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Research Article

On the analytic extension of the Horn's confluent function H_6 on domain in the space $\mathbb{C}^2$

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. The paper considers the problem of representation and extension of Horn's confluent functions by a special family of functions – branched continued fractions. An estimate of the rate of convergence for the branched continued fraction expansions of the ratios of Horn's confluent functions H_6 with real parameters is established in a new region. Here, the region refers to a domain (an open connected set) which may include all, part, or none of its boundary. Additionally, a new domain of the analytical continuation of the above-mentioned ratios is established, using their branched continued fraction expansions whose elements are polynomials in the space $\mathbb{C}^2$. These expansions can approximate solutions to certain differential equations and analytic functions represented by Horn's confluent functions H_6.

Keywords: Horn's hypergeometric function, branched continued fraction, holomorphic functions of several complex variables, analytic continuation, convergence

2020 Mathematics Subject Classification: 33C65, 32A17, 32A10, 30B40, 40A99.

1. INTRODUCTION

Hypergeometric functions have been and continue to be the subject of research for a century, surprisingly appearing in various applications in many sciences [8, 20, 28, 27, 29, 31]. Among the problems associated with the study of these functions, one of the most interesting and difficult is the representation and analytical expansion through a special family of functions – branched continued fractions [7, 14, 19]. A generalization of the classical Gaussian method is used to construct formal branched continued fraction expansions of hypergeometric functions [13, 21, 30], and the PC and PF methods are used to establish domains of analytical continuation of these functions [4, 16, 18].

In this paper, we consider the Horn's confluent function H_6, which is defined by the following double power series (see, [25])

$$\mathrm{H}_6(\alpha,\beta;\mathbf{z})=\sum_{p,q=0}^{+\infty}\frac{(\alpha)_{2p+q}}{(\beta)_{p+q}}\frac{z_1^p z_2^q}{p!q!},\quad |z_1|<1/4,\ |z_2|<+\infty,$$

where $\alpha,\beta\in\mathbb{C}$, $\beta\notin\{0,-1,-2,\ldots\}$, $(\cdot)_k$ is the Pochhammer symbol, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$.

Let $\mathfrak{I}_0=\{1,2,3\}$, $i(k)=(i_0,i_1,i_2,\ldots,i_k)$, and

$$\mathfrak{I}_k=\left\{i(k):\ i_0\in\mathfrak{I}_0,\ 2-\left[\frac{i_{r-1}-1}{2}\right]\le i_r\le 3-\left[\frac{i_{r-1}-1}{2}\right],\ 1\le r\le k\right\},\quad k\ge 1,$$

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where $[\cdot]$ denotes an integer part.

In [5], it is formally established that for each $i_0 \in \mathfrak{I}_0$

$$\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})} =1-\frac{\alpha}{2\beta}\delta_{i_0}^3+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}+\sum_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\frac{u_{i(2)}(\mathbf{z})}{v_{i(2)}+\ddots}}, \tag{1.1}$$

where δ_i^j denotes the Kronecker delta, and, for $i(1)\in\mathfrak{I}_1$,

$$u_{i(1)}(\mathbf{z})=\begin{cases}-2\dfrac{\alpha+1}{\beta}z_1, \text{ if } i_0=1,\ i_1=2,\\ -\dfrac{z_2}{\beta}, \text{ if } i_0=1,\ i_1=3,\\ -\dfrac{(2\beta-\alpha)(\alpha+1)}{\beta(\beta+1)}z_1, \text{ if } i_0=2,\ i_1=2,\\ -\dfrac{\beta-\alpha}{\beta(\beta+1)}z_2, \text{ if } i_0=2,\ i_1=3,\\ \dfrac{\alpha}{2\beta}, \text{ if } i_0=3,\ i_1=1,\\ \dfrac{\alpha}{2\beta(\beta+1)}z_2, \text{ if } i_0=3,\ i_1=2,\end{cases} \tag{1.2}$$

for $i(k+1)\in\mathfrak{I}_{k+1}$, $k\ge 1$,

$$u_{i(k+1)}(\mathbf{z}) =\begin{cases}-\dfrac{2(\alpha+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^3)}{\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}z_1, \text{ if } i_k=1,\ i_{k+1}=2,\\ -\dfrac{z_2}{\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1}, \text{ if } i_k=1,\ i_{k+1}=3,\\ \dfrac{(\alpha-2\beta-k-\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(\alpha+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^3)}{(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}z_1, \text{ if } i_k=2,\ i_{k+1}=2,\\ -\dfrac{\beta-\alpha+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}z_2, \text{ if } i_k=2,\ i_{k+1}=3,\\ \dfrac{\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}, \text{ if } i_k=3,\ i_{k+1}=1,\\ \dfrac{\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}z_2, \text{ if } i_k=3,\ i_{k+1}=2,\end{cases} \tag{1.3}$$

and, for $i(k)\in\mathfrak{I}_k$, $k\ge 1$,

$$v_{i(k)}=1-\frac{\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}\delta_{i_k}^3. \tag{1.4}$$

Here it is shown that if

$$\alpha\ge 0,\quad \beta\ge\alpha+1+\delta_{i_0}^1\quad\text{for}\quad i_0\in\mathfrak{I}_0, \tag{1.5}$$

and there exist the positive numbers $\nu_1, \nu_2, \nu_3, \mu_1, \mu_2$, and μ_3 such that

$$\frac{2\nu_1}{\mu_2} \leq \min\left\{1-\mu_1-\frac{\nu_2}{\beta\mu_3},\ 1-\mu_2-\frac{\nu_2}{(\beta+1)\mu_3}\right\}, \quad \frac{\nu_3}{(\beta+1)\mu_2}+\mu_3 \leq \frac{1}{2},$$

then

$$\Omega_{\nu_1,\nu_2,\nu_3} = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_2| - \mathrm{Re}(z_2) < 2\nu_3,\ |z_k| + \mathrm{Re}(z_k) < 2\nu_k,\ k=1,2,\}$$

is the domain of the analytical continuation of the function on the left side of (1.1).

The results of the study of branched continued fraction expansions of the other Horn's hypergeometric functions can be found in [1, 17, 23, 24].

