New Banach Sequence Spaces That Is Defined By The Aid Of Lucas Numbers

Murat KARAKAŞ[1], Ayşe Metin KARAKAŞ[1]

ABSTRACT: In this work, we establish a new matrix by using Lucas numbers and define a new sequence space. Besides, we give some inclusion relations and investigate the geometrical properties such as Banach-Saks type , weak fixed point property for this space.

Keywords: Banach-Saks property, difference matrix, difference sequence spaces, lucas numbers, weak fixed point property

Lucas Sayıları Yardımıyla Tanımlanan Yeni Banach Dizi Uzayları

ÖZET: Bu makalede, Lucas sayılarını kullanarak yeni bir matris oluşturuyoruz ve yeni bir dizi uzayı tanımlıyoruz. Ayrıca bu uzay için bazı kapsama bağıntıları veriyoruz ve uzayın p tipi Banach-Saks, zayıf sabit nokta gibi geometrik özelliklerini araştırıyoruz.

Anahtar Kelimeler: Banach-Saks özelliği, fark matrisi, fark dizi uzayları, lucas sayıları, zayıf sabit nokta özelliği

[1]   Murat KARAKAŞ (0000-0003-1968-2040), Ayşe Metin KARAKAŞ (0000-0003-3552-0105), Bitlis Eren Üniversitesi, Fen Edebiyat Fakültesi, İstatistik, Bitlis, Türkiye
Sorumlu yazar/Corresponding Author: Murat KARAKAŞ, mkarakas@beu.edu.tr

INTRODUCTION

Recently, (Kara, 2013) has defined and studied Fibonacci difference sequence spaces. According to his study, we have examined the Lucas difference sequence space and have shown that this space is a Banach -space and also is linear isomorphic to ℓ_p for $1 \leq p \leq \infty$..

All real and complex valued sequences are represented by w. A sequence space is linear subspace of w. Throughout the paper $\ell_\infty, \ell_p (1 \leq p < \infty), c, c_0$

$$B_n(x) = \sum_k b_{nk} x_k \tag{1}$$

The sequence Bx is said to be the B-transform of x by the matrix B. The notation (X,Y) shows the class of matrices B such that $B: X \to Y$. Therefore $B \in (X, Y)$ iff the series on the equality (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Bx \in Y$ for all $x \in X$.

The matrix domain of B for a sequence space X is given by

$$X_B = \{x \in w: Bx \in X\} \tag{2}$$

which is a sequence space.

For the sequence whose n^{th} term is 1 and others are 0 for each $n \in \mathbb{N}$, we'll write $e^{(n)}$ and also use $e = (1,1, \dots)$.

characterise the spaces of all bounded, p-absolutely summable, convergent and null sequences.

Let $B = (b_{nk})$ be an infinite matrix of real numbers $b_{nk} (n, k = 1,2, \dots)$ and X, Y be two sequence spaces. It is said that the matrix B describes a matrix transformation from X into Y, and we symbolize it $B: X \to Y$, if the sequence $Bx = (B_n(x))$ is in Y for every $x = (x_k) \in X$ where

Recently, several authors have made use of the view of constituting sequence space by the help of matrix domain of an infinite triangle matrix, e.g., (Mursaleen and Noman, 2010; Mursaleen and Noman, 2011), (Mursaleen et al., 2006), (Altay et al., 2006), (Başar and Altay, 2003), (Altay and Başar, 2005), (Savaş et al., 2009), (Karakaş, 2015), (Kara and Başarır, 2012).

