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Source of semiprimeness of $*$-prime rings

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Abstract — This study constructs a structure S_R^* that had never been studied before and obtained new results by defining a subset S_R^* of R as $S_R^* = \{a \in R |\, aRa = aRa^* = (0)\}$ where $*$ is an involution and it is called as the source of $*$-semiprimeness of R. Moreover, it investigates some properties of the subset S_R^* in any ring R. Additionally, the relation between the prime radical, which provides the opportunity to work on prime rings, has been studied in many ways, and the set $\mathcal{S}_R^\sigma$, the motivation of this study, is provided. Furthermore, it is proved that $\mathcal{S}_R^\sigma = \{0\}$ in the case where the ring R is a reduced (σ-semiprime) ring and $f(\mathcal{S}_R^\sigma) = \mathcal{S}_{f(R)}^\sigma$ under certain conditions for a ring homomorphism f. Besides, it is presented that for the idempotent element e, the inclusion $e\mathcal{S}_R^\sigma e \subseteq \mathcal{S}_{eRe}^\sigma$ is provided, and for the right ideal (ideal) I of the ring R, $\mathcal{S}_R^\sigma(I)$ is a left semigroup ideal (semigroup ideal) of the multiplicative semigroup R. In addition, it is analyzed that the set $\mathcal{S}_R^\sigma$ is contained by the intersection of all semiprime ideals of the ring R.

Keywords: Involution, $*$-prime ring, $*$-semiprime ring, source of semiprimeness

Subject Classification (2020): 16N60, 16W10

1. Introduction

An additive mapping $x \to x^*$ on a ring R said to be an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$ [1]. The $*$-ring is a ring with a $*$-involution. A ring with an involution is said to be $*$-prime (resp. $*$-semiprime) if $xRy = x\ Ry^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$ (resp. $xRx = x^*Rx = 0$ implies that $x = 0$). Generally, we know that every prime ring with an involution is $*$ -prime, but the converse need not hold. An example analyzed by Oukhtite and Salhi [2], R is a prime ring and $S = R \times R^o$, where R^o is the opposite ring of R. They show that $*(x, y) = (y, x)$ is involution $*$ on S and S is $*$-prime, but not prime, therein. Their basic work has become a clue to study $*$-prime rings compose an overall class of prime rings. This work ignited the fire for the study of $*$-prime rings over time. Henceforth, involution $*$ will be denoted by σ. Let I be an ideal of ring R. If $\sigma(I) \subseteq I$, then I is said to be a σ-ideal of R [3]. In [4], it is clarified that an ideal I of R may not be a σ-ideal with the following example. Let $R = \mathbb{Z} \times \mathbb{Z}$ and $\sigma : R \to R$ defined by $\sigma(a, b) = (b, a)$, for all $a, b \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a σ-ideal of R since $\sigma(I) = \{0\} \times Z = I$.

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In [5], McCoy extensively devoted that ring R is isomorphic to a subdirect sum of prime rings iff $\beta(R) = (0)$, such that $\beta(R)$ is a prime radical of R. Recently, several studies have been conducted on prime rings and prime radicals.

Since each prime ring is a semiprime ring, semiprime rings play a crucial role in more general results. Inspired by the search work in this field over the last decades, multiple authors have demonstrated commutativity for prime and semiprime rings [1,6–8]. Posner [9] presented that if a prime ring has a nontrivial derivation that is centralizing on the entire ring, then the ring is commutative. In [10], the same result is proved for a prime ring with a nontrivial centralizing automorphism. Some authors have generalized these results by considering only mappings assumed to be centralized on a convenient ideal of the ring. Inspired by research studies carried out in this field, [11] and [12] introduced the source of semiprimeness of the nonempty subset A in R, where $S_R(A) = \{a \in R | aAa = (0)\}$. S_R is written in place of $S_R(R)$ for a ring R.

In the present paper, section 2 generalizes some results provided in literature. Afterward, section 3 discusses the need for further research.

2. Results

This section generalizes some of the literature results using the involution σ and the source of the semiprimeness in the rings.

Definition 2.1. Let R be a ring, $\sigma : R \to R$ be an involution, and A be a nonempty subset of R. The set $\mathcal{S}_R^\sigma(A) = \{ a \in R |\, aAa = aA\sigma(a) = (0)\}$ is called *the source of σ-semiprimeness* of A in R and $\mathcal{S}_R^\sigma$ is written in place of $\mathcal{S}_R^\sigma(R)$ for a ring R. It is obvious that $\mathcal{S}_R^\sigma \subseteq \mathcal{S}_R$.

