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Research Article

Approximation of the Hilbert transform on $(0, +\infty)$ by using discrete de la Vallée Poussin filtered polynomials

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. In the present paper, is proposed a method to approximate the Hilbert transform of a given function f on $(0, +\infty)$ employing truncated de la Vallée discrete polynomials recently studied in [25]. The method generalizes and improves in some sense that introduced in [24] based on a truncated Lagrange interpolating polynomial, since is faster convergent and simpler to apply. Moreover, the additional parameter defining de la Vallée polynomials helps to attain better pointwise approximations. Stability and convergence are studied in weighted uniform spaces and some numerical tests are provided to asses the performance of the procedure.

Keywords: Hilbert transform, discrete de la Vallèe-Poussin approximation, generalized Laguerre polynomials, approximation by polynomials.

2020 Mathematics Subject Classification: 65D30, 41A05.

1. INTRODUCTION

Let $\mathcal{H}(f,t)$ be the Hilbert transform of f

$$\mathcal{H}(f,t) = \int_0^\infty \frac{f(x)}{x-t} w(x)dx, \tag{1.1}$$

where $w(x) = e^{-x}x^\alpha$ is a Laguerre weight, and the integral in (1.1) is understood in the Cauchy principal value sense. In numerical analysis and in approximation theory, the approximation of Hilbert transforms over bounded or unbounded regions, represents a relevant topic, since it arises in several problems of the applied sciences, such as image analysis, optics, signal processing, fluid mechanics, electrodynamics. A collection of problems can be found in [13, Vol I , II]. In addition, Hilbert transforms and their derivatives can appear in singular and hypersingular integral equations, which in turn are possible tools to model several physics problems [20, 14, 15]. The literature dealing with numerical methods to approximate Hilbert transforms is rich. We cite among them [9, 21, 1, 12, 3, 5, 22, 7, 8, 10, 23] and the references contained in it. Concerning the approximation of $\mathcal{H}(f,t)$ by global methods based on Laguerre zeros, we recall two product-type integration rules, one obtained by approximating f by truncated Lagrange polynomials [5], the other by discrete de la Vallée Poussin polynomials [25]. In both these rules, the coefficients are obtained by recurrence relations depending on t, and hence requiring a considerable computational effort when $\mathcal{H}(f,t)$ is needed for a large number of points. So they are efficient, but "expensive". Another significant and reliable approach is given by the truncated Gauss-Laguerre rule, suitable "modified" to overcome numerical instability due to

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the closeness of some Gaussian node to the singularity t [8]. Such rule largely applied also in other contexts (see e.g. [4, 8, 2]) is simpler than the previous ones, but not employable in methods for solving integral equations, since it requires the choice between two sequence of Gauss-Laguerre nodes, according to the position of the singularity t. To overcome in some sense the aforesaid issues, in [24] it was introduced a method to approximate the function

$$\mathcal{F}(f,t)=\int_0^\infty \frac{f(x)-f(t)}{x-t}w(x)dx,$$

by means of suitable truncated Lagrange polynomials based on Laguerre zeros (say it L-method), and to compute $\mathcal{H}(f,t)=\mathcal{F}(f,t)+f(t)\mathcal{H}(\mathbf{1},t)$. By this way, for any t are required always the same samples of $\mathcal{F}(f)$. In the present paper, we want to generalize the L-method and improve it in some sense, proposing to approximate $\mathcal{F}(f)$ by the sequence of discrete de la Vallée Poussin (VP) polynomials $V_n^m(w,\mathcal{F}(f))$, recently introduced and studied in [25]. Analogously to the L-method, the polynomial $V_n^m(w,\mathcal{F}(f))$ requires the samples of the function $\mathcal{F}(f)$ at $j << n$ zeros of the Laguerre polynomial $p_n(w)$. Moreover, $V_n^m(w,\mathcal{F}(f))$ depends on the additional parameter $1\le m\le n-1$, which in turns can be fruitfully used to reduce possible Gibbs phenomenon. As it is known, the latter affects Lagrange interpolating approximation, especially when the interpolated function presents isolated "pathologies" (peaks, cusps, etc.). In addition, the Lebesgue constants associated to VP polynomials, are uniformly bounded in weighted spaces of continuous functions, whereas those of the Lagrange processes diverge logarithmically at least.

The outline of the paper is as follows. In Section 2 are collected some notations and preliminary results useful to introduce the proposed numerical method. The latter is stated in Section 3, accompanied by the study of the stability and convergence, and error estimates in suitable spaces of functions. Finally, in Section 4, a selection of numerical tests is proposed.

2. Notations and preliminary results

Along all the paper the notation $\mathcal{C}$ will be used several times to denote a positive constant having different values in different formulas. We will write $\mathcal{C}\neq\mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is independent of the parameters $a,b,\ldots$, and $\mathcal{C}=\mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a,b,\ldots$. Moreover, if $A,B>0$ are quantities depending on some parameters, we will write $A\sim B$, if there exists an absolute constant $\mathcal{C}>0$, independent on such parameters, such that $\mathcal{C}^{-1}B\le A\le\mathcal{C}B$.

Denote by $\mathbb{P}_m$ the space of all algebraic polynomials of degree at most m.