The matrix domain λ_Δ is called the difference sequence space if λ is a normed or paranormed sequence space where Δ symbolizes the backward difference matrix $\Delta = (\Delta_{nk})$ and $\Delta' = (\Delta'_{nk})$ symbolizes the transpoze of the matrix Δ, the forward difference matrix, which are identified by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n \geq k \geq n-1 \\ 0, & k > n \text{ or } 0 \leq k < n-1 \end{cases}$$

and

$$\Delta'_{nk} = \begin{cases} (-1)^{n-k}, & n \leq k \leq n+1 \\ 0, & k > n+1 \text{ or } 0 \leq k < n \end{cases}$$

for all $k, n \in \mathbb{N}$; ; respectively. (Kızmaz, 1981) first presented the concept of difference sequence spaces and determined the following sequence spaces:

$$X(\Delta) = \{x \in w : (x_k - x_{k+1}) \in X\}, \qquad X = \ell_\infty, c, c_0.$$

(Altay and Başar, 2007) examined the difference sequence space bv_p which involves sequences (x_k) such that $(x_k - x_{k-1})$ in the case $0 < p < 1$. For $1 \le p \le \infty$., (Çolak et al., 2004) examined this difference space. In addition, authors analyzed certain difference sequence spaces, see (Et, 1993), (Mursaleen, 1996), (Gaur and Mursaleen, 1998), (Et and Esi, 2000), (Et and Çolak, 1995), (Et and Başarır, 1997), (Karakaş et al., 2013).

The primary aim of this note is to establish the Lucas difference matrix $\hat{E}$ and introduce new sequence spaces $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ connected with matrix domain of $\hat{E}$ for sequence spaces ℓ_p and ℓ_∞ where $1 \le p < \infty$.

In the second part of this paper, we give some formulas and basic notions about Lucas sequence and a BK-space. In the third part, by using Lucas numbers, we introduce the sequence spaces $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$, and establish some inclusion relations. Also, we construct the basis of $\ell_p(\hat{E})$ for $1 \le p < \infty$. Finally, we analyze some geometric properties for the space $\ell_p(\hat{E}), 1 \le p < \infty$.

$$\sum_{i=0}^{n} L_i = L_{n+2} - 1,$$

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2,$$

MATERIAL AND METHODS

A K-space is a sequence space with a linear topology provided each maps $\tau_k : X \to \mathbb{C}$, $\tau_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$. If a K-space X is a complete linear metric space, then X is called a FK-space. A Banach K-space is a normed FK-space. The sequence spaces ℓ_∞, c, c_0 are Banach -spaces with the norm $\|x\|_\infty = sup_k |x_k|$. Also ℓ_p, is a Banach -space with $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}, 1 \le p < \infty$.

For a normed linear space X, the sequence $(b^{(n)})_{n=0}^{\infty}$ is called Schauder basis if there is a unique sequence $(\gamma_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \gamma_n b^{(n)}$ for every $x \in X$, that is,

$$\lim_{m \to \infty} \left\| x - \sum_{n=0}^{m} \gamma_n b^{(n)} \right\|_X = 0$$

The Lucas sequence (L_n) is defined by using Fibonacci recurrence relation and different initial conditions as:

$$L_0 = 2, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2}, n \ge 2.$$

Lucas numbers are closely related to Fibonacci numbers. In spite of this relation, they exhibit distinct properties. Some fundamental characteristics of Lucas sequences are given as follows (Koshy, 2001), (Vajda, 2008):

$$\lim_{n \to \infty} \frac{L_n}{L_{n-1}} = \varphi \text{ (golden section)},$$

$$L_{n-1}L_{n+1} - L_n^2 = -5.(-1)^n.$$

It can be easily derived by placing in the last equality that $L_{n-1}^2 + L_n L_{n-1} - L_n^2 = -5(-1)^n$.

RESULTS AND DISCUSSION

In the present section, we'll firstly give the following new double band matrix $\hat{E} = (\hat{L}_{nk})$ by means of the sequence (L_n) of Lucas numbers for all $n, k \in \mathbb{N} - \{0\}$:

$$\hat{L}_{nk} = \begin{cases} -\dfrac{L_n}{L_{n-1}}, & k = n-1 \\ \dfrac{L_{n-1}}{L_n}, & k = n \\ 0, & other \end{cases}$$

The inverse of the above Lucas matrix can be easily calculated and it is given by

$$\hat{L}^{-1} = \begin{cases} \dfrac{L_n^2}{L_{k-1}L_k}, & n \geq k > 0 \\ 0, & n < k \end{cases}.$$

