$TraceH^2 = a(constant), \tag{2.4}$$

$$R(Y,Z)\xi = \eta(Y)H^2Z - \eta(Z)H^2Y,$$

$$S(X,Y) = -\lambda g(X,Y),$$

$$QX = -\lambda X,$$

$$S(X,\xi) = \lambda\eta(X), \tag{2.5}$$

where λ: M(manifold) $\rightarrow$ R(real number) is a function.

$$S(\phi Y,Z) = S(Y,\phi Z),$$

$$\phi Q = Q\phi,$$

$$S(\phi Y,\phi Z) = S(Y,Z) + \eta(Y)\eta(Z)(TraceH^2).$$

A contact metric structure (η, g) on M is η-Einstein if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are constants. If $b = 0$, then the manifold M is an Einstein manifold [7].

A quasi-Einstein manifold is defined as one whose Ricci tensor S of type (0,2) is not identically zero and meets the condition [19]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.6}$$

for all vector fields X,Y, where a,b are scalars, $b \neq 0$, and η is a non-zero 1-form.

In the tangent bundle TM of M, there are two naturally determined distributions, U = Ker η and V = Span ξ, such that TM = U$\oplus$V, U $\cap$ V= 0 and U$\perp$V. For this decomposition the SVK connection can be defined over a nearly contact metric structure.

Concerning the Levi-Civita Connection $\triangledown$, the Schouten Van Kampen Connection $\tilde{\triangledown}$ on a nearly contact metric manifold is defined by [20]

$$\tilde{\triangledown}_X Y = \triangledown_X Y - \eta(Y)\triangledown_X \xi + (\triangledown_X \eta)(Y)\xi. \tag{2.7}$$

3. Curvature Properties of NC Manifold concerning SVK Connection $\tilde{\triangledown}$

Let $\tilde{M}$ be an NC manifold, then using (2.1), (2.3) and (2.6), in (2.7) we have

$$\tilde{\triangledown}_X Y = \triangledown_X Y - \eta(Y)HX - g(Y,HX)\xi. \tag{3.1}$$

Moreover,

$$\tilde{\triangledown}_X \xi = 0. \tag{3.2}$$

If R and $\tilde{R}$ are curvature tensors with respect to $\triangledown$ and $\tilde{\triangledown}$, then

$$R(X,Y)Z = \triangledown_X \triangledown_Y Z - \triangledown_Y \triangledown_X Z - \triangledown_{[X,Y]}Z, \tag{3.3}$$

and

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{3.4}$$

By using (2.1), (2.4), (3.1), (3.2), (3.3) in (3.4), we have

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + g(Z,HX)HY - g(Z,HY)HX - X\eta(Z) + \eta(X)\eta(Z)\xi \\ &- \eta(Y)g(Z,X)\xi + Y\eta(Z) - \eta(Y)\eta(Z)\xi + \eta(X)g(Z,Y)\xi.\end{aligned}$$

Using the above equation, the Ricci tensor of NC Manifold with SVK connection can be obtained as

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-1)\eta(Z). \tag{3.5}$$

The Ricci operator $\widetilde{Q}$ for NC Manifolds with respect to the connection $\widetilde{\nabla}$ is given by

$$\widetilde{S}(Y,Z) = g(\widetilde{Q}Y,Z). \tag{3.6}$$

From equations (3.5) and (3.6), we have

$$\widetilde{Q}Y = QY - (n-1)\xi.$$

The scalar curvature with respect to the connection $\widetilde{\nabla}$ is given by

$$\widetilde{r} = r - (n-1). \tag{3.7}$$

4. Ricci Soliton Types on an n-dimensional NC Manifold with SVK Connection

In this section, we explore types of Ricci soliton kinds on NC manifold M having SVK connection $\widetilde{\nabla}$.

A SVK connection $\widetilde{\nabla}$ in an NC manifold $\widetilde{M}$, is said to be metric if $(\widetilde{\nabla} g)$ = 0 torsion tensor $\widetilde{T} \neq 0$, where $\widetilde{T}$ is torsion tensor with respect to $\widetilde{\nabla}$.