Lemma 2.2. Let R be a ring. Then, $a \in \mathcal{S}_R$ if and only if $\sigma(a) \in \mathcal{S}_R$.

Proof. Let $a \in \mathcal{S}_R$. Then, $aRa = (0)$. Using the additivity of σ, $\sigma(aRa) = \sigma(0) = 0$. Thus, $\sigma(a)\sigma(R)\sigma(a) = (0)$. Since σ is bijective, $\sigma(a) \in \mathcal{S}_R$. On the other hand, it is observable since σ is injective. □

The following proposition is easily obtained from the definitions of the $\mathcal{S}_R^\sigma$ and σ-semiprime ring.

Proposition 2.3. Let R be a σ-semiprime ring. Then, $\mathcal{S}_R^\sigma = \{0\}$.

Lemma 2.4. Let R be a ring and A be a subring of R. Then, $\mathcal{S}_A^\sigma = \mathcal{S}_R^\sigma(A) \cap A$.

Proof. Let $x \in \mathcal{S}_A^\sigma$. In this case, $x \in A$ and $xAx = xA\sigma(x) = (0)$. Since $x \in R$, $x \in \mathcal{S}_R^\sigma(A)$. Therefore, $x \in \mathcal{S}_R^\sigma(A) \cap A$. Conversely, assume that $y \in \mathcal{S}_R^\sigma(A) \cap A$. Thus, $y \in A$ and $yAy = yA\sigma(y) = (0)$. Hence, $y \in \mathcal{S}_A^\sigma$ is provided. □

Lemma 2.5. Let R be a ring and A and B be nonempty subsets of R. If $A \subseteq B$, then $\mathcal{S}_R^\sigma(B) \subseteq \mathcal{S}_R^\sigma(A)$.

Proof. Let $x \in \mathcal{S}_R^\sigma(B)$. Then, $xBx = xB\sigma(x) = (0)$. Hence, $xAx \subseteq xBx = (0)$ and $xA\sigma(x) \subseteq xB\sigma(x) = (0)$. Consequently, $x \in \mathcal{S}_R^\sigma(A)$ is provided. □

Theorem 2.6. Let R and T be two rings, $A \subseteq R$ and $B \subseteq T$, and $\sigma : T \to T$ and $\beta : R \to R$ be two involutions. In this case, the mapping

$$\alpha : R \times T \to R \times T,\ \ \alpha(a, b) = (\beta(a), \sigma(b))$$

is an involution. Furthermore, the equation $\mathcal{S}_{R\times T}^\alpha(A \times B) = \mathcal{S}_R^\beta A) \times \mathcal{S}_T^\sigma(B)$ is provided.

Proof. Let R and T be two rings, $A \subseteq R$ and $B \subseteq T$, and $\sigma : T \to T$ and $\beta : R \to R$ be two involutions. In this case, *i.*

$$
\begin{aligned}
\alpha((a,b)+(c,d)) &= \alpha(a+c,b+d) \\
&= (\beta(a+c),\sigma(b+d)) \\
&= (\beta(a)+\beta(c),\sigma(b)+\sigma(d)) \\
&= (\beta(a),\sigma(b))+(\beta(c),\sigma(d)) \\
&= \alpha(a,b)+\alpha(c,d)
\end{aligned}
$$

ii.

$$
\begin{aligned}
\alpha((a,b)(c,d)) &= \alpha(ac,bd) \\
&= (\beta(ac),\sigma(bd)) \\
&= (\beta(c)\beta(a),\sigma(d)\sigma(b)) \\
&= (\beta(c),\sigma(d))(\beta(a),\sigma(b)) \\
&= \alpha(c,d)\alpha(a,b)
\end{aligned}
$$

iii.

$$
\begin{aligned}
\alpha^2(a,b) &= \alpha(\alpha(a,b)) \\
&= \alpha(\beta(a),\sigma(b)) \\
&= (\beta^2(a),\sigma^2(b)) \\
&= \alpha(a,b)
\end{aligned}
$$

Therefore, the mapping α is an involution.