In the next section, we give the formula for the difference of two approximants of branched continued fraction expansions of the ratios of Horn's confluent functions H_6 with real parameters and prove the auxiliary Theorem 2.1 on the estimation of the rate of convergence for these expansions in a new region of the space $\mathbb{R}^2$ (that is, a domain (an open connected set) which may include all, part, or none of its boundary). In Section 3, we prove Theorem 3.2 on the new domain of the analytical extension of the above-mentioned ratios in the space $\mathbb{C}^2$ and give an important Corollary 3.1 from it.

2. Auxiliary Results

Let i_0 be an arbitrary index in $\mathfrak{I}_0$. We set, for $i(n) \in \mathfrak{I}_n$, $n \geq 1$,

$$F_{i(n)}^{(n)}(\mathbf{z}) = v_{i(n)}, \tag{2.6}$$

and for $i(k) \in \mathfrak{I}_k$, $1 \leq k \leq n-1$, $n \geq 2$,

$$F_{i(k)}^{(n)}(\mathbf{z}) = v_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{u_{i(k+1)}(\mathbf{z})}{v_{i(k+1)} + \ddots + \sum_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \frac{u_{i(n)}(\mathbf{z})}{v_{i(n)}}}.$$

This gives the following recurrence relations

$$F_{i(k)}^{(n)}(\mathbf{z}) = v_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{u_{i(k+1)}(\mathbf{z})}{F_{i(k+1)}^{(n)}(\mathbf{z})}, \quad i(k) \in \mathfrak{I}_k,\ 1 \leq k \leq n-1,\ n \geq 2, \tag{2.7}$$

and also the following expressions

$$f_n^{(i_0)}(\mathbf{z}) = 1 - \frac{\alpha}{2\beta}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{u_{i(1)}(\mathbf{z})}{F_{i(1)}^{(n)}(\mathbf{z})}, \quad n \geq 1. \tag{2.8}$$

Suppose that $F_{i(k)}^{(n)}(\mathbf{z}) \neq 0$ for all $i(k) \in \mathfrak{I}_k$, $1 \leq k \leq n$, $n \geq 1$. Using the method suggested in ([11, p. 28]), we show that for $n \geq 1$ and $k \geq 1$

$$f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \cdots \sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]} \frac{(-1)^n \prod_{r=1}^{n+1} u_{i(r)}(\mathbf{z})}{\prod_{r=1}^{n+1} F_{i(r)}^{(n+k)}(\mathbf{z}) \prod_{r=1}^{n} F_{i(r)}^{(n)}(\mathbf{z})}. \tag{2.9}$$

On the first step we obtain

$$\begin{aligned}
&f^{(i_0)}_{n+k}(\mathbf{z})-f^{(i_0)}_{n}(\mathbf{z})\\
=&1-\frac{\alpha}{2\beta}\delta^3_{i_0}+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{F^{(n+k)}_{i(1)}(\mathbf{z})}-\left(1-\frac{\alpha}{2\beta}\delta^3_{i_0}+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{F^{(n)}_{i(1)}(\mathbf{z})}\right)\\
=&-\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{F^{(n+k)}_{i(1)}(\mathbf{z})F^{(n)}_{i(1)}(\mathbf{z})}(F^{(n+k)}_{i(1)}(\mathbf{z})-F^{(n)}_{i(1)}(\mathbf{z})).
\end{aligned}$$

For an arbitrary integer r such that $1\le r\le n-1$, we have

$$\begin{aligned}
&F^{(n+k)}_{i(r)}(\mathbf{z})-F^{(n)}_{i(r)}(\mathbf{z})\\
=&v_{i(r)}+\sum_{i_{r+1}=2-[(i_r-1)/2]}^{3-[(i_r-1)/2]}\frac{u_{i(r+1)}(\mathbf{z})}{F^{(n+k)}_{i(r+1)}(\mathbf{z})}-\left(v_{i(r)}+\sum_{i_{r+1}=2-[(i_r-1)/2]}^{3-[(i_r-1)/2]}\frac{u_{i(r+1)}(\mathbf{z})}{F^{(n)}_{i(r+1)}(\mathbf{z})}\right)\\
=&-\sum_{i_{r+1}=2-[(i_r-1)/2]}^{3-[(i_r-1)/2]}\frac{u_{i(r+1)}(\mathbf{z})}{F^{(n+k)}_{i(r+1)}(\mathbf{z})F^{(n)}_{i(r+1)}(\mathbf{z})}\left(F^{(n+k)}_{i(r+1)}(\mathbf{z})-F^{(n)}_{i(r+1)}(\mathbf{z})\right).
\end{aligned}\tag{2.10}$$

Since

$$F^{(n+k)}_{i(n)}(\mathbf{z})-F^{(n)}_{i(n)}(\mathbf{z})=\sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]}\frac{u_{i(n+1)}(\mathbf{z})}{F^{(n+k)}_{i(n+1)}(\mathbf{z})},$$

using the recurrence relation (2.10), at the nth step we obtain (2.9). For convenience, we rewrite (2.9) as follows

$$\begin{aligned}
f^{(i_0)}_{n+k}(\mathbf{z})-f^{(i_0)}_{n}(\mathbf{z})=(-1)^n&\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\cdots\sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{F^{(q)}_{i(1)}(\mathbf{z})}\\
&\times\prod_{k=1}^{[(n+1)/2]}\frac{u_{i(2k)}(\mathbf{z})}{F^{(p)}_{i(2k-1)}(\mathbf{z})F^{(p)}_{i(2k)}(\mathbf{z})}\prod_{k=1}^{[n/2]}\frac{u_{i(2k+1)}(\mathbf{z})}{F^{(q)}_{i(2k)}(\mathbf{z})F^{(q)}_{i(2k+1)}(\mathbf{z})},
\end{aligned}\tag{2.11}$$

where $q=n+k$, $p=n$, if n is even, and $q=n$, $p=n+k$, if n is odd.

The following theorem is true:

Theorem 2.1. *Suppose that α and β are real constants such that satisfy inequalities (1.5). Then for each $i_0\in\mathfrak{I}_0$:*

(A) *The branched continued fraction (1.1) converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z}\in\Theta_{l_1,l_2}$, where*

$$\Theta_{l_1,l_2}=\{\mathbf{z}\in\mathbb{R}^2:\ -l_1\le z_1\le 0,\ 0\le z_2\le l_2\},\quad l_1>0,\ 0<l_2<\frac{\beta}{4},\tag{2.12}$$

in addition, it converges uniformly on every compact subset of an interior of the region Θ_{l_1,l_2}.