With the help of equation (3.1), we can easily find the value

$$(\widetilde{L}_Y g)(X,Z) = g(\nabla_X Y, Z) + g(X, \nabla_Z Y) = (L_Y g)(X,Z), \tag{4.1}$$

where L and $\widetilde{L}$ are Lie derivatives on NC manifold with respect to Levi-Civita connection ∇ and SVK connection $\widetilde{\nabla}$ respectively.

Now, for an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection $\widetilde{\nabla}$, the almost Ricci soliton is given by

$$\widetilde{L}_Y g + 2\widetilde{S} + 2\alpha g = 0. \tag{4.2}$$

Using (4.1) and (4.2), we get

$$g(\nabla_X Y, Z) + g(X, \nabla_Z Y) + 2\widetilde{S}(X,Y) + 2\alpha g(X,Y) = 0.$$

Therefore,

$$2\widetilde{S}(X,Y) = -g(\nabla_X Y, Z) - g(X, \nabla_Z Y) - 2\alpha g(X,Y).$$

By substituting Y = ξ in previous equation and using (2.2) and (2.3), we have

$$\widetilde{S}(X,Z) = -\alpha g(X,Z). \tag{4.3}$$

In view of (3.5) we can write above equation as

$$S(X,Z) = -(n-1)\eta(X)\eta(Z) - g(Z,\phi X) trace\phi - \alpha g(X,Z).$$

Conversely, consider that an n-dimensional NC manifold $\widetilde{M}$ with respect to SVK connection $\widetilde{\nabla}$ is an Einstein manifold. For Y = ξ, we have

$$\widetilde{S}(X,Z) = -\lambda g(X,Z)$$

and

$$(\widetilde{L_\xi g})(X,Z) = 0,$$

where λ is a constant.

Using the above results, we can easily find the value of

$$(\widetilde{L}_\xi g)(X,Z) + 2\widetilde{S}(X,Z) + 2\alpha g(X,Z) = 2(\alpha - \lambda) g(X,Z). \tag{4.4}$$

Therefore, it is evident from (4.4) that if $\alpha - \lambda = 0$, then the manifold $\widetilde{M}$ admits a Ricci soliton. Thus, we make the following statement:

Theorem 4.1. *An n-dimensional NC manifold $\widetilde{M}$ admits a Ricci soliton with respect to SVK connection $\widetilde{\bigtriangledown}$ iff $\widetilde{M}$ is an Einstein manifold with respect to SVK connection $\underset{\sim}{\bigtriangledown}$.*

Corollary 4.2. *A Ricci Soliton on NC manifold with SVK connection is Einstien Manifold.*

Proof. Put $X = \xi$ in (4.2), we have

$$(\widetilde{L}_\xi g)(X,Z) + 2\widetilde{S}(X,Z) + 2\alpha g(X,Z) = 0.$$

Using (4.1),

$$(\widetilde{L}_\xi g)(X,Z) = 0.$$

Thus

$$\widetilde{S}(X,Z) = -\alpha g(X,Z). \tag{4.5}$$

Using (4.5) and (3.6), we have

$$S(X,Z) = -(n-1)\eta(X) - \alpha g(X,Z).$$

Hence the theorem. □

Theorem 4.3. *The scalar curvature for an n-dimensional NC manifold $\widetilde{M}$ with SVK connection having an almost Ricci soliton, is $\widetilde{r}$ = -αn.*

Proof. In view of (4.3), we have

$$\widetilde{r} = -\alpha n.$$

Hence the theorem. □

Theorem 4.4. *An n-dimensional NC manifold $\widetilde{M}$ with SVK connection will be an Einstien manifold if $\widetilde{M}$ with SVK connection enabling a η-Ricci soliton.*

Proof. Currently, based on (1.2), the η-Ricci soliton on an NCM of dimension n with SVK connection is given by

$$(\widetilde{L}_Y g)(X,Z) + 2\widetilde{S}(X,Z) + 2\alpha g(X,Z) + 2\beta(\eta \otimes \eta)(X,Z) = 0. \tag{4.6}$$

From (4.1) and (4.6), we have

$$g(\bigtriangledown_X Y, Z) + g(X, \bigtriangledown_Z Y) + 2\widetilde{S}(X,Z) + 2\alpha g(X,Z) + 2\beta\eta(X)\eta(Z) = 0. \tag{4.7}$$