2.1. **Orthogonal Polynomials.** For $w(x)=e^{-x}x^\alpha,\quad \alpha>-1$, let $\{p_n(w)\}_n$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_n(w,x)=\gamma_n(w)x^n+\text{ terms of lower degree},\quad \gamma_n(w)>0.$$

Denoted by $\{x_{n,k}(w)\}_{k=1}^n$ the zeros of $p_n(w)$, it is known that [27]

$$\frac{\mathcal{C}}{n}<x_{n,1}(w)<x_{n,2}(w)<\ldots<x_{n,n}(w)<4n+2\alpha-\mathcal{C}n^{\frac{1}{3}},\qquad \mathcal{C}\neq\mathcal{C}(n). \tag{2.2}$$

For any fixed $0<\rho<1$ the node $x_{n,j}$, $j=j(n)$, is defined as

$$x_{n,j}(w)=\min\left\{x_{n,k}(w):x_{n,k}(w)\ge 4n\rho,\quad k=1,2,..,n\right\}. \tag{2.3}$$

As it is known, the zeros of $p_n(w)$ interlace those of $p_{n+1}(w)$, i.e.

$$x_{n+1,k}(w)<x_{n,k}(w)<x_{n+1,k+1}(w),\qquad k=1,2,\ldots,n. \tag{2.4}$$

Moreover, inside the interval $[0, x_{n+1,j_1}(w)]$ where j_1 is defined as

$$x_{n+1,j_1}(w) = \min\{x_{n+1,k}(w) : x_{n+1,k}(w) \geq 4(n+1)\rho, \quad k = 1, 2, .., n+1\}, \tag{2.5}$$

the distance between two consecutive zeros of $p_{n+1}(w)p_n(w)$ can be estimated as [4, Lemma 2.1]

$$x_{n,k}(w) - x_{n+1,k}(w) \geq \mathcal{C}\sqrt{\frac{x_{n+1,k}(w)}{n}}, \quad k = 1, 2, \ldots, j, \tag{2.6}$$

uniformly in $n \in \mathbb{N}$.

Finally, we recall the "truncated" Gauss-Laguerre rule introduced in [18] and based on the first j zeros of $p_n(w)$, j defined in (2.3),

$$\int_0^\infty f(x)w(x)dx = \sum_{k=1}^{j} f(x_{n,k}(w))\lambda_{n,k}(w) + R_m(f), \tag{2.7}$$

where $\{\lambda_{n,k}(w)\}_{k=1}^n$ are the Christoffel numbers w.r.t. w and $R_n(f)$ is the remainder term.

2.2. **Function Spaces.** Introducing the weight $u(x) = e^{-\frac{x}{2}}x^\gamma, \quad \gamma \geq 0$, we consider the space C_u of the functions f continuous in any closed subset of $]0, \infty[$, such that

$$\lim_{x\to+\infty}(fu)(x) = 0, \qquad \text{and, if } \gamma > 0, \text{ also} \qquad \lim_{x\to 0^+}(fu)(x) = 0,$$

endowed with the norm $\|f\|_{C_u} = \sup_{x\geq 0}|f(x)|u(x)$. The error of best approximation of $f \in C_u$ by algebraic polynomials of degree $\leq n$ is defined as

$$E_n(f)_u = \inf_{P\in\mathbb{P}_n}\|(f-P)u\|_\infty.$$

For $s \in \mathbb{N}$, $s \geq 1$, let $W_s(u)$ be the Sobolev-type space

$$W_s(u) = \left\{f \in C_u : f^{(s-1)} \in AC(\mathbb{R}^+), \|f^{(s)}\varphi^s u\|_\infty < \infty\right\}, \qquad \varphi(x) = \sqrt{x},$$

where $AC(\mathbb{R}^+)$ denotes the set of the functions which are absolutely continuous on every closed subset of $\mathbb{R}^+$, equipped with the norm

$$\|f\|_{W_s(u)} = \|f\|_{C_u} + \|f^{(s)}\varphi^s u\|_\infty. \tag{2.8}$$

In order to deal with more refined subspaces of C_u, for any $\lambda \in \mathbb{R}^+$ let $Z_\lambda(u)$ be the Zygmund-type space

$$Z_\lambda(u) = \left\{f \in C_u \ : \ \sup_{t>0}\frac{\Omega^r_\varphi(f,t)_u}{t^\lambda} < +\infty\right\}$$

of parameter $0 < \lambda < r$, $r \in \mathbb{N}$, where

$$\Omega^r_\varphi(f,t)_u = \sup_{0<h\leq t}\|u\Delta^r_{h\varphi}f\|_{I_{rh}}, \quad t > 0$$

is the main part of the $r-$th $\varphi-$modulus of smoothness, $I_{rh} = \left[4r^2h^2, \frac{\mathcal{C}}{h^2}\right]$, being $\mathcal{C}$ a fixed positive constant, and

$$\Delta^r_{h\varphi}f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f\left(x + h\varphi(x)(r-k)\right),$$

equipped with the norm

$$\|f\|_{Z_\lambda(u)} = \|f\|_{C_u} + \sup_{t>0}\frac{\Omega^r_\varphi(f,t)_u}{t^\lambda}.$$

$E_n(f)_u$ can be estimated in Zygmund and Sobolev subspaces as follows [6, 16]

$$E_n(f)_u \le \mathcal{C}\frac{\|f\|_{W_s(u)}}{\sqrt{n^s}}, \quad \forall f \in W_s(u), \tag{2.9}$$

$$E_n(f)_u \le \frac{\mathcal{C}}{\sqrt{n^\lambda}}\|f\|_{Z_\lambda(u)}, \quad \forall f \in Z_\lambda(u), \tag{2.10}$$

where in both the cases $\mathcal{C} \neq \mathcal{C}(n,f)$.