(B) *If $f^{(i_0)}_n(\mathbf{z})$ denotes the nth approximant of the branched continued fraction (1.1), then for each $\mathbf{z}\in\Theta_{l_1,l_2}$*

$$|f^{(i_0)}(\mathbf{z})-f^{(i_0)}_n(\mathbf{z})|\le C^{(i_0)}\left(\frac{\varrho}{\varrho+1}\right)^n,\quad n\ge 1,$$

where

$$C^{(i_0)} = \begin{cases} \dfrac{2l_1(\alpha+1)(\beta+1)}{\beta(\beta-2l_2+1)} + \dfrac{2l_2}{\beta}, \text{ if } i_0 = 1, \\ \dfrac{l_1(2\beta-\alpha)(\alpha+1)}{\beta(\beta-2l_2+1)} + \dfrac{2l_2(\beta-\alpha)}{\beta(\beta+1)}, \text{ if } i_0 = 2, \\ \dfrac{\alpha}{2(\beta-2l_2)} + \dfrac{l_2\alpha}{2\beta(\beta-2l_2+1)}, \text{ if } i_0 = 3, \end{cases} \tag{2.13}$$

and

$$\varrho = \max\left\{\left(\frac{2l_1(\beta+1)}{\beta-2l_2+1} + \frac{2l_2}{\beta}\right)\frac{\beta}{\beta-4l_2}, \frac{\beta}{\beta-2l_2} + \frac{l_2}{2(\beta-2l_2+1)}\right\}. \tag{2.14}$$

Proof. We will carry out the proof in the same way as in [5, Theorem 3.2]. In [5, Formula (3.22)], it is shown that

$$\begin{aligned} v_{i(k-1),3} &= 1 - \frac{\alpha + k - \sum_{r=0}^{k-1}\delta_{i_r}^{3}}{2\beta + 2k - 2\sum_{r=0}^{k-1}\delta_{i_r}^{1}} \\ &\geq \frac{1}{2}, \quad i(k-1) \in \mathfrak{I}_k,\ k \geq 2, \end{aligned} \tag{2.15}$$

are valid under the condition (1.5).

Let n be an arbitrary natural number and $\mathbf{z}$ be an arbitrary fixed point in (2.12). We set

$$g_1 = 1 - \frac{2l_2}{\beta},\ g_2 = 1 - \frac{2l_2}{\beta+1},\ g_3 = \frac{1}{2}.$$

By induction on k, we prove that

$$F_{i(k)}^{(n)}(\mathbf{z}) \geq g_{i_k}, \tag{2.16}$$

where $i(k) \in \mathfrak{I}_k$, $1 \leq k \leq n$.

Taking into account (1.4), (2.6), (2.12), and (2.15), it is clear that for $k = n$ inequalities (2.16) hold for $i_n = 1, 2, 3$. Let (2.16) hold for $k = r+1$ such that $r+1 \leq n$ and for all $i(r+1) \in \mathfrak{I}_r$. Then from (2.7) for $k = r$ and for any $i(r-1) \in \mathfrak{I}_{r-1}$, we have

$$\begin{aligned} F_{i(r)}^{(n)}(\mathbf{z}) &= v_{i(r)} + \sum_{i_{r+1}=2}^{3} \frac{u_{i(r+1)}(\mathbf{z})}{F_{i(r+1)}^{(n)}(\mathbf{z})} \\ &\geq v_{i(r)}(\mathbf{z}) - 2\frac{l_2}{\beta} \\ &= 1 - \frac{2l_2}{\beta} \\ &= g_1, \end{aligned}$$

if $i_r = 1$,

$$\begin{aligned} F_{i(r)}^{(n)}(\mathbf{z}) &\geq v_{i(r)} - 2\frac{l_2}{\beta+1} \\ &= 1 - \frac{2l_2}{\beta+1} \\ &= g_2, \end{aligned}$$

if $i_r=2$, and

$$\begin{aligned} F^{(n)}_{i(r)}(\mathbf{z}) &\ge v_{i(r)} \\ &\ge \frac{1}{2} \\ &= g_3, \end{aligned}$$

if $i_r=3$.

Let now n be an arbitrary integer such that $n\ge 2$. We show that

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})F^{(n)}_{i(k)}(\mathbf{z})|} \le \frac{\varrho}{\varrho+1}, \tag{2.17}$$

where $i(k)\in\mathfrak{I}_k$, $1\le k\le n-1$, and ϱ is defined by (2.14), or the same as

$$\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|} \le \varrho\left(|F^{(n)}_{i(k)}(\mathbf{z})| - \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|}\right),$$

where $i(k)\in\mathfrak{I}_k$, $1\le k\le n-1$.

Using (1.2)–(1.4), (2.12), (2.15), and (2.16), for any $i(k)\in\mathfrak{I}_k$, $1\le k\le n-1$, and for any $\mathbf{z}\in\Theta_{l_1,l_2}$ we have

$$\begin{aligned} |F^{(n)}_{i(k)}(\mathbf{z})| - \sum_{i_{k+1}=2}^{3} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|} &\ge 1-2\frac{|u_{i(k),3}(\mathbf{z})|}{g_3} \\ &= 1-\frac{2}{\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}\frac{|z_2|}{g_3} \\ &\ge 1-\frac{4l_2}{\beta} \end{aligned}$$

and

$$\begin{aligned} \sum_{i_{k+1}=2}^{3} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|} &\le 2\frac{\alpha+k-\sum_{r=0}^{k-1}\delta^3_{i_r}+1}{\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}\frac{|z_1|}{g_2} + \frac{1}{\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r}}\frac{|z_2|}{g_3} \\ &\le \frac{2l_1(\beta+1)}{\beta+1-2l_2}+\frac{2l_2}{\beta} \\ &\le \varrho\left(1-\frac{4l_2}{\beta}\right), \end{aligned}$$

if $i_k=1$,

$$\begin{aligned} |F^{(n)}_{i(k)}(\mathbf{z})| - \sum_{i_{k+1}=2}^{3} \frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|} &\ge 1-2\frac{|u_{i(k),3}(\mathbf{z})|}{g_3} \\ &= 1-2\frac{\beta-\alpha+\sum_{r=0}^{k-1}(\delta^3_{i_r}-\delta^1_{i_r}}{(\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(\beta+k+1-\sum_{r=0}^{k-1}\delta^1_{i_r})}\frac{|z_2|}{g_3} \\ &\ge 1-\frac{4l_2}{\beta+1} \end{aligned}$$