Now, we define the $\hat{E}$-transform of a sequence $x = (x_n)$ in the form:

$$y_n = \hat{E}_n(x) = \frac{L_{n-1}}{L_n} x_n - \frac{L_n}{L_{n-1}} x_{n-1}, \; n \geq 1. \tag{3}$$

And now, let's introduce the Lucas difference sequence spaces $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ in the form of

$$\ell_p(\hat{E}) = \left\{ x \in w : \sum_n \left| \frac{L_{n-1}}{L_n} x_n - \frac{L_n}{L_{n-1}} x_{n-1} \right|^p < \infty \right\}, 1 \leq p < \infty$$

$$\ell_\infty(\hat{E}) = \left\{ x \in w : \sup_n \left| \frac{L_{n-1}}{L_n} x_n - \frac{L_n}{L_{n-1}} x_{n-1} \right| < \infty \right\}.$$

These spaces can be redefined with the help of equality (2) by

$$\ell_P(\hat{E}) = (\ell_P)_{\hat{E}}, 1 \leq p < \infty \text{ and } \ell_\infty(\hat{E}) = (\ell_\infty)_{\hat{E}}. \tag{4}$$

106

Iğdır Üni. Fen Bilimleri Enst. Der. / *Iğdır Univ. J. Inst. Sci. & Tech.*

Theorem 3.1. The sets $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ are Banach K-spaces with

$$\|x\|_{\ell_p(\hat{E})} = \left(\sum_n |\hat{E}_n(x)|^p\right)^{1/p}, 1 \le p < \infty \text{ and } \|x\|_{\ell_\infty(\hat{E})} = sup_n|\hat{E}_n(x)|.$$

Proof. By the Theorem 4.3.12 of (Wilansky, 1984), we obtain that $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ are Banach K-spaces with the above norms since the matrix E is triangle and equality (4) holds.

$\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ are the sequence spaces of non-absolute type. So indeed,

$$\|x\|_{\ell_p(\hat{E})} \ne \||x|\|_{\ell_p(\hat{E})} \text{ and } \|x\|_{\ell_\infty(\hat{E})} \ne \||x|\|_{\ell_\infty(\hat{E})}.$$

This means that the absolute property is not valid on the spaces $\ell_p(\hat{E})$ and $\ell_\infty(\hat{E})$ for fewest one sequence where $|x| = (|x_k|)$.

Theorem 3.2. The Lucas difference sequence space $\ell_p(\hat{E})$ of non-absolute type is linear isomorphic to ℓ_p for $1 \le p \le \infty$.

Proof. Let $1 \le p \le \infty$ and consider the transformation Z from $\ell_p(\hat{E})$ to ℓ_p defined by $x \to y = Zx$ with the notation of equality (3). Then, we have $Zx = y = \hat{E}x \in \ell_p$ for every $x \in \ell_p(\hat{E})$. It is trivial that Z is linear. Additionally, it is easy to see that $x = 0$ if $Zx = 0$ is injective.

Besides, let $y = (y_k) \in \ell_p$ and consider the sequence $x = (x_k)$ by

$$x_k = \sum_{j=1}^k \frac{L_k^2}{L_{j-1}L_j} y_j. \tag{5}$$

Next, we obtain that

$$\|x\|_{\ell_p(\hat{E})} = \left(\sum_k \left|\frac{L_{k-1}}{L_k}x_k - \frac{L_k}{L_{k-1}}x_{k-1}\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_k \left|\frac{L_{k-1}}{L_k}\sum_{j=1}^k \frac{L_k^2}{L_{j-1}L_j}y_j - \frac{L_k}{L_{k-1}}\sum_{j=1}^{k-1} \frac{L_{k-1}^2}{L_{j-1}L_j}y_j\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_k |y_k|^p\right)^{\frac{1}{p}} = \|y\|_{\ell_p} < \infty \text{ and}$$

$$\|x\|_{\ell_\infty(\hat{E})} = sup_k \left|\frac{L_{k-1}}{L_k}x_k - \frac{L_k}{L_{k-1}}x_{k-1}\right| = sup_k|y_k| = \|y\|_{\ell_\infty} < \infty.$$

This means that $x \in \ell_p(\hat{E}), 1 \leq p \leq \infty$. Hence, we see that Z is surjective and preserves the norm. Therefore, Z is a linear bijection and so $\ell_p(\hat{E})$ and ℓ_p are linear isomorphic.