2.3. **VP filtered approximation.** For a given $n \in \mathbb{N}$ and with $m \in \mathbb{N}$ s.t. $1 \le m \le n-1$, the discrete VP filtered polynomial $V_n^m(w,f)$ approximating a given function $f \in C_u$ is defined as [25]

$$V_n^m(w,f,x) = \sum_{k=1}^{j} f(x_{n,k}(w))\Phi_{n,k}^m(x), \qquad x \ge 0 \tag{2.11}$$

with j defined in (2.3), and the fundamental VP polynomials defined as

$$\Phi_{n,k}^m(x) = \lambda_{n,k}(w) \sum_{i=0}^{n+m-1} \mu_{n,i}^m p_i(w, x_{n,k}(w)) p_i(w,x), \tag{2.12}$$

where $\mu_{n,i}^m$ are called "filter coefficients"

$$\mu_{n,i}^m := \begin{cases} 1 & \text{if} \quad i = 0,\dots,n-m, \\ \dfrac{n+m-i}{2m} & \text{if} \quad n-m+1 \le i \le n+m-1. \end{cases} \tag{2.13}$$

The polynomial $V_n^m(w,f) \in \mathbb{P}_{n+m-1}$ and does not interpolate f.

One of the main features proved in [25] is the boundedness of the map $V_n^m(w) : f \in C_u \to C_u$ under proper assumptions on w and u. This means to deal with a polynomial sequence uniformly convergent to any function $f \in C_u$, behaving as a near–best approximation polynomial sequence. This property similarly holds in $[-1,1]$ for filtered de la Vallée Poussin polynomials w.r.t. Jacobi polynomials, introduced and studied in [28].

Theorem 2.1 ([25]). *For any $f \in C_u$, under the assumption*

$$\max\left\{\frac{\alpha}{2} - \frac{1}{4},\ 0\right\} < \gamma < \min\left\{\frac{\alpha}{2} + \frac{7}{6},\ \alpha + 1\right\} \tag{2.14}$$

then, fixing $\theta \in (0,1)$ and choosing $m = \lfloor n\theta \rfloor$, the map $V_n^m(w) : C_u \to C_u$ is uniformly bounded w.r.t. n, i.e.

$$\|V_n^m(w,f)\|_{C_u} \le \mathcal{C}\|f\|_{C_u}, \quad \forall f \in C_u, \qquad \mathcal{C} \neq \mathcal{C}(n,m,f). \tag{2.15}$$

Moreover, with $0 < \rho < 1$ fixed to define the index j in (2.3),

$$\|(f - V_n^m(w,f))\|_{C_u} \le \mathcal{C}\left(E_q(f)_u + e^{-An}\|f\|_{C_u}\right), \quad q = \min\left\{n-m,\ \left\lfloor n\frac{\rho}{(1+\rho)}\right\rfloor\right\}, \tag{2.16}$$

with the positive constants $\mathcal{C}, A$ independent of n, m, f.

We point out that Theorem 2.1 has been proved in [25], under more general relationships between n and m.

2.4. **Truncated Lagrange interpolation.** With j defined in (2.3), in [17] it was introduced the truncated Lagrange polynomial $L^*_{n+1}(w,f)$ defined as

$$L^*_{n+1}(w,f,x) := \sum_{k=1}^{j} f(x_{n,k}(w))\ell_{n,k}(x), \tag{2.17}$$

where

$$\ell_{n,k}(x) = \frac{p_n(w,x)}{p_n'(x_k)(x-x_k)}\frac{4n-x}{4n-x_k}, \qquad k=1,2,\dots,j, \tag{2.18}$$

are fundamental Lagrange polynomials based on the zeros of $p_n(w,x)(4n-x)$. $L^*_{n+1}(w,f)$ interpolates the function at the first j zeros of $p_n(w)$ and vanishes at the remaining nodal points $x_{n,j+1}(w),\dots,x_{n,n}(w),4n$. In particular, for $\rho = 1$ $L^*_{n+1}(w,f)$ coincides with the Lagrange polynomial $L_{n+1}(w,f)$, interpolating f at all the zeros of $p_n(w,x)(4n-x)$. It is known that the norm of the operator $L^*_{n+1}(w) : f \in C_u \to C_u$, i.e. the weighted Lebesgue constant

$$\|L^*_{n+1}(w)\|_{C_u} = \sup_{\|fu\|=1}\|L^*_{n+1}(w,f)u\| = \sup_{x\geq 0}\sum_{k=1}^{j}|\ell_{n,k}(x)|\frac{u(x)}{u(x_{n,k}(w))}, \tag{2.19}$$

as $n\to\infty$ diverge at least as $\log n$. To be more precise, the following result holds

Theorem 2.2 ([17]). *Let $w(x)=e^{-x}x^\alpha$, $\alpha>-1$ and $u(x)=e^{-\frac{x}{2}}x^\gamma$, $\gamma\geq 0$. Under the assumption*

$$\min\left(0,\frac{\alpha}{2}+\frac{1}{4}\right)\leq\gamma\leq\frac{\alpha}{2}+\frac{5}{4}, \tag{2.20}$$

we have

$$\|L^*_{n+1}(w,f)\|_{C_u}\leq\mathcal{C}\|f\|_{C_u}\log n, \quad \forall f\in C_u, \qquad \mathcal{C}\neq\mathcal{C}(n,f). \tag{2.21}$$