and

$$\sum_{i_{k+1}=2}^{3}\frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|}\le\frac{(2\beta-\alpha+k+\sum_{r=0}^{k-1}(\delta^3_{i_r}-2\delta^1_{i_r}))(\alpha+k+1-\sum_{r=0}^{k-1}\delta^3_{i_r})}{(\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(\beta+k-1-\sum_{r=0}^{k-1}\delta^1_{i_r})}\frac{|z_1|}{g_3}$$
$$+\frac{\beta-\alpha+\sum_{r=0}^{k-1}(\delta^3_{i_r}-\delta^1_{i_r}}{(\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(\beta+k+1-\sum_{r=0}^{k-1}\delta^1_{i_r})}\frac{|z_2|}{g_3}$$
$$\le\frac{2l_1(\beta+1)}{\beta+1-2l_2}+\frac{2l_2}{\beta+1}$$
$$<\frac{2l_1(\beta+1)}{\beta+1-2l_2}+\frac{2l_2}{\beta}$$
$$\le\varrho\left(1-\frac{4l_2}{\beta}\right)$$
$$<\varrho\left(1-\frac{4l_2}{\beta+1}\right),$$

if $i_k=2$, and last

$$|F^{(n)}_{i(k)}(\mathbf{z})|-\sum_{i_{k+1}=1}^{2}\frac{|u_{i(k+1)}(\mathbf{z})|}{|F^{(n)}_{i(k+1)}(\mathbf{z})|}\ge v_{i(k)}$$
$$\ge g_3$$
$$=\frac{1}{2}$$

and

$$\sum_{i_{k+1}=1}^{2}\frac{|P_{i(k+1)}(\mathbf{z})|}{|G^{(n)}_{i(k+1)}(\mathbf{z})|}\le\frac{\alpha+k-\sum_{r=0}^{k-1}\delta^3_{i_r}}{2(\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r})}\frac{1}{g_1}$$
$$+\frac{\alpha+k-\sum_{r=0}^{k-1}\delta^3_{i_r}}{(\beta+k-\sum_{r=0}^{k-1}\delta^1_{i_r})(\beta+k+1-\sum_{r=0}^{k-1}\delta^1_{i_r})}\frac{|z_2|}{g_2}$$
$$\le\frac{\beta}{2(\beta-2l_2)}+\frac{l_2}{2(\beta+1-2l_2)}$$
$$\le\frac{1}{2}\varrho,$$

if $i_k=3$. Hence, due to the arbitrariness of $i(k)$, the validity of inequalities (2.17) follows.

Now from (1.2), (2.12), (2.13), and (2.16) it follows that

$$\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{|u_{i(1)}(\mathbf{z})|}{|F^{(q)}_{i(1)}(\mathbf{z})|}\le C^{(i_0)}\quad\text{for}\quad i_0\in\mathfrak{I}_0\quad\text{and}\quad q\ge1. \tag{2.18}$$

Indeed,

$$\frac{|u_{1,2}(\mathbf{z})|}{|F^{(q)}_{1,2}(\mathbf{z})|}+\frac{|u_{1,3}(\mathbf{z})|}{|F^{(q)}_{1,3}(\mathbf{z})|}\le\frac{2l_1(\alpha+1)(\beta+1)}{\beta(\beta-2l_2+1)}+\frac{2l_2}{\beta}$$
$$=C^{(1)},$$

if $i_0=1$,

$$\begin{aligned}\frac{|u_{2,2}(\mathbf{z})|}{|F^{(q)}_{2,2}(\mathbf{z})|}+\frac{|u_{2,3}(\mathbf{z})|}{|F^{(q)}_{2,3}(\mathbf{z})|}&\le\frac{l_1(2\beta-\alpha)(\alpha+1)(\beta+1)}{\beta(\beta+1)(\beta-2l_2+1)}+\frac{2l_2(\beta-\alpha)}{\beta(\beta+1)}\\&=\frac{l_1(2\beta-\alpha)(\alpha+1)}{\beta(\beta-2l_2+1)}+\frac{2l_2(\beta-\alpha)}{\beta(\beta+1)}\\&=C^{(2)},\end{aligned}$$

if $i_0=2$,

$$\begin{aligned}\frac{|u_{3,1}(\mathbf{z})|}{|F^{(q)}_{3,1}(\mathbf{z})|}+\frac{|u_{3,2}(\mathbf{z})|}{|F^{(q)}_{3,2}(\mathbf{z})|}&\le\frac{\alpha\beta}{2\beta(\beta-2l_2)}+\frac{l_2\alpha(\beta+1)}{2\beta(\beta+1)(\beta-2l_2+1)}\\&=\frac{\alpha}{2(\beta-2l_2)}+\frac{l_2\alpha}{2\beta(\beta-2l_2+1)}\\&=C^{(3)},\end{aligned}$$

if $i_0=3$.

From (2.12) and (2.16) it is clear that $F^{(q)}_{i(k)}(\mathbf{z})\neq 0$ for $i(k)\in\mathfrak{I}_k$, $1\le k\le q$, $q\ge 1$, and for all $\mathbf{z}\in\Theta_{l_1,l_2}$. Thus, using (2.17) and (2.18), from (2.11) for any $\mathbf{z}\in\Theta_{l_1,l_2}$ we have

$$\begin{aligned}|f^{(i_0)}_{n+k}(\mathbf{z})-f^{(i_0)}_n(\mathbf{z})|&\le\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{|u_{i(1)}(\mathbf{z})|}{|F^{(q)}_{i(1)}(\mathbf{z})|}\left(\frac{\varrho}{\varrho+1}\right)^n\\&\le C^{(i_0)}\left(\frac{\varrho}{\varrho+1}\right)^n,\quad n\ge1,\ k\ge1,\end{aligned}$$

where $q=n+k$, if n is even, and $q=n$, if n is odd. Finally, (A) and (B) follow when $n\to\infty$ and $k\to\infty$, respectively. □

3. Analytical Continuation

In this section, we prove the following result:

Theorem 3.2. *Let α and β be real constants satisfying the inequalities (1.5), and $\nu_1,\nu_2,\nu_3,\mu_1,\mu_2,\mu_3$ be positive numbers such that*

$$\frac{2\nu_1}{\mu_2}\le\min\left\{1-\mu_1-\frac{\nu_2}{\beta\mu_3},\ 1-\mu_2-\frac{\nu_2}{(\beta+1)\mu_3}\right\},\quad\frac{\kappa}{2\mu_1}+\frac{\nu_3}{(\beta+1)\mu_2}\le\frac{1}{2}-\mu_3,\tag{3.19}$$

where $0\le\kappa<1$. Then for each $i_0\in\mathfrak{I}_0$:

(**A**) *The branched continued fraction (1.1) converges uniformly on every compact subset of the domain*

$$\Xi_{\nu_1,\nu_2,\nu_3}=\bigcup_{\substack{\varphi\in(-\pi/4,\pi/4)\\ \mathrm{tg}(\varphi)\le\sqrt{\kappa}}}\Xi_{\nu_1,\nu_2,\nu_3,\varphi},\tag{3.20}$$

where

$$\Xi_{\nu_1,\nu_2,\nu_3,\varphi}=\left\{\mathbf{z}\in\mathbb{C}^2:\ \frac{|z_2|-\mathrm{Re}(z_2e^{-2\varphi})}{\cos^2(\varphi)}<2\nu_3,\ \frac{|z_k|+\mathrm{Re}(z_ke^{-2\varphi})}{\cos^2(\varphi)}<2\nu_k,\ k=1,2\right\},\tag{3.21}$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in the domain Ξ_{ν_1,ν_2,ν_3}.

(**B**) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of branched continued fraction (1.1) in the domain (3.20).*

Proof. To prove (A), we use the convergence continuation theorem (see, [3, Theorem 3] and also [11, Theorem 2.17], [34, Theorem 24.2]), which extends the domain of convergence of a branched continued fraction, which is already known for a small domain, to a larger domain.

Let i_0 be an arbitrary index in $\mathfrak{I}_0$. Let us show that $\{f_n^{(i_0)}(\mathbf{z})\}$, where $f_n^{(i_0)}(\mathbf{z})$, $n \geq 1$, are defined by (2.8), is a sequence of functions holomorphic in the domain (3.20). Since each approximant of the branched continued fraction (1.1) is an entire function, it suffices to show that

$$F_{i(1)}^{(n)}(\mathbf{z}) \neq 0 \quad \text{for} \quad i(1) \in \mathfrak{I}_1,\ n \geq 1, \quad \text{and} \quad \text{for} \quad \mathbf{z} \in \Xi_{\nu_1,\nu_2,\nu_3}.$$

Let φ be an arbitrary real in $(-\pi/4, \pi/4)$ such that $\mathrm{tg}(\varphi) \leq \sqrt{\kappa}$ and $\mathbf{z}$ be an arbitrary fixed point in the domain (3.21). Under the condition (1.5) and (2.15), from (1.4) for any $i(k-1) \in \mathfrak{I}_k$, $k \geq 2$, we have

$$\begin{aligned} \mathrm{Re}(v_{i(k-1),3}e^{-i\varphi}) &= \left(1 - \frac{\alpha + k - \sum_{r=0}^{k-1}\delta_{i_r}^3}{2\beta + 2k - 2\sum_{r=0}^{k-1}\delta_{i_r}^1}\right)\mathrm{Re}(e^{-i\varphi}) \\ &\geq \frac{1}{2}\cos(\varphi), \end{aligned}$$

and for $i_k = 1$ or $i_k = 2$ we get

$$\begin{aligned} \mathrm{Re}(v_{i(k)}e^{-i\varphi}) &= \mathrm{Re}(e^{-i\varphi}) \\ &= \cos(\varphi). \end{aligned}$$

Now, using (1.4), (1.5), (2.15), and (3.21), from (1.3) for any $i(k) \in \mathfrak{I}$, $k \geq 2$, herewith $i_k = 1$ we obtain

$$\begin{aligned} |u_{i(k),2}(\mathbf{z})| - \mathrm{Re}(u_{i(k),2}(\mathbf{z})e^{-2i\varphi}) &= \frac{2(\alpha + k - \sum_{r=0}^{k-1}\delta_{i_r}^3 + 1)}{\beta + k - \sum_{r=0}^{k-1}\delta_{i_r}^1}(|z_1| + \mathrm{Re}(z_1 e^{-2i\varphi})) \\ &< 4\nu_1\cos^2(\varphi), \\ |u_{i(k),3}(\mathbf{z})| - \mathrm{Re}(u_{i(k),3}(\mathbf{z})e^{-2i\varphi}) &= \frac{|z_2| + \mathrm{Re}(z_2 e^{-2i\varphi})}{\beta + k - \sum_{r=0}^{k-1}\delta_{i_r}^1} \\ &< \frac{2\nu_2}{\beta}\cos^2(\varphi), \end{aligned}$$

and, thus,

$$\begin{aligned} \sum_{i_{k+1}=2}^{3} \frac{|u_{i(k+1)}(\mathbf{z})| - \mathrm{Re}(u_{i(k+1)}(\mathbf{z})e^{-2i\varphi})}{\mu_{i_{k+1}}\cos(\varphi)} &< \frac{4\nu_1}{\mu_2}\cos(\varphi) + \frac{2\nu_2}{\beta\mu_3}\cos(\varphi) \\ &\leq 2(1-\mu_1)\cos(\varphi) \\ &= 2(\mathrm{Re}(v_{i(k)}e^{-i\varphi}) - \mu_1\cos(\varphi)). \end{aligned}$$

When $i_k=2$ we have

$$\begin{aligned}&|u_{i(k),2}(\mathbf{z})|-\mathrm{Re}(u_{i(k),2}(\mathbf{z})e^{-2i\varphi})\\=&\frac{(2\beta-\alpha+k+\sum_{r=0}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3+1)}{(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_1|+\mathrm{Re}(z_1e^{-2i\varphi}))\\<&4\nu_1\cos^2(\varphi),\end{aligned}$$

$$\begin{aligned}&|u_{i(k),3}(\mathbf{z})|-\mathrm{Re}(u_{i(k),3}(\mathbf{z})e^{-2i\varphi})\\=&\frac{\beta-\alpha+\sum_{r=0}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1+1)}(|z_2|+\mathrm{Re}(z_2e^{-2i\varphi}))\\<&\frac{2\nu_2}{\beta+1}\cos^2(\varphi),\end{aligned}$$

and, thus,

$$\begin{aligned}\sum_{i_{k+1}=2}^{3}\frac{|u_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(u_{i(k+1)}(\mathbf{z})e^{-2i\varphi})}{\mu_{i_{k+1}}\cos(\varphi)}&<\frac{4\nu_1}{\mu_2}\cos(\varphi)+\frac{2\nu_2}{(\beta+1)\mu_3}\cos(\varphi)\\&\le 2(\mathrm{Re}(v_{i(k)}e^{-i\varphi})-\mu_2\cos(\varphi)).\end{aligned}$$