Moreover, if $(x,y) \in \mathcal{S}^{\alpha}_{R\times T}(A\times B)$, then $(x,y)(A\times B)(x,y) = (x,y)(A\times B)(\beta(x),\sigma(y)) = (0,0)$. From here, $xAx = yBy = xA\beta(x) = yB\sigma(y) = 0$. Thus, $x \in \mathcal{S}^{\beta}_{R}(A)$ and $y \in \mathcal{S}^{\sigma}_{T}(B)$. This implies that, $(x,y) \in \mathcal{S}^{\beta}_{R}(A)\times\mathcal{S}^{\sigma}_{T}(B)$. Conversely, suppose that $(x,y) \in \mathcal{S}^{\beta}_{R}(A)\times\mathcal{S}^{\sigma}_{T}(B)$. Then, $x \in \mathcal{S}^{\beta}_{R}(A)$ and $y \in \mathcal{S}^{\sigma}_{T}(B)$. This requires that, $xAx = yBy = xA\beta(x) = yB\sigma(y) = 0$. Subsequently, $(x,y)(A\times B)(x,y) = (x,y)(A\times B)(\beta(x),\sigma(y)) = (0,0)$ is obtained. Therefore, $(x,y) \in \mathcal{S}^{\alpha}_{R\times T}(A\times B)$. $\square$

Corollary 2.7. According to Remark 2.6, if $R = T$ and $\beta = \sigma$ is taken, then $\mathcal{S}^{\alpha}_{R\times R}(A\times B) = \mathcal{S}^{\sigma}_{R}(A)\times\mathcal{S}^{\sigma}_{R}(B)$ is provided.

Theorem 2.8. Let R be a ring. Then, the following conditions hold:

i. If I is a right ideal of R, then $\mathcal{S}^{\sigma}_{R}(I)$ is a semigroup left ideal of multiplicative semigroup R.

ii. If I is an ideal of R, then $\mathcal{S}^{\sigma}_{R}(I)$ is a semigroup ideal of multiplicative semigroup R.

Proof. *i.* Let $a \in \mathcal{S}^{\sigma}_{R}(I)$ and $r \in R$. Thus, $aIa = aI\sigma(a) = (0)$. Since I is a right ideal, $raIra \subseteq r(aIa) = (0)$ and $raI\sigma(ra) = raI\sigma(a)\sigma(r) = (0)$. Hence, $ra \in \mathcal{S}^{\sigma}_{R}(I)$.

ii. Since I is an ideal, in view of (i), $\mathcal{S}^{\sigma}_{R}(I)$ is a semigroup left ideal. Therefore, it remains to prove that $\mathcal{S}^{\sigma}_{R}(I)$ is a semigroup right ideal. Let $a \in \mathcal{S}^{\sigma}_{R}(I)$ and $r \in R$. Hence, $aIa = aI\sigma(a) = (0)$. Since I is an ideal,

$$
\begin{gathered}
arIar \subseteq aIar = (0) \\
arI\sigma(ar) = arI\sigma(r)\sigma(a) \subseteq aI\sigma(a) = (0)
\end{gathered}
$$

Thus, $ar \in \mathcal{S}^{\sigma}_{R}(I)$. $\square$

Lemma 2.9. Let R be a ring. If Q is a semiprime ideal of R, then $\mathcal{S}_R^\sigma \subseteq Q$. Consequently, if $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a family of semiprime ideals of R, then $\mathcal{S}_R^\sigma \subseteq \bigcap_{\lambda\in\Lambda} Q_\lambda$.

Proof. Let Q be a semiprime ideal of R and $a \in \mathcal{S}_R^\sigma$. Then, $aRa = aR\sigma(a) = (0) \subseteq Q$. Since Q is a semiprime ideal, $a \in Q$. Moreover if $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a family of semiprime ideals of R then $a \in Q_\lambda$ for all $\lambda \in \Lambda$. □

It is an immediate corollary that $\mathcal{S}_R^\sigma$ is contained in the prime radical $\beta(R)$ of R.

Proposition 2.10. If $e \in R$ is an idempotent element, then $e\mathcal{S}_R^\sigma e \subseteq \mathcal{S}_{eRe}^\sigma$.

Proof. Let $a \in \mathcal{S}_R^\sigma$ and assume that $x = eae \in e\mathcal{S}_R^\sigma e$. Then, $aRa = aR\sigma(a) = (0)$. From here, $x(eRe)x = eaeReae \subseteq eaRae = (0)$ and $x(eRe)\sigma(x) = eaeRe\sigma(e)\sigma(a)\sigma(e) \subseteq eaR\sigma(a)\sigma(e) = (0)$ holds. Consequently, $x \in \mathcal{S}_{eRe}^\sigma$. □

However, the inclusion $e\mathcal{S}_R^\sigma e \subseteq e\mathcal{S}_R^\sigma(eRe)e$ follows from Lemma 2.5**.**

Remark 2.11. Let R be a ring, $T_n(R)$ be a ring of all $n \times n$ diagonal matrices over R, and $\sigma : R \to R$ be an involution. Then,

$$\gamma : T_n(R) \to T_n(R), [\gamma(A)]_{ij} = \begin{cases} \sigma(a_{ij}), & i = j \\ 0, & i \neq j \end{cases}$$

is an involution.