Remark 2.1. *We point out that in* [19] *it was proved that*

$$\|L_{n+1}(w,f)\|_{C_u}\leq\mathcal{C}\|f\|_{C_u}\log n, \quad \forall f\in C_u, \qquad \mathcal{C}\neq\mathcal{C}(n,f), \tag{2.22}$$

if and only if the assumption (2.20) *holds.*

3. The method

Let us start from the relation

$$\mathcal{H}(f,t)=\mathcal{F}(f,t)+f(t)\mathcal{H}(\mathbf{1},t), \quad \mathcal{F}(f,t)=\int_0^\infty\frac{f(x)-f(t)}{x-t}w(x)dx, \tag{3.23}$$

and assume from now on $\alpha<1$, since in the case $\alpha\geq 1$ we deal equivalently with $\widetilde{\mathcal{H}}(g,t):=\int_0^\infty\frac{g(x)}{x-t}\tilde{w}(x)dx$, $g(x):=f(x)x^{[\alpha]}$, $\tilde{w}(x)=e^{-x}x^{\alpha-[\alpha]}$. Now, taking into account that $\mathcal{H}(\mathbf{1},t)$ can be computed by [11, p. 1086, 9.213]

$$\mathcal{H}(\mathbf{1},t)=\begin{cases}-e^{-t}\mathrm{Ei}(t), & \alpha=0\\ -\pi t^\alpha e^{-t}\cot((1+\alpha)\pi)+\Gamma(\alpha){}_1\mathrm{F}_1(1,1-\alpha,-t), & \alpha\neq 0,\end{cases} \tag{3.24}$$

where $\mathrm{Ei}(t)$ and ${}_1\mathrm{F}_1(a,b,x)$ are the exponential integral function and the Confluent Hypergeometric function, respectively, we focus on the approximation of $\mathcal{F}(f)$. To this end, we recall a result proved in [24], which relates the smoothness of $\mathcal{F}(f)$ to that of f.

Lemma 3.1. *For any $f \in Z_{\lambda+1}(u)$, under the assumption $0 \le \gamma < \alpha + \frac{1}{4}$, the function $\mathcal{F}(f) \in Z_\lambda(u)$, and*

$$E_n(\mathcal{F}(f))_u \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \quad \mathcal{C} \ne \mathcal{C}(n,f). \tag{3.25}$$

Belonging $\mathcal{F}(f)$ to a subspace of C_u allows to approximate $\mathcal{F}(f)$ by the sequence of VP polynomials defined in (2.11), i.e. to consider

$$\mathcal{F}(f,t) \sim V_n^m(w,\mathcal{F}(f),t) = \sum_{k=1}^{j} \Phi_{n,k}^m(t)\mathcal{F}(f,x_{n,k}(w)). \tag{3.26}$$

By Lemma 3.1, and Theorem 2.1 combined with estimate (2.10), under the assumption

$$\max\left\{\frac{\alpha}{2}-\frac{1}{4},\,0\right\} \le \gamma < \alpha + \frac{1}{4},$$

$\mathcal{F}(f)$ can be uniformly approximated by $V_n^m(w,\mathcal{F}(f))$, and the error is

$$\|\mathcal{F}(f) - V_n^m(w,\mathcal{F}(f))\|_{C_u} \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \quad \forall f \in Z_{\lambda+1}(u). \tag{3.27}$$

However, the computational problem in constructing (3.26) is the lack of the samples $\{\mathcal{F}(f,x_{n,k}(w))\}_{k=1}^j$, unknown in the general case. To overcome this problem, the integrals $\mathcal{F}(f,x_{n,k}(w))$, $k=1,2,\dots,j$ are approximated by the $(n+1)$-th truncated Gauss-Laguerre rules w.r.t. the weight w, i.e.,

$$\mathcal{F}(f,x_{n,k}(w)) \sim \sum_{i=1}^{j_1} \lambda_{n+1,i}(w)\frac{f(x_{n+1,i}(w)) - f(x_{n,k}(w))}{x_{n+1,i}(w) - x_{n,k}(w)}, \quad k=1,\dots,j,$$

being for every fixed $\rho \in (0,1)$ the index j_1 defined in (2.5). Note that possible numerical cancellation arising in case t is "too close" to any Gauss-Laguerre nodes is avoided, since the zeros of $p_n(w)$ are far enough from the Gaussian nodes $x_{n+1,k}(w)$ in view of (2.6). Now we prove that these further approximations $\mathcal{F}(f,x_{n,k}(w)) \sim \mathcal{F}_{n+1}(f,x_{n,k}(w))$, induce errors of the same orders as $\|\mathcal{F}(f) - V_n^m(w,\mathcal{F}(f))\|_{C_u}$. This is stated in the next lemma.

Lemma 3.2. *For any $f \in Z_{\lambda+1}(u)$, under the assumption $0 \le \gamma < \alpha + \frac{1}{4}$, the following error estimate holds true*

$$\|\mathcal{F}(f) - \mathcal{F}_{n+1}(f)\|_{C_u} \le \mathcal{C}\,\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \quad \mathcal{C} \ne \mathcal{C}(n,f). \tag{3.28}$$

Proof. First we recall that under the assumption $0 \le \gamma < \alpha + \frac{1}{4}$, for any $f \in Z_{\lambda+1}(u)$ in [24, Lemmas 5.5, 5.7] there were proved $\mathcal{F}_{n+1}(f) \in Z_{\lambda+1}(u)$ and

$$E_n(\mathcal{F}_n(f))_u \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}. \tag{3.29}$$