If $i_k=3$, we obtain

$$\begin{aligned}|u_{i(k),1}(\mathbf{z})|-\mathrm{Re}(u_{i(k),1}(\mathbf{z})e^{-2i\varphi})&=\frac{\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}-\frac{\alpha+k-\sum_{p=r}^{k-1}\delta_{i_r}^3}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)}\cos(2\varphi)\\&\le\frac{1}{2}(1-\cos(2\varphi))\\&\le\kappa\cos^2(\varphi),\\|u_{i(k),2}(\mathbf{z})|-\mathrm{Re}(u_{i(k),2}(\mathbf{z})e^{-2i\varphi})&=\frac{(\alpha+k-\sum_{r=0}^{k-1}\delta_{i_r}^3)(|z_2|-\mathrm{Re}(z_2e^{-2i\varphi}))}{2(\beta+k-\sum_{r=0}^{k-1}\delta_{i_r}^1)(\beta+k+1-\sum_{r=0}^{k-1}\delta_{i_r}^1)}\\&<\frac{2\nu_3}{\beta+1}\cos(\varphi),\end{aligned}$$

and, thus,

$$\begin{aligned}\sum_{i_{k+1}=1}^{2}\frac{|u_{i(k+1)}(\mathbf{z})|-\mathrm{Re}(u_{i(k+1)}(\mathbf{z})e^{-2i\varphi})}{\mu_{i_{k+1}}\cos(\varphi)}&<\frac{\kappa}{\mu_1}\cos(\varphi)+\frac{2\nu_3}{(\beta+1)\mu_2}\cos(\varphi)\\&\le 2(\mathrm{Re}(v_{i(k)}e^{-i\varphi})-\mu_3\cos(\varphi)).\end{aligned}$$

From [6, Proposition 2], with $g_{i(k)}=\mu_{i_k}$, $i(k)\in\mathfrak{I}_k$, $k\ge1$, it follows that

$$\mathrm{Re}(F_{i(k)}^{(n)}(\mathbf{z})e^{-i\varphi})\ge\mu_k\cos(\varphi)>0 \tag{3.22}$$

for $i(k)\in\mathfrak{I}_k$, $1\le k\le n$, $n\ge1$, and for $\mathbf{z}\in\Xi_{\nu_1,\nu_2,\nu_3,\varphi}$. Hence,

$$F_{i(1)}^{(n)}(\mathbf{z})\neq0\quad\text{for}\quad i(1)\in\mathfrak{I}_1,\ n\ge1,\quad\text{and}\quad\text{for}\quad \mathbf{z}\in\Xi_{\nu_1,\nu_2,\nu_3,\varphi},$$

and, therefore, due to the arbitrariness of φ and for $\mathbf{z}\in\Xi_{\nu_1,\nu_2,\nu_3}$. Thus, each approximants of the branched continued fraction (1.1) is a function holomorphic in the domain (3.20).

Again, let φ be an arbitrary real in $(-\pi/4, \pi/4)$ such that $\operatorname{tg}(\varphi) \le \sqrt{\kappa}$ and let $\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}$ is an arbitrary compact subset of the domain (3.21). Then there exists an open ball with center at the origin and radius ρ containing $\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}$. Using (3.22), from (2.8) for any $\mathbf{z} \in \Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}$ we have

$$|f_n^{(i_0)}(\mathbf{z})| \le 1 + \frac{\alpha}{2\beta}\delta_{i_0}^3 + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \frac{|u_{i(1)}(\mathbf{z})|}{\mu_{i(1)}\cos(\varphi)}$$
$$= M^{(i_0)}(\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}) \quad \text{for} \quad n \ge 1,$$

where

$$M^{(i_0)}(\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}) = \begin{cases} 1 + \dfrac{2(\alpha+1)\rho}{\beta\mu_2\cos(\varphi)} + \dfrac{\rho}{\beta\mu_3\cos(\varphi)}, & \text{if } i_0 = 1, \\ 1 + \dfrac{(2\beta-\alpha)(\alpha+1)\rho}{\beta(\beta+1)\mu_2\cos(\varphi)} + \dfrac{(\beta-\alpha)\rho}{\beta(\beta+1)\mu_3\cos(\varphi)}, & \text{if } i_0 = 2, \\ 1 + \dfrac{\alpha}{2\beta} + \dfrac{\alpha}{2\beta\mu_1\cos(\varphi)} + \dfrac{\alpha\rho}{2\beta(\beta+1)\mu_2\cos(\varphi)}, & \text{if } i_0 = 3, \end{cases}$$

that is, the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ is uniformly bounded on $\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi}$, and, at the same time, uniformly bounded on every compact subset of the domain (3.21).

Now, let $\Upsilon_{\nu_1,\nu_2,\nu_3}$ is an arbitrary compact subset of the domain (3.20). Let us cover $\Upsilon_{\nu_1,\nu_2,\nu_3}$ with domains of form $\Xi_{\nu_1,\nu_2,\nu_3,\varphi}$. From this cover we choose the finite subcover

$$\Xi_{\nu_1,\nu_2,\nu_3,\varphi^{(1)}},\ \Xi_{\nu_1,\nu_2,\nu_3,\varphi^{(2)}},\ \ldots,\ \Xi_{\nu_1,\nu_2,\nu_3,\varphi^{(k)}}.$$

We set

$$M^{(i_0)}(\Upsilon_{\nu_1,\nu_2,\nu_3}) = \max_{1\le r\le k} M^{(i_0)}(\Upsilon_{\nu_1,\nu_2,\nu_3,\varphi^{(r)}}).$$

Then for any $\mathbf{z} \in \Upsilon_{\nu_1,\nu_2,\nu_3}$ we have

$$|f_n^{(i_0)}(\mathbf{z})| \le M^{(i_0)}(\Upsilon_{\nu_1,\nu_2,\nu_3}) \quad \text{for} \quad n \ge 1,$$

that is, the sequence $\{f_n^{(i_0)}(\mathbf{z})\}$ is uniformly bounded on $\Upsilon_{\nu_1,\nu_2,\nu_3}$, and, hence, it is uniformly bounded on every compact subset of the domain (3.20).