Theorem 3.3. If $1 \leq p \leq \infty$, then the inclusion $\ell_p \subset \ell_p(\hat{E})$ strictly holds.

Proof. Let $x \in \ell_p, 1 < p \leq \infty$. It can be easily seen that the inequalities $\frac{L_{k-1}}{L_k} \leq 2$ and $\frac{L_k}{L_{k-1}} \leq 3$ hold. By means of these inequalities and equality (3), we have

$$\sum_k |\hat{E}_k(x)|^p \leq \sum_k 6^{p-1}(|2x_k|^p + |3x_{k-1}|^p) \leq 6^{2p-1}(\sum_k |x_k|^p + \sum_k |x_{k-1}|^p),$$

$$sup_k |\hat{E}_k(x)| \leq 5 sup_k |x_k|.$$

Hence, for $1 < p \leq \infty$, we obtain

$$\|x\|_{\ell_p(\hat{E})} \leq 36\|x\|_p \text{ and } \|x\|_{\ell_\infty(\hat{E})} \leq 5\|x\|_\infty. \tag{6}$$

Additionally, the sequence $x = (x_k) = (L_k^2) = (1, 3^2, 4^2, 7^2, \dots)$ is in $\ell_p(\hat{E}) - \ell_p$. This gives that the inclusion $\ell_p \subset \ell_p(\hat{E})$ is strict in the case $1 < p \leq \infty$. Also, the inequality (6) holds for $p = 1$.

Theorem 3.4. $\ell_p(\hat{E}) \subset \ell_s(\hat{E})$ for $1 \leq p < s$.

Proof. Let y be the sequence given by equality (3). If we take $x \in \ell_p(\hat{E})$, then we have $y \in \ell_p$. Since the inclusion $\ell_p \subset \ell_s$ holds, we obtain that $y \in \ell_s$. This yields us $x \in \ell_s(\hat{E})$ and so, the inclusion $\ell_p(\hat{E}) \subset \ell_s(\hat{E})$ holds.

Theorem 3.5. The sequence $\left(b^{(k)}\right)_{k=1}^{\infty}$ which is established by

$$\left(b^{(k)}\right)_n = \begin{cases} \frac{L_n^2}{L_{k-1}L_k}, & n \geq k \\ 0 & , & n < k \end{cases} \tag{7}$$

forms a basis for $\ell_p(\hat{E})$ in the case $1 \le p < \infty$. Also, every $x \in \ell_p(\hat{E})$ has a unique representation of the form

$$x = \sum_k \hat{E}_k(x) b^{(k)}. \tag{8}$$

Proof. From equality (7), $\hat{E}(b^{(k)}) = e^{(k)} \in \ell_p$ and this means that $b^{(k)} \in \ell_p(\hat{E})$. Now, let's take $x \in \ell_p(\hat{E})$ and put $x^{(m)} = \sum_{k=1}^m \hat{E}_k(x) b^{(k)}$ for every non-negative integer m. Thus, it is obtained that

$$\hat{E}(x^{(m)}) = \sum_{k=1}^m \hat{E}_k(x) \hat{E}(b^{(k)}) = \sum_{k=1}^m \hat{E}_k(x) e^{(k)}$$

and also

$$\hat{E}_n(x - x^{(m)}) = \begin{cases} \hat{E}_n(x), & m < n \\ 0, & m \ge n \ge 0 \end{cases}.$$