As a consequence, for any $P_n \in \mathbb{P}_n$

$$\|\mathcal{F}(f) - \mathcal{F}_n(f)\|_{C_u} \le \mathcal{C}\,(E_n(\mathcal{F}(f))_u + E_n(\mathcal{F}_n(f))_u)$$

and in view of estimates (3.25) and (3.29), (3.28) follows. □

In conclusion, we approximate the function $\mathcal{F}(f)$ by the sequence $\{V_n^m(w,\mathcal{F}_{n+1}(f))\}_n$, i.e.,

$$\mathcal{F}(f,t) = \Sigma_n(\mathcal{F},t) + e_{n,m}(f,t),, \tag{3.30}$$

being

$$\Sigma_n(f,t) := \sum_{k=1}^{j} \Phi_{n,k}^m(t)\mathcal{F}_{n+1}(f, x_{n,k}(w)) = \sum_{k=1}^{j} \Phi_{n,k}^m(t) \sum_{i=1}^{j_1} \lambda_{n+1,i}(w) \frac{f(x_{n+1,i}(w)) - f(x_{n,k}(w))}{x_{n+1,i}(w) - x_{n,k}(w)}.$$

Next Theorem states conditions under which formula (3.30) is stable and convergent

Theorem 3.3. *For any $f \in Z_{\lambda+1}(u),\ \lambda > 0$, under the assumption*

$$\max\left(0, \frac{\alpha}{2} - \frac{1}{4}\right) \le \gamma < \alpha + \frac{1}{4}, \tag{3.31}$$

$$\|\Sigma_n(f)\|_{C_u} \le \mathcal{C}\|f\|_{Z_{\lambda+1}(u)}, \tag{3.32}$$

and

$$\|e_{n,m}(f)\|_{C_u} \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \tag{3.33}$$

$\mathcal{C} \ne \mathcal{C}(n, f)$.

Proof. First we prove (3.32). Using $\Sigma_n(f,t) = V_n^m(w, \mathcal{F}_{n+1}(f), t)$, by Theorem 2.1, under the assumption (3.31),

$$\|\Sigma_n(f)\|_{C_u} \le \mathcal{C}\|\mathcal{F}_{n+1}(f)\|_{C_u}$$

and by Lemma 3.2 and (3.25), (3.32) follows. To estimate (3.33), start from

$$\begin{aligned} |e_{n,m}(f,t)|u(t) &= |\mathcal{F}(f,t) - V_n^m(w, \mathcal{F}_{n+1}(f), t)|u(t) \\ &\le |\mathcal{F}(f,t) - \mathcal{F}_{n+1}(f,t)|u(t) \\ &+ |\mathcal{F}_{n+1}(f,t) - V_n^m(w, \mathcal{F}_{n+1}(f), t)|u(t) \\ &=: A_1(t) + A_2(t). \end{aligned} \tag{3.34}$$

Under the assumption $f \in Z_{\lambda+1}(u)$ by Lemma 3.2

$$A_1(t) \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \tag{3.35}$$

and by Theorem 2.1 combined with estimate (3.29), we get

$$A_2(t) \le \mathcal{C}\ E_q(\mathcal{F}_{n+1}(f))_u \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+1}(u)}}{\sqrt{n^\lambda}}, \tag{3.36}$$

and (3.33) follows combining (3.35),(3.36) with (3.34) □

3.1. **Comparison between VP-method and L-method.** As previously said, in [24] the function $\mathcal{F}(f,t)$ has been approximated by using truncated Lagrange polynomials interpolating $\mathcal{F}(f,t)$ and based on Laguerre zeros. To be more precise, denoted by $w^+(x) = e^{-x}x^{\alpha+1}$, $w^-(x) = e^{-x}x^{\alpha-1}$ in the case $\alpha > 0$, the following work-scheme has been considered:

$$\mathcal{F}(f,t) \sim \begin{cases} L_{n+1,1}^*(w^+, \mathcal{F}_{n+1}(f), t) & -\frac{1}{4} < \alpha \le 0 \\ L_{n+2}^*(w^-, \mathcal{F}_n(f), t), & \alpha > 0, \end{cases} \tag{3.37}$$

where $\mathcal{F}_{n+1}(f,t) = \sum_{i=1}^{j} \lambda_{n+1,i}(w)\frac{f(x_{n+1,i}(w))-f(t)}{x_{n+1,i}(w)-t}$, $L_{n+1,1}^*(w^+, \mathcal{F}_{n+1}(f), x)$ is the truncated Lagrange polynomial based on the knots $\{x_{n,k}(w^+)\}_{k=1}^n \cup \{4n\} \cup \{t_1\}$, with $t_1 = \frac{x_{n,1}(w^+)}{2}$, and

$L^*_{n+2}(w^-,\mathcal{F}_n(f),t)$ is the truncated polynomial based on the knots $\{x_{n+1,k}(w^-)\}_{k=1}^n \cup \{4n\}$. It was necessary introduce two different paths, in order to consider interpolation processes having Lebesgue constants behaving always as $\log n$, for any choice of α,γ satisfying (3.31). The VP-method, unlike the L-method, is simpler to construct, since involves only one approximant, without distinguishing two cases, whatever are the values of α and γ satisfying (3.31).

About the rate of convergence, the L-method results a little bit slower than the VP-method, due to the presence of the extra factor $\log n$ (see [24, Theorems 3.2-3.3]. In conclusion from the theoretical and computational point of view, the VP-method represents a simpler strategy to obtain a little bit faster convergence.