Proposition 2.12. Let $\sigma : R \to R$ be an involution and γ be the involution defined in the Remark above. Then,

i. $\mathcal{S}_{T_n(R)}^\gamma \subseteq T_n(\mathcal{S}_R^\sigma)$

ii. If $\mathcal{S}_R^\sigma$ is a principal ideal of R, then $\mathcal{S}_{T_n(R)}^\gamma = T_n(\mathcal{S}_R^\sigma)$.

Proof. *i.* Here,

$$\mathcal{S}_{T_n(R)}^\gamma = \{\, A \in T_n(R) \mid AT_n(R)A = AT_n(R)\gamma(A) = [0]\}$$

and

$$\begin{aligned} T_n(\mathcal{S}_R^\sigma) &= \{\, A = [a_{ij}] \in T_n(R) \mid a_{ij} \in \mathcal{S}_R^\sigma,\ i,j = 1,\dots,n\} \\ &= \{\, A = [a_{ij}] \mid a_{ij}Ra_{ij} = a_{ij}R\sigma(a_{ij}) = (0),\ i,j = 1,\dots,n\} \end{aligned}$$

Assume that $A \in \mathcal{S}_{T_n(R)}^\gamma$. Let $E_{ij}(x) = xE_{ij}$ be scalar matrices, for any $x \in R$. In this case, $AE_{ij}(x)A = AE_{ij}(x)\gamma(A) = [0]$. Respectively, $AE_{ij}(x)A = [0]$ and $AE_{ij}(x)\gamma(A) = [0]$ gives that $a_{ij}Ra_{ij} = (0)$ and $a_{ij}R\sigma(a_{ij}) = (0)$. Hence, $\mathcal{S}_{T_n(R)}^\gamma \subseteq T_n(\mathcal{S}_R^\sigma)$.

ii. Assume that $\mathcal{S}_R^\sigma = (u)$ is a principal ideal generated by $u \in R$. Let $A = [a_{ij}] \in T_n(\mathcal{S}_R^\sigma)$. It is written that $[ABA]_{ii} = a_{ii}b_{ii}a_{ii}$ and $[AB\gamma(A)]_{ii} = a_{ii}b_{ii}\sigma(a_{ii})$, for any $B = [b_{ij}] \in T_n(R)$ where $1 \leq i \leq n$. Adopting $a_{ii} \in \mathcal{S}_R^\sigma = (u)$ and $u \in \mathcal{S}_R^\sigma$, $[ABA]_{ii} = (0)$ and $[AB\gamma(A)]_{ii} = (0)$, for all $1 \leq i \leq n$. Therefore, $ABA = [0]$ and $AB\gamma(A) = [0]$. Thus, $A \in \mathcal{S}_{T_n(R)}^\gamma$. Consequently, $T_n(\mathcal{S}_R^\sigma) \subseteq \mathcal{S}_{T_n(R)}^\gamma$. In view of (i), the equality $\mathcal{S}_{T_n(R)}^\gamma = T_n(\mathcal{S}_R^\sigma)$ holds.

□

Theorem 2.13. Let R and T be two rings, $f : R \to T$ be a ring homomorphism, and $\sigma f = f\sigma$. Then, $f(\mathcal{S}_R^\sigma) \subseteq \mathcal{S}_{f(R)}^\sigma$. Moreover, if f is injective, then $f(\mathcal{S}_R^\sigma) = \mathcal{S}_{f(R)}^\sigma$.

Proof. Let $x \in f(\mathcal{S}_R^\sigma)$. In this case, there is an element $a \in \mathcal{S}_R^\sigma$ such that $x = f(a)$. Thus, $aRa = aR\sigma(a) = (0)$. Hence, $xf(R)x = f(aRa) = f((0)) = (0)$ and $xf(R)\sigma(x) = f(aR\sigma(a)) = f((0)) = (0)$. Therefore, $x \in \mathcal{S}_{f(R)}^\sigma$.

Let f be injective. Assume that $y \in \mathcal{S}^{\sigma}_{f(R)}$. Then, $yf(R)y = yf(R)\sigma(y) = (0)$ where $y = f(a)$, for $a \in R$. This means that $(0) = yf(R)y = f(aRa)$ and $(0) = yf(R)\sigma(y) = f(aR\sigma(a))$. Since f is injective, $aRa = aR\sigma(a) = (0)$. As a result, $a \in \mathcal{S}^{\sigma}_{R}$. This means that $y = f(a) \in f(\mathcal{S}^{\sigma}_{R})$. $\square$

Lemma 2.14. Let R be a ring. If $a \in R$ is neither a right zero divisor nor a left zero divisor, then $a \in R - \mathcal{S}^{\sigma}_{R}$.