Putting the values from (2.2) and (2.3) in (4.7), we get

$$\widetilde{S}(X,Z) = -\alpha g(X,Z) - \beta\eta(X)\eta(Z). \tag{4.8}$$

Hence the theorem. □

Corollary 4.5. *If an η-Ricci soliton on an NC manifold $\widetilde{M}$ with SVK connection is defined, then ($\widetilde{M}$, g) is Quasi Einstien.*

Proof. Using (1.2), (4.8) and (4.9), we can easily find the required result. □

Theorem 4.6. *A SVK connection on an NC manifold $\widetilde{M}$ admitting Ricci soliton is invariant iff it satisfies*

$$g(Y,HX)\eta(Z)+g(Y,HZ)\eta(X)+2(n-1)\eta(Z)=0.$$

Proof. Using (3.1), (4.1), (4.2), we get

$$(\widetilde{L}_Y g)(X,Z)=(L_Y g)(X,Y)-g(Y,HX)\eta(Z)-g(Y,HZ)\eta(Y). \tag{4.9}$$

By putting the values from (3.6) and (4.9) in (4.2), we obtain

$$g(Y,HX)\eta(Z)+g(Y,HZ)\eta(Y)+2(n-1)\eta(Z)=0, \tag{4.10}$$

hence the theorem. □

Theorem 4.7. *A Ricci soliton on NC maniold $\widetilde{M}$ with SVK connection is steady if λ = (n-1), shrinking if λ < (n-1) and expanding if λ > (n-1).*

Proof. Using (3.5), (2.5) and (4.3), we have the required result. □

5. Yamabe Soliton on NC Manifold with SVK Connection

The almost Yamabe soliton on an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection is studied within this segment.

We now examine an n-dimensional NC manifold that allows the SVK connection to deal with an almost Yamabe soliton, as described in (1.3). Hence, we have

$$\frac{1}{2}(\widetilde{L}_Y g)(X,Z)=(\widetilde{r}-\gamma)g(X,Z). \tag{5.1}$$

Using (3.7), (4.1) and (5.1), we have

$$\frac{1}{2}(L_Y g)(X,Z)=(r-n+1-\gamma)g(X,Z). \tag{5.2}$$

The subsequent theorem may thus be stated from (5.2).

Theorem 5.1. *If n = 1, then an almost Yamabe soliton (M,Y,γ,g) on an n-dimensional NC manifold $\widetilde{M}$ is invariant concerning SVK connection.*

According to (5.1) and (4.1), we have

$$\frac{1}{2}(g(\triangledown_X Y,Z)+g(X,\triangledown_Z Y))(X,Z)=(\widetilde{r}-\gamma)g(X,Z).$$

If we put $Y=\xi$ in the above equation, we obtain

$$\frac{1}{2}(g(\triangledown_X \xi,Z)+g(X,\triangledown_Z \xi))(X,Z)=(\widetilde{r}-\gamma)g(X,Z). \tag{5.3}$$

In view of (2.2) and (2.3), from (5.3), we have

$$\widetilde{r}=\gamma.$$

Thus, we may deduce the conclusion as mentioned below:

Theorem 5.2. *If an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection, admits an almost Yamabe soliton then the scalar curvature $\widetilde{r}$ of $\widetilde{M}$ is equal to γ iff Y and ξ are pairwise collinear in TM.*

6. Example

Let us consider a 3-dimensional manifold $\widetilde{M} = \{(x,y,z) \in R^3\}$, where (x, y, z) represent the standard coordinates in R^3. Suppose

$$\tau_1 = e^{z^2}\frac{\partial}{\partial x}, \tau_2 = e^{z^2}\frac{\partial}{\partial y}, \tau_3 = \frac{\partial}{\partial z},$$

are linearly independent vector fields of $\widetilde{M}$. Then

$$[\tau_1, \tau_2] = 0,\ [\tau_2, \tau_3] = -2z\tau_2,\ [\tau_1, \tau_3] = -2z\tau_1.$$