So, there is a non-negative integer m_0 such that $\sum_{n=m_0+1}^\infty |\hat{E}_n(x)|^p \le \left(\frac{\varepsilon}{2}\right)^p$ for any $\varepsilon > 0$. Hence, we have that

$$\left\| x - x^{(m)} \right\|_{\ell_p(\hat{E})} = \left(\sum_{n=m+1}^\infty |\hat{E}_n(x)|^p \right)^{1/p} \le \left(\sum_{n=m_0+1}^\infty |\hat{E}_n(x)|^p \right)^{1/p} \le \frac{\varepsilon}{2} < \varepsilon$$

for every $m \ge m_0$, that is,

$$\lim_{m \to \infty} \left\| x - x^{(m)} \right\|_{\ell_p(\hat{E})} = 0.$$

Last, let's assume that $x = \sum_k \beta_k(x) b^{(k)}$ to demostrate the uniqueness of equality (8) for $x \in \ell_p(\hat{E})$. By using the continuity of the linear transformation Z, we have

$$\hat{E}_n(x) = \sum_k \beta_k(x) \hat{E}_n(b^{(k)}) = \sum_k \beta_k(x) \delta_{nk} = \beta_n(x).$$

Therefore, the proof is completed.

Theorem 3.6. The spaces $\ell_p(\hat{E})$ and bv_p do not include each other for $1 \leq p < \infty$.

Proof. If we take $x = (x_k) = (L_k^2) = (1, 3^2, 4^2, 7^2, \dots)$ and $e = (1,1,1,\dots)$, then we result that $x \in \ell_p(\hat{E})$ and $x \notin bv_p$ by reason of $\hat{E}x = (2,0,0,\dots) \in \ell_p$ and $\Delta x = (1, L_0 L_3, L_1 L_4, \dots, L_{k-2} L_{k+1}, \dots) \notin \ell_p$. Now, we take the equation

$$\left| \frac{L_{k-1}}{L_k} - \frac{L_k}{L_{k-1}} \right| = \frac{|L_{k-1}^2 - L_k^2|}{L_{k-1} L_k} = \frac{|(-5)(-1)^k - L_{k-1} L_k|}{L_{k-1} L_k}$$

into consideration. If k is even, then $L_{k-1} L_k < |(-5)(-1)^k - L_{k-1} L_k|$. Hence, the series $\sum_k \left| \frac{L_{k-1}}{L_k} - \frac{L_k}{L_{k-1}} \right|^p$ is not convergent for $1 \leq p < \infty$. Thus, $\hat{E}e = \left(\frac{L_{k-1}}{L_k} - \frac{L_k}{L_{k-1}} \right) \notin$

ℓ_p in the case $1 \leq p < \infty$ and it is clear that $e \in \ell_p$. In conclusion, the spaces $\ell_p(\hat{E})$ and bv_p coincide but they do not contain one another.

A Banach space Y possess Banach-Saks property if any bounded sequence in Y approves a subsequence whose arithmetic mean converges in norm. Similarly, a Banach space Y holds weak Banach-Saks property if any weakly null sequence in X admits a subsequence whose arithmetic mean strongly converges in norm.

Let X be a Banach space. (Garcia-Falset, 1994) defined the coefficient $R(X)$ as:

$$R(X) = sup\left(\lim_{n \to \infty} inf \|x_n + x\| \right)$$

He also proved that a Banach space X with $R(X) < 2$ has the weak fixed point property.

Of late years, some studies about geometrical properties of a sequence space can be seen in (Et and Karakaya, 2014), (Karakaya and Altun, 2014), (Mursaleen et al., 2007).

Theorem 3.7. $\ell_p(\hat{E})$ holds the Banach-Saks property of type p.

Proof. It can be proved by standard technic which is available in (Karakaş et al., 2013).

Remark 3.1. As $\ell_p(\hat{E})$ is linearly isomorphic to space ℓ_p, we take in consideration .

$$R\left(\ell_p(\hat{E}) \right) = R(\ell_p) = 2^{1/p}$$

Since $R\left(\ell_p(\hat{E}) \right) < 2$, it can be given the below theorem:

Theorem 3.9. The space $\ell_p(\hat{E})$ possess the weak fixed point property for $1 < p < \infty$.

REFERENCES

Altay B, Başar F, 2005. Some Euler sequence spaces of nonabsolute type. *Ukrainian Math. J*, 57: 1-17.

Altay B, Başar F, 2007. The fine spectrum and the matrix domain of the difference operator Δ on the sequence space , . *Commun. Math. Anal.*, 2: 1-11.