4. Numerical Examples

We have tested the VP-method and compared the results with those taken by using the L-method on some test functions. Here we go to propose a selection of three examples which seem to be more exhaustive to highlight the performance of the VP-method in comparison with the L-method. In each tests we have considered the approximation only of the function $\mathcal{F}(f)$, this being the main topic we are dealing with. We precise that:

- $$e^{VP}_{n,m}(\mathcal{F},t) = |V^m_n(w,\mathcal{F}_{n+1}(f),t) - \mathcal{F}(f,t)|u(t),$$
  $$e^{Lag}_{n}(\mathcal{F},t) = |L^*_{n+1}(\mathcal{F}_{n+1}(f),t) - \mathcal{F}(f,t)|u(t),$$
  are the weighted pointwise errors related to the approximation of $\mathcal{F}(f)$ only.
- $$\|e^{VP}_{n,m}(\mathcal{F})\| = \max_{t\in Y} e^{VP}_{n,m}(\mathcal{F},t),$$
  $$\|e^{Lag}_{n}(\mathcal{F})\| = \max_{t\in Y} e^{Lag}_{n}(\mathcal{F},t),$$
  are the maximum weighted errors on Y, where Y is a sufficiently large mesh of equispaced points in the range $(0,a)$ with $a>0$ sufficiently large.
- The parameter $\theta\in(0,1)$ defining $m=\lfloor n\theta\rfloor$ in $V^m_n(w,\mathcal{F}_{n+1}(f),t)$, has been selected as that giving the minimal absolute error $e^{VP}_{n,m}(\mathcal{F},t)$.
- The exact values, always unknown, have been computed for $n=2048,\ m=n/2$.
- All the computations have been performed in double-machine precision ($eps\sim 2.22044e-16$).
- In each test we have selected two values of $t>0$ providing the pointwise absolute errors $|e^{VP}_{n,m}(\mathcal{F},t)|u(t)$ (first column), the values of n and θ in the first and second columns, respectively. The errors $|e^{Lag}_{n}(\mathcal{F},t)|u(t)$ are reported in the fifth column, while the third column contains the number j of functions evaluations in both the VP-rule and the L-rule. Moreover, in each tests are given also the maximum absolute errors taken over proper ranges of t.
- Some graphs are stated to highlight the benefits offered by the VP-method, by suitably modulating the parameter m to reduce pointwise errors, especially in case f presents isolated "pathologies" (peaks, cusps, etc.), i.e. reducing Gibbs phenomenon, which affects Lagrange interpolating polynomials not only around the localized "pathological" point, but also along subinterval "far" from the point itself.

Example 4.1.

$$\mathcal{F}(f,t) = \int_0^\infty \frac{f(x)-f(t)}{x-t}e^{-x}x^{0.6}dx, \quad f(x)=\frac{1}{100+10(x-3)^2},\ \alpha=0.6.$$

According to the conditions stated for both the VP-method and L-method, we take the parameter $\gamma = 0.05$. *In Tables 1, are reported the results related to the pointwise approximation for* $t = 3$, *and* $t = 10$. *The approximate values are in this case*

$$\mathcal{F}(f,3)u(3) \sim 2.0516e-04, \quad \mathcal{F}(f,10)u(10) \sim -5.0389e-06.$$

In this case $f(x) \in Z_\lambda(u),\ \forall \lambda > 0$, *and presents a peak for* $x = 3$. *As Table 1 shows, better results are attained on average by VP-method, and the machine precision is catched for some* n.

n	θ	j	$\|e_{n,m}^{VP}(\mathcal{F},t)\|u(t)$	$\|e_n^{Lag}(\mathcal{F},t)\|u(t)$
20	1.0e-01	19	1.00e-07	9.23e-08
50	1.0e-01	32	3.52e-09	2.20e-09
150	3.0e-01	56	1.47e-14	1.35e-12
250	1.0e-01	73	6.62e-15	6.02e-15
350	2.0e-01	87	3.36e-17	1.07e-16

n	θ	j	$\|e_{n,m}^{VP}(\mathcal{F},t)\|u(t)$	$\|e_n^{Lag}(\mathcal{F},t)\|u(t)$
20	5.0e-01	19	1.12e-08	3.23e-08
50	3.0e-01	32	2.18e-11	1.43e-09
150	1.0e-01	56	1.09e-13	2.47e-13
250	2.0e-01	73	1.30e-16	4.84e-16
350	1.0e-01	87	1.65e-18	1.25e-17

TABLE 1. Ex.1: $t = 3$ (up), $t = 10$ (down)

In Table 2 are given the maximum absolute errors attained for increasing values of n *in* $[0, 10]$, *and a moderate better performance of the VP-method w.r.t. the L-method is confirmed.*

n	$\|e_{n,m}^{VP}(\mathcal{F})u\|$	$\|e_n^{Lag}(\mathcal{F})u\|$
20	6.66e-07	6.27e-07
50	6.98e-09	1.87e-08
150	1.98e-12	1.02e-11
250	8.11e-15	1.49e-14
350	1.04e-15	2.76e-15

TABLE 2. Ex.1: maximum errors in $[0, 10]$

We conclude proposing the graph of the pointwise absolute errors attained for $n = 150$, *choosing the "optimal"* θ *for any point* t *(Fig. 1), and as expected, and the pointwise errors by the VP-method are better than those attained by the L-method. In addition, we have produced in Fig. 2 also the graph of the pointwise absolute errors for* $\theta = 0.1$, *to highlight that also for* θ *fixed, the previous trend is confirmed.*

Example 4.2.

$$\mathcal{F}(f,t) = \int_0^\infty \frac{f(x)-f(t)}{x-t}\frac{e^{-x}}{x^{\frac{1}{8}}}dx, \quad f(x) = \frac{e^{x/4}}{(1+x^2)^4},\ \alpha = -1/8,\ \gamma = 0.5.$$