If g represent the Riemannian metric, then we have

$$g(\tau_1, \tau_1) = g(\tau_2, \tau_2) = g(\tau_3, \tau_3) = 1,$$

$$g(\tau_1, \tau_2) = g(\tau_2, \tau_3) = g(\tau_1, \tau_3) = 0.$$

Let η be the 1-form defined by η(X) = g(X, τ_3), $\forall$ X $\in \widetilde{M}$, and let ϕ be the (1,1) tensor field defined by

$$\phi(\tau_1) = \tau_2,\ \phi(\tau_2) = -\tau_1,\ \phi(\tau_3) = 0.$$

Using the above relations, following results holds:

$$\phi^2 X = -X + \eta(X)\xi,$$

$$\eta(\tau_3) = 1,$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

where ξ = τ_3 and X,Y is arbitrary vector field on $\widetilde{M}$. Hence $\widetilde{M}$ fulfills all the condition for an NC manifold.
Using the Koszul formula, we get

$$\triangledown_{\tau_1}\tau_1 = 2z\tau_3,\ \triangledown_{\tau_2}\tau_1 = 0,\ \triangledown_{\tau_3}\tau_1 = 0,$$

$$\triangledown_{\tau_1}\tau_2 = 0,\ \triangledown_{\tau_2}\tau_2 = 2z\tau_3,\ \triangledown_{\tau_3}\tau_2 = 0,$$

$$\triangledown_{\tau_1}\tau_3 = -2z\tau_1,\ \triangledown_{\tau_2}\tau_3 = -2z\tau_2,\ \triangledown_{\tau_3}\tau_3 = 0,$$

and

$$\widetilde{\triangledown}_{\tau_1}\tau_1 = 2z\tau_3,\ \widetilde{\triangledown}_{\tau_2}\tau_1 = \tau_3,\ \widetilde{\triangledown}_{\tau_3}\tau_1 = 0,$$

$$\widetilde{\triangledown}_{\tau_1}\tau_2 = -\tau_3,\ \widetilde{\triangledown}_{\tau_2}\tau_2 = 2z\tau_3,\ \widetilde{\triangledown}_{\tau_3}\tau_2 = 0,$$

$$\widetilde{\triangledown}_{\tau_1}\tau_3 = -2z\tau_1 - \tau_2,\ \widetilde{\triangledown}_{\tau_2}\tau_3 = -2z\tau_2 + \tau_1,\ \widetilde{\triangledown}_{\tau_3}\tau_3 = 0.$$

We can easily deduce the following identities using above results.

$$\widetilde{R}(\tau_1, \tau_2)\tau_3 = 8z\tau_3;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_3 = (2 - 4z^2)\tau_2 + 2z\tau_1;$$

$$\widetilde{R}(\tau_1, \tau_2)\tau_2 = -4z^2\tau_1 - 4z\tau_2 + \tau_1;$$

$$\widetilde{R}(\tau_1, \tau_2)\tau_1 = -4z\tau_1 - \tau_2 + 4z^2\tau_2;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_2 = (-2 + 4z^2)\tau_3;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_1 = 2z\tau_3;$$

$$\widetilde{R}(\tau_1, \tau_3)\tau_2 = -2z\tau_3;$$

$$\widetilde{R}(\tau_1, \tau_3)\tau_3 = (2 - 4z^2)\tau_1 - 2z\tau_2;$$

$$\widetilde{R}(\tau_1,\tau_3)\tau_1 = (-2+4z^2)\tau_3;$$

and

$$\widetilde{S}(\tau_1,\tau_1) = \widetilde{S}(\tau_2,\tau_2) = 3-8z^2,\ \widetilde{S}(\tau_3,\tau_3) = 4-8z^2.$$

Hence $\widetilde{r}$ = 10 - 24z^2. Let

$$V = (x+y)e^{-z^2}z_1 + (-x+y)e^{-z^2}z_2$$

and

$$\sum_{i=1}^{3}(\widetilde{L}_V g)(\tau_i,\tau_i) = 4.$$

Now, we put X = Y = τ_i in (4.2), summing over $i = 1,2,3$ and using above results, we get α = 8z^2 - 4, also using (4.10), we obtain
Case I: for $z^2 = \frac{1}{2}$,the Ricci soliton is steady.
Case II: for $z^2 \neq \frac{1}{2}$,the Ricci soliton is shrinking.

7. Conclusion

The study provides new insights beyond the usual Levi-Civita framework and highlights the versatility of the SVK connection as a tool for studying geometric structures with torsion. These contributions enhance the way for further research in theoretical physics and mathematics while also improving our understanding of solitons in NC manifolds.

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