Altay B, Başar F, Mursaleen M, 2006. On the Euler sequence spaces which include the spaces and I, *Inform. Sci*, 176: 1450-1462.

Başar F, Altay B, 2003. On the space of sequences of -bounded variation and related matrix mappings. *Ukrainian Math. J*, 55: 136-147.

Çolak R, Et M, Malkowsky E, 2004. *Some topics of sequence spaces, Lecture notes in Mathematics. Fırat Univ. Press, 1-63*.

Et M, 1993. On some difference sequence spaces. *Doğa Mat.*, 17: 18-24.

Et M, Başarır M, 1997. On some new generalized difference sequence spaces. *Period. Math. Hungar.*, 35: 169-175.

Et M, Çolak R, 1995. On some generalized difference sequence spaces. *Soochow J. Math.*, 21: 377-386.

Et M, Esi A, 2000. On Köthe-Toeplitz duals of generalized difference sequence spaces. *Bull. Malays. Math. Sci. Soc. (2)*, 23: 25-32.

Et M, Karakaya V, 2014. A new difference sequence set of order and its geometrical properties. *Abstr. Appl. Anal.*, Art. ID 278907: 4 p.

García-Falset J, 1994. Stability and fixed points for nonexpansive mappings. *Houston J. Math.*, 20: 495-506.

Gaur AK, Mursaleen M, 1998. Difference sequence spaces. *Internat. J. Math. Math. Sci.*, 21: 701-706.

Kara EE, 2013. Some topological and geometrical properties of new Banach sequence spaces. *J. Inequal. Appl.*, 38: 15 pp.

Kara EE, Başarır M, 2012. An application of Fibonacci numbers into infinite Toeplitz matrices. Caspian Journal of Mathematics Sciences, 1: 1-6.

Karakaş M, 2015. A new regular matrix defined by Fibonacci numbers and its applications. BEU Journal of Science, 4: 205-210.

Karakaş M, Çınar M, Et M, 2013. Some geometric properties of a new sequence space. *J. Comput. Anal. Appl.*, 15: 23-31.

Karakaş M, Et M, Karakaya V, 2013. Some geometric properties of a new difference sequence space involving lacunary sequences. *Acta Math. Sci. Ser. B Engl. Ed.*, 33: 1711-1720.

Karakaya V, Altun F, 2014. On some geometric properties of a new paranormed sequence space. *J. Funct. Spaces,* Art. ID 685382: 8 p.

Kızmaz H, 1981. On certain sequence spaces. *Canad. Math. Bull.*, 24: 169-176.

Koshy T, 2001. Fibonacci and Lucas Numbers with Applications. John Wiley&Sons, Canada. 672 p.

Mursaleen M, 1996. Generalized spaces of difference sequences. *J. Math. Anal. Appl.*, 203: 738-745.

Mursaleen M, Başar F, Altay B, 2006. On the Euler sequence spaces which include the spaces and II. *Nonlinear Analysis,* 65: 707-717.

Mursaleen M, Çolak R, Et M, 2007. Some geometric inequalities in a new Banach sequence space. *J. Inequal. Appl.*, Art. ID 86757: 6 p.

Mursaleen M, Noman AK, 2010. On some new difference sequence spaces of non-absolute type. *Math. Comput. Modelling*, 52: 603-617.

Mursaleen M, Noman AK, 2010. On the spaces of -convergent and bounded sequences. *Thai J. Math.*, 8: 311-329.

Mursaleen M, Noman AK, 2011. On some new sequence spaces of non-absolute type related to the spaces and I. *Filomat*, 25: 33-51.

Savaş E, Karakaya V, Şimşek N, 2009. Some -type new sequence spaces and their geometric properties. *Abstr. Appl. Anal.*, Art. ID 696971: 12 pp.

Vajda S, 2008. Fibonacci and Lucas Numbers, and the Golden Section Theory and Applications. Dover Edition, NewYork, USA. 189 p.

Wilansky A, 1984. Summability Through Functional Analysis. *Elseiver Science Publishers BV, Amsterdam, The Netherlands. 317 p.*