In this test $f \in Z_\lambda(u), \forall \lambda > 0$, *and grows exponentially as* $x \to +\infty$. *In Table 3 are reported the pointwise errors for* $t = 1$ *and* $t = 15$. *The approximate values are*

$$\mathcal{F}(f,1)u(1) \sim -3.9662e-01, \quad \mathcal{F}(f,15)u(15) \sim -7.0825e-05.$$

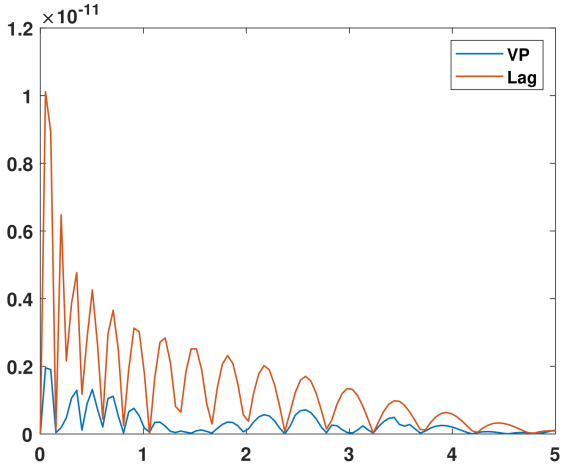

FIGURE 1. Ex. 1: pointwise absolute errors for $n = 150$, θ optimal

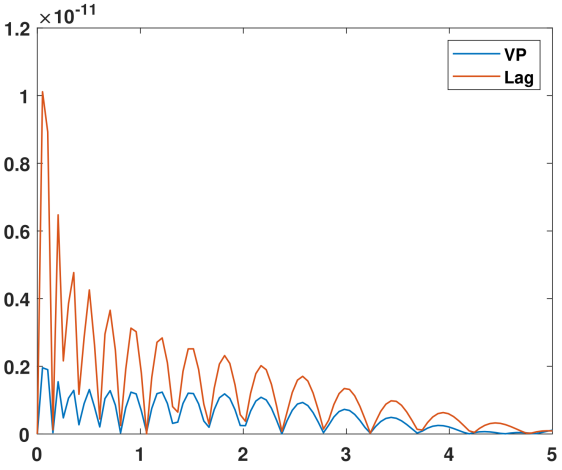

FIGURE 2. Ex. 1: pointwise absolute errors for $n = 150$, $\theta = 0.1$

In Table 4 are given the maximum absolute errors attained for increasing values of n in $[0, 10]$, and a moderate better performance of the VP-method w.r.t. the L-method is confirmed.

Example 4.3.

$$\mathcal{F}(f,t) = \int_0^\infty \frac{f(x) - f(t)}{x - t} e^{-x} dx, \quad f(x) = |x-1|^{1.5}|x-1.5|^{1.9}\ \alpha = 0,\ \gamma = 0.$$

The approximate values for $t = 1$ and $t = 1.5$ are

$$\mathcal{F}(f,1)u(1) \sim 5.6531e-02, \quad \mathcal{F}(f,1.5)u(1.5) \sim 2.4085e-01.$$

n	θ	j	$\|e_{n,m}^{VP}(\mathcal{F},t)\|u(t)$	$\|e_{n}^{Lag}(\mathcal{F},t)\|u(t)$
50	3.0e-01	34	2.20e-05	1.41e-04
150	5.0e-01	61	4.38e-07	7.81e-07
250	2.0e-01	79	4.46e-09	7.36e-09
350	1.0e-01	93	2.67e-10	3.03e-10
450	1.0e-01	106	1.51e-11	1.42e-11
550	1.0e-01	117	8.56e-13	7.58e-13
650	1.0e-01	127	4.38e-13	3.56e-13

n	θ	j	$\|e_{n,m}^{VP}(\mathcal{F},t)\|u(t)$	$\|e_{n}^{Lag}(\mathcal{F},t)\|u(t)$
50	5.0e-01	34	1.92e-05	7.87e-05
150	1.0e-01	61	3.69e-08	1.64e-07
250	2.0e-01	79	9.14e-10	3.30e-09
350	1.0e-01	93	1.05e-11	1.02e-11
450	1.0e-01	106	2.93e-12	1.81e-12
550	1.0e-01	117	1.56e-14	3.78e-14
650	2.0e-01	127	1.92e-14	2.03e-13

TABLE 3. Ex.2: $t = 1$ (up), $t = 15$ (down)

n	$\|e_{n,m}^{VP}(\mathcal{F})u\|$	$\|e_{n}^{Lag}(\mathcal{F})u\|$
50	4.17e-04	6.8490e-04
150	2.64e-06	2.5247e-06
250	1.82e-08	1.27e-08
350	5.34e-10	6.18e-10
450	3.19e-11	3.06e-11
550	2.16e-12	2.08e-12
650	1.40e-12	2.46e-12

TABLE 4. Ex.2: maximum errors in $[0, 10]$

The graph of the function $\mathcal{F}(f,t)u(t)$, in a range including the critical points 1 and 1.5 is shown in Figure 3. In Table 6 are given the maximum absolute errors attained for increasing values of n in $[0, 10]$.

We conclude with the pointwise absolute errors for $n = 200$, in the case of θ chosen "optimal" for each t (Figure 4), and for θ fixed, namely $\theta = 0.5$ (Figure 5).