Next, let

$$\tau = \min\left\{l_1,\ l_2,\ \nu_2\cos^2(\varphi)\right\}.$$

Then, according to Theorem 2.1, the branched continued fraction (1.1) converges in the domain

$$\Delta_\chi = \{\mathbf{z} \in \mathbb{R}^2 :\ -\tau < -\chi < \operatorname{Re}(z_1) < 0,\ 0 < \operatorname{Re}(z_2) < \chi < \tau\}.$$

It is clear that Δ_χ is contained in the domain (3.20) for each $0 < \chi < \tau$, in particular, $\Delta_{\tau/2} \subset \Xi_{\nu_1,\nu_2,\nu_3}$. Therefore, (A) follows by Theorem 3 [3].

To prove (B), we use the PC method (see, [2, 3]). Let i_0 be an arbitrary index in $\mathfrak{I}_0$. We set

$$R_{i(n)}^{(n)}(\mathbf{z}) = \frac{\mathrm{H}_6(\alpha+n-\sum_{r=0}^{n-1}\delta_{i_r}^3, \beta+n-\sum_{r=0}^{n-1}\delta_{i_r}^1; \mathbf{z})}{\mathrm{H}_6(\alpha+n+1-\sum_{r=0}^{n}\delta_{i_r}^3, \beta+n+1-\sum_{r=0}^{n}\delta_{i_r}^1; \mathbf{z})},\ i(n) \in \mathfrak{I},\ n \ge 1, \tag{3.23}$$

and

$$R_{i(k)}^{(n)}(\mathbf{z}) = v_{i(k)} + \sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]} \cfrac{u_{i(k+1)}(\mathbf{z})}{v_{i(k+1)} + \ddots + \sum\limits_{i_n=2-[(i_{n-1}-1)/2]}^{3-[(i_{n-1}-1)/2]} \cfrac{u_{i(n)}(\mathbf{z})}{R_{i(n)}^{(n)}(\mathbf{z})}},$$

where $i(k)\in\mathfrak{I}$, $1\le k\le n-1$, $n\ge 2$. Then it is clear that

$$R_{i(k)}^{(n)}(\mathbf{z})=v_{i(k)}+\sum_{i_{k+1}=2-[(i_k-1)/2]}^{3-[(i_k-1)/2]}\frac{u_{i(k+1)}(\mathbf{z})}{R_{i(k+1)}^{(n)}(\mathbf{z})}, \tag{3.24}$$

where $i(k)\in\mathfrak{I}$, $1\le k\le n-1$, $n\ge 2$. It follows that for $n\ge 1$

$$\begin{aligned}&\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}\\ =&1-\frac{\alpha}{2\beta}\delta_{i_0}^3+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}+\ddots+\sum_{i_2=2-[(i_{n+1}-1)/2]}^{3-[(i_{n+1}-1)/2]}\frac{u_{i(n+1)}(\mathbf{z})}{R_{i(n+1)}^{(n+1)}(\mathbf{z})}}\\ =&1-\frac{\alpha}{2\beta}\delta_{i_0}^3+\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\frac{u_{i(1)}(\mathbf{z})}{R_{i(1)}^{(n+1)}(\mathbf{z})}.\end{aligned}$$

Since $F_{i(k)}^{(n)}(\mathbf{0})=1$ and $R_{i(k)}^{(n)}(\mathbf{0})=1$ for any $i(k)\in\mathfrak{I}$, $1\le k\le n$, $n\ge 1$, then there exist $\Lambda(1/F_{i(k)}^{(n)})$ and $\Lambda(1/R_{i(k)}^{(n)})$ (here, $\Lambda(\cdot)$ is the Taylor expansion of a function holomorphic in some neighborhood of the origin). Moreover, it is clear that $F_{i(k)}^{(n)}(\mathbf{z})\not\equiv 0$ and $R_{i(k)}^{(n)}(\mathbf{z})\not\equiv 0$ for all indices. Using (2.6), (2.8), (3.23), and (3.24) from (2.9) for $n\ge 1$, we have

$$\begin{aligned}&\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}-f_n^{(i_0)}(\mathbf{z})\\ =&(-1)^n\sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]}\ldots\sum_{i_{n+1}=2-[(i_n-1)/2]}^{3-[(i_n-1)/2]}\frac{\prod_{r=1}^{n+1}u_{i(r)}(\mathbf{z})}{\prod_{r=1}^{n+1}R_{i(r)}^{(n+1)}(\mathbf{z})\prod_{r=1}^{n}F_{i(r)}^{(n)}(\mathbf{z})}.\end{aligned}$$

From this formula in a neighborhood of origin for any $n\ge 1$, we have

$$\Lambda\left(\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}\right)-\Lambda(f_n^{(i_0)})=\sum_{\substack{k+l\ge[n+\delta_{i_0}^1+\delta_{i_0}^2]/2\\ k\ge 0,\ l\ge 0}}c_{k,l}^{(n)}z_1^kz_2^l,$$

where $c_{k,l}^{(n)}$, $k\ge 0$, $l\ge 0$, $k+l\ge[n+\delta_{i_0}^1+\delta_{i_0}^2]/2$, are some coefficients. It follows that

$$\begin{aligned}\nu_n&=\lambda\left(\Lambda\left(\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}\right)-\Lambda(f_n^{(i_0)})\right)\\ &=[n+\delta_{i_0}^1+\delta_{i_0}^2]/2\end{aligned}$$

(here, $\lambda(\cdot)$ is a function defined on the set of all formal double power series $L(\mathbf{z})$ at the origin as follows: if $L(\mathbf{z})\equiv 0$ then $\lambda(L)=\infty$; if $L(\mathbf{z})\not\equiv 0$ then $\lambda(L)=k$, where k is the smallest degree of homogeneous terms for which at least one coefficient is different from zero) tends monotonically to ∞ as $n\to\infty$.