Proof. If $a \in R$ is neither a right zero divisor nor a left zero divisor, then $a \in R - \mathcal{S}_{R}$. Given the inclusion $\mathcal{S}^{\sigma}_{R} \subseteq \mathcal{S}_{R}$, $R - \mathcal{S}_{R} \subseteq R - \mathcal{S}^{\sigma}_{R}$ is obtained. Therefore, $a \in R - \mathcal{S}^{\sigma}_{R}$. $\square$

Lemma 2.15. If R is a unitary and commutative ring, then

$$\mathcal{S}^{\sigma}_{R} = \left\{ a \in R \mid \ a^2 = a\sigma(a) = (0) \right\}$$

Proof. Let R be unity and commutative ring. Consider the set $K = \{a \in R |\, a^2 = a\sigma(a) = 0\}$. Assume that $a \in \mathcal{S}^{\sigma}_{R}$. Then, $aRa = aR\sigma(a) = (0)$. Adopting the fact that R is unity, $a^2 = a\sigma(a) = 0$. Thus, $a \in K$. To prove the converse, let $b \in K$. Then, $b^2 = b\sigma(b) = 0$. Therefore, $b^2x = b\sigma(b)x = 0$, for all $x \in R$. Since R is commutative, $bxb = bx\sigma(b) = 0$. Hence, $bRb = bR\sigma(b) = (0)$. Consequently, $b \in \mathcal{S}^{\sigma}_{R}$ yields that $K = \mathcal{S}^{\sigma}_{R}$. $\square$

Proposition 2.16. If R is a reduced ring, then $\mathcal{S}^{\sigma}_{R} = \{0\}$.

Proof. If R is a reduced ring, then $\mathcal{S}_{R} = \{0\}$ (see [11], [12]). Using the inclusion $\mathcal{S}^{\sigma}_{R} \subseteq \mathcal{S}_{R}$, $\mathcal{S}^{\sigma}_{R} = \{0\}$. $\square$

3. Conclusion

In this study, the set $\mathcal{S}^{\sigma}_{R}$ was defined using the involution defined by a ring and the relation between set $\mathcal{S}_{R}$ found in the literature was obtained. Afterward, the relationship between $\mathcal{S}^{\sigma}_{R}$ and the prime radical is investigated. Towards the end of this study, the behavior of the set $\mathcal{S}^{\sigma}_{R}$ under a homomorphism and its characteristic property in a reduced ring are investigated. Many studies exist on $*$-prime rings, $*$-semiprime rings, and their ideals. These studies can be generalized by using the structure of the source of semiprimeness in $*$-prime rings and $*$-semiprime rings. As a result, this set, which provides our motivation, will serve as a reference for other studies in the field.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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References

[1] I. N. Herstein, Rings with involution, Chicago lectures in mathematics, The University of Chicago Press, Chicago, 1976.

[2] L. Oukhtite, S. Salhi, *Centralizing automorphisms and Jordan left derivations on σ-prime rings*, Advances in Algebra 1 (1) (2008) 19–26.

[3] L. Oukhtite, S. Salhi, *On commutativity of σ-prime rings*, Glasnik Matematicki Series III 41 (61) (2006) 57–64.

[4] N. U. Rehman, R. M. Al-Omary, A. Z. Ansari, *On Lie ideals of $*$-prime rings with generalized derivations*, Boletin de la Sociedad Matematica Mexicana 21 (2015) 19–26.

[5] N. H. McCoy, The theory of rings, The Macmillan & Co LTD., New York, 1964.

[6] M. Ashraf, S. Ali, *On left multipliers and commutativity of prime rings*, Demonstratio Mathematica 4 (41) (2008) 764–771.

[7] H. E. Bell, W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canadian Mathematical Bulletin 30 (1) (1987) 92–101.

[8] L. Molnár, *On centralizers of an H*-algebra*, Publicationes Mathematicae Debrecen 46 (1-2) (1995) 89–95.

[9] E. C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society 8 (1957) 1093–1100.

[10] J. Mayne, *Centralizing automorphisms of prime rings*, Canadian Mathematical Bulletin 19 (1976) 113–115.

[11] N. Aydın, Ç. Demir, D. Karalarlıoğlu Camcı, *The source of semiprimeness of rings*, Communications of the Korean Mathematical Society 33 (4) (2018) 1083–1096.

[12] D. Karalarlıoğlu Camcı, *Source of semiprimeness and multiplicative (generalized) derivations in rings*, Doctoral Dissertation, Çanakkale Onsekiz Mart University (2017) Çanakkale.