CONCLUSIONS

We have proposed a method to approximate the Hilbert transform with a Laguerre weight. It employs filtered VP approximation of the function $\mathcal{F}(f)$, and improves a previous method based on the interpolation of the same function by truncated Lagrange polynomials. Indeed, w.r.t. to this, the new method converges a little bit faster. Moreover, while the Lagrange based method requires two different approaches, according to $\alpha > 0$ or not, the proposed rule does not. The algorithm, easier to implement, essentially requires zeros and weights of the Gauss-Laguerre rule, efficiently computable by the Golub-Welsh algorithm. Moreover, differently from the modified Gauss-Laguerre rule [8], we use always the same samples of the density function f at the Laguerre zeros, whatever are the values of t for which to compute $\mathcal{F}(f), t)$.

n	θ	j	$\lvert e_{n,m}^{VP}(\mathcal{F},t)\rvert u(t)$	$\lvert e_{n}^{Lag}(\mathcal{F},t)\rvert u(t)$
50	*9.00e-01*	*41*	*7.21e-04*	*3.76e-03*
100	*8.00e-01*	*60*	*2.87e-04*	*1.44e-03*
200	*1.00e-01*	*86*	*1.10e-03*	*1.06e-03*
400	*9.00e-01*	*122*	*4.16e-04*	*5.16e-04*
800	*1.00e-01*	*173*	*1.63e-04*	*1.51e-04*
1000	*9.00e-01*	*194*	*1.21e-04*	*1.55e-04*

n	θ	j	$\lvert e_{n,m}^{VP}(\mathcal{F},t)\rvert u(t)$	$\lvert e_{n}^{Lag}(\mathcal{F},t)\rvert u(t)$
50	*1.00e-01*	*41*	*2.41e-03*	*2.28e-03*
100	*8.00e-01*	*60*	*2.10e-05*	*4.86e-04*
200	*7.00e-01*	*86*	*6.85e-05*	*3.13e-04*
400	*6.00e-01*	*122*	*6.83e-05*	*9.85e-05*
800	*4.00e-01*	*173*	*2.75e-06*	*6.14e-05*
1000	*2.00e-01*	*194*	*4.74e-07*	*4.90e-06*

TABLE 5. Ex.3: $t = 1$ (up), $t = 1.5$ (down)

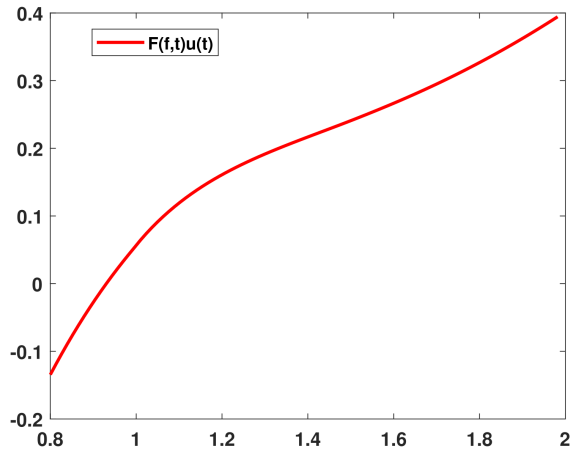

FIGURE 3. Graph of $\mathcal{F}(f,t)u(t),\ t \in [0.8, 2.1]$

n	$\lVert e_{n,m}^{VP}(\mathcal{F})u\rVert$	$\lVert e_{n}^{Lag}(\mathcal{F})u\rVert$
50	*2.6824e-03*	*6.4905e-03*
100	*6.9861e-04*	*2.6261e-03*
200	*2.3962e-03*	*2.3353e-03*
400	*1.0617e-03*	*1.0656e-03*
800	*7.7203e-04*	*7.5605e-04*

TABLE 6. Ex.3: maximum errors in $[0, 10]$

None recurrence relation is required as in product integration type rules [26]. In addition, due to the presence of the localizing parameter $0 < m < n$, oscillations and overshoots around

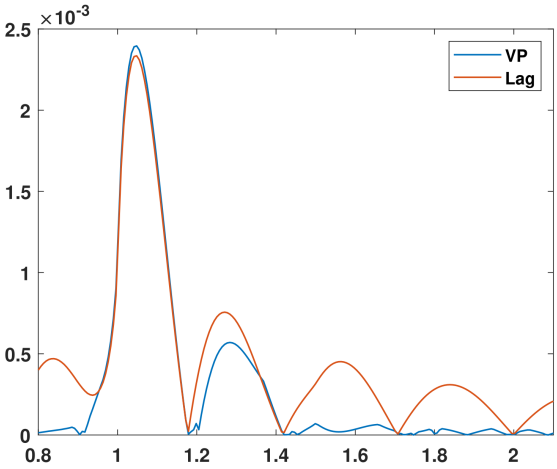

FIGURE 4. Ex. 3: pointwise absolute errors for $n = 200$, θ optimal

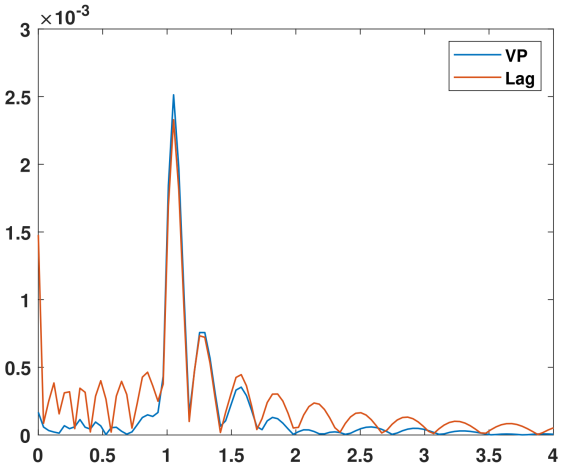

FIGURE 5. Ex. 3: pointwise absolute errors for $n = 200$, $\theta = 0.5$

"pathological" points of the function to be approximated, more present in Lagrange interpolation, are reduced.

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