Therefore, the branched continued fraction (1.1) corresponds at the origin to

$$\Lambda\left(\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta_{i_0}^1+\delta_{i_0}^2,\beta+\delta_{i_0}^2+\delta_{i_0}^3;\mathbf{z})}\right).$$

Let Θ be a neighborhood of the origin, which is contained in the domain (3.20), and in which

$$\Lambda\left(\frac{\mathrm{H}_6(\alpha,\beta;\mathbf{z})}{\mathrm{H}_6(\alpha+\delta^1_{i_0}+\delta^2_{i_0},\beta+\delta^2_{i_0}+\delta^3_{i_0};\mathbf{z})}\right)=\sum_{k,l=0}^{\infty}c_{k,l}z_1^kz_2^l. \tag{3.25}$$

According to (A) and Weierstrass's theorem (see, [32, p. 288]) for arbitrary $k+l$, $k\geq 0$, $l\geq 0$, we have

$$\frac{\partial^{k+l}f_n^{(i_0)}(\mathbf{z})}{\partial z_1^k\partial z_2^l}\to\frac{\partial^{k+l}f^{(i_0)}(\mathbf{z})}{\partial z_1^k\partial z_2^l}\quad \text{as}\quad n\to\infty$$

on each compact subset of (3.20), and to the above proven, the expansion of each approximant $f_n^{(i_0)}(\mathbf{z})$, $n\geq 1$, into formal double power series and (3.25) agree for all homogeneous terms up to and including degree $([n+\delta^1_{i_0}+\delta^2_{i_0}]/2-1)$. Then for any $k+l$, $k\geq 0$, $l\geq 0$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\partial^{k+l}f_n^{(i_0)}}{\partial z_1^k\partial z_2^l}(\mathbf{0})\right)&=\frac{\partial^{k+l}f^{(i_0)}}{\partial z_1^k\partial z_2^l}(\mathbf{0})\\&=k!l!c_{k,l}.\end{aligned}$$

For all $\mathbf{z}\in\Theta$, it follows that

$$\begin{aligned}f^{(i_0)}(\mathbf{z})&=\sum_{k,l=0}^{\infty}\frac{1}{k!l!}\left(\frac{\partial^{k+l}f^{(i_0)}}{\partial z_1^k\partial z_2^l}(\mathbf{0})\right)z_1^kz_2^l\\&=\sum_{k,l=0}^{\infty}c_{k,l}z_1^kz_2^l.\end{aligned}$$

Finally, by the principle of analytic continuation (see, [33, p. 53]) follows (B). □

Note that if

$$\beta=2,\ \nu_1=\nu_2=\nu_3=\frac{1}{20},\ \mu_1=\mu_2=\mu_3=\frac{1}{5},\ \kappa=\frac{13}{150},$$

then it is clear that the inequalities (3.19) hold.

Setting $\alpha=0$ and $i_0=1$ (or $i_0=2$ and replacing β by $\beta-1$) in Theorem 3.2, we obtain the following result:

Corollary 3.1. *Let β be real constant such that $\beta\geq 2$, and $\nu_1,\nu_2,\nu_3,\mu_1,\mu_2,\mu_3$ be positive numbers satisfying (3.19), where $0\leq\kappa<1$. Then for $i_0=1$ (or $i_0=2$):*

(**A**) *The branched continued fraction*

$$\cfrac{1}{1+\sum\limits_{i_1=2}^{3}\cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}+\sum\limits_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]}\cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}+\ddots+\sum\limits_{i_k=2-[(i_{k-1}-1)/2]}^{3-[(i_{k-1}-1)/2]}\cfrac{u_{i(k)}(\mathbf{z})}{v_{i(k)}+\ddots}}}},$$

where for $i(1)\in\mathfrak{I}_1$

$$u_{i(1)}(\mathbf{z})=\begin{cases}-\dfrac{2}{\beta}z_1, & \text{if } i_1=2,\\ -\dfrac{z_2}{\beta}, & \text{if } i_1=3,\end{cases}$$

for $i(k+1) \in \mathfrak{I}_{k+1}$, $k \geq 1$,

$$u_{i(k+1)}(\mathbf{z}) = \begin{cases} -\dfrac{2(k+1-\sum_{r=1}^{k-1}\delta_{i_r}^3)}{\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1} z_1, \ \textit{if } i_k = 1,\ i_{k+1} = 2, \\ -\dfrac{z_2}{\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1}, \ \textit{if } i_k = 1,\ i_{k+1} = 3, \\ \dfrac{(2(1-\beta)-k-\sum_{r=1}^{k-1}(\delta_{i_r}^3-2\delta_{i_r}^1))(k-\sum_{r=1}^{k-1}\delta_{i_r}^3+1)}{(\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1)(\beta+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)} z_1, \ \textit{if } i_k = 2,\ i_{k+1} = 2, \\ -\dfrac{\beta-1+\sum_{r=1}^{k-1}(\delta_{i_r}^3-\delta_{i_r}^1)}{(\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1)(\beta+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)} z_2, \ \textit{if } i_k = 2,\ i_{k+1} = 3, \\ \dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1)}, \ \textit{if } i_k = 3,\ i_{k+1} = 1, \\ \dfrac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1)(c+k-\sum_{r=1}^{k-1}\delta_{i_r}^1)} z_2, \ \textit{if } i_k = 3,\ i_{k+1} = 2, \end{cases}$$

and for $i(k) \in \mathfrak{I}_k$, $k \geq 1$,

$$v_{i(k)} = 1 - \frac{k-\sum_{r=1}^{k-1}\delta_{i_r}^3}{2(\beta+k-1-\sum_{r=1}^{k-1}\delta_{i_r}^1)}\delta_{i_k}^3,$$

converges uniformly on every compact subset of (3.20) to the function $f^{(i_0)}(\mathbf{z})$ *holomorphic in (3.20).*

(**B**) *The function* $f^{(i_0)}(\mathbf{z})$ *is an analytic continuation of* $\mathrm{H}_6(1, c; \mathbf{z})$ *in the domain (3.20).*

Note that Theorem 3.2 and Corollary 3.1 are generalizations of Theorem 3.3 and Corollary 3.1 in [5], respectively.

4. Conclusions

In the paper, a new domain of analytical expansion of the ratios of Horn's confluent functions H_6 due to their branched continued fraction expansions is obtained. These expansions can be used as an efficient approximation tool for approximating special functions and solutions of differential equations represented by the Horn's confluent functions H_6. Recent studies of branched continued fractions (see, [9, 10, 15, 22]) open up good prospects for establishing new domains of analytical continuation of hypergeometric functions and conducting analysis of truncation errors and computational stability of their branched continued fraction expansions. Our further investigation will be devoted to the development of the approach proposed in [22] to the study of numerical stability for the above-mentioned expansions.

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