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A STUDY ON NON-HOMOGENEOUS MULTIPLICATIVE BOUNDARY VALUE PROBLEMS

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Abstract. The paper deals with the non-homogeneous boundary value problems established in the multiplicative calculus. For this problem, the multiplicative meanings of concepts such as adjoint operator, self-adjoint operator, Lagrange identity, Green's formula are obtained. Then, a criterion, called multiplicative Fredholm alternative, is given for the existence of solutions to non-homogeneous multiplicative boundary value problems.

1. Introduction

The concept of the classical calculus was developed independently by Newton and Leibniz in the late 17th century when modern science was born [3, 8]. Subsequent studies, including the creation of the idea of limits, placed these developments on a more solid conceptual basis. The classical calculus is widely used today in engineering, science and social sciences [19]. Since the operations of addition and subtraction are the basis of all concepts defined and all theorems established in classical calculus, this calculus is also called additive calculus or Newtonian calculus.

The roles of the operations used between the elements are important in establishing different calculus. Therefore, the various alternative calculus to the classical calculus have been defined with the help of different arithmetic operations. These calculus are defined as Non-Newtonian calculus by Grossman and Katz [15, 16]. Geometric calculus, bigeometric calculus and anaquadratic calculus can be given as examples of these calculus. The concept of geometric calculus was studied widely and related to multiplication, is called multiplicative calculus [9, 24]. In different areas such as biomedical applications [13], differential and integral equations [2,25–31] , finance [10–12], geometry [1,17,18], machine learning [7], numerical applications [21,22,32,35], social sciences [5] and spectral theory [14,23,33,34], significant contributions have been made by multiplicative calculus [4, 6, 24].

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In the literature, some calculus are called multiplicative calculus because the elements in the domain and/or range sets are defined by exponential function arithmetic. Moreover, multiplicative derivatives and integrals are defined in this calculus. Due to the existence of various application areas mentioned above and the existence of provable properties, the multiplicative calculus shown with the help of the multiplicative derivative, whose definition is given in the next section, is preferred.

2. Preliminaries

Definition 2.1. [4, 24] Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$. *The multiplicative derivative* of the function f at x is given by

$$f^*(x) = \lim_{\varepsilon \to 0} \left[\frac{f(x+\varepsilon)}{f(x)} \right]^{\frac{1}{\varepsilon}}, \tag{2.1}$$

if the limit exists and positive.

Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$ be differentiable in usual case. Then, there is a relation between classical and derivatives as follows:

$$f^*(x) = e^{(\ln o f)'(x)}.$$

Repeating this procedure n times, it can be obtained the relation between the $n-$th order classical derivative and $n-$th *derivative as

$$f^{*(n)}(x) = e^{(\ln o f)^{(n)}(x)}.$$

Theorem 2.2. *[4, 24] Suppose that f, g be multiplicative differentiable and h be usual differentiable at x. If $c \in \mathbb{R}^+$ is an arbitrary constant, then the functions cf, fg, $\frac{f}{g}$, f^h, foh and $f+g$ have multiplicative derivatives given by*

- **i.** $(cf)^*(x) = f^*(x)$,
- **ii.** $(fg)^*(x) = f^*(x) g^*(x)$,
- **iii.** $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$,
- **iv.** $\left(f^h\right)^*(x) = f^*(x)^{h(x)} f(x)^{h'(x)}$,
- **v.** $(foh)^*(x) = f^*(h(x))^{h'(x)}$,
- **vi.** $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$.

Definition 2.3. [4, 24] Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$ bounded on $[a, b]$. *The multiplicative integral* of the function f on $[a, b]$ is given by $\int_a^b f(x)^{dx}$.

If f is positive and Riemann integrable on $[a, b]$, then it is also multiplicative integrable on $[a, b]$. Additionally, the following equality is satisfied.

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln o f)(x) dx}.$$

On the contrary,

$$\int_a^b f(x) dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}$$

if the function f is multiplicative integrable on $[a, b]$.

Theorem 2.4. *[4, 24] Suppose that f, g be bounded functions on $[a, b]$. If f, g are *integrable on $[a, b]$, then the functions f^c, fg, $\frac{f}{g}$ have multiplicative integrals on $[a, b]$ given by*

i. $\int_a^b [f(x)^c]^{dx} = \left[\int_a^b f(x)^{dx}\right]^c, c \in \mathbb{R},$

ii. $\int_a^b [f(x)g(x)]^{dx} = \int_a^b f(x)^{dx} \int_a^b g(x)^{dx},$

iii. $\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}},$

iv. $\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \int_c^b f(x)^{dx}, a \leq c \leq b,$

v.

$$\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = f(x)^{g(x)}\Big|_a^b \frac{1}{\int_a^b \left[f(x)^{g'(x)}\right]^{dx}},$$

where, $f(x)^{g(x)}\Big|_a^b = \frac{f(b)^{g(b)}}{f(a)^{g(a)}}$.

This equality (v) is known as *multiplicative integration by parts method.*

Lemma 2.5. *[14, 20]* $L_2^*[a, b] = \left\{f : \int_a^b \left[f(x)^{f(x)}\right]^{dx} < \infty\right\}$ *is an multiplicative inner product space with*

$$<, >_*: L_2^*[a, b] \times L_2^*[a, b] \to \mathbb{R}^+, < f, g >_*= \int_a^b \left[f(x)^{g(x)}\right]^{dx},$$

where $f, g \in L_2^*[a, b]$ are positive functions.

Definition 2.6. [6, 27–30] $n-$th order linear multiplicative differential equation is defined by

$$\left(y^{*(n)}\right)^{a_n(x)} \left(y^{*(n-1)}\right)^{a_{n-1}(x)} \cdots (y^{**})^{a_2(x)} (y^*)^{a_1(x)} y^{a_0(x)} = \phi(x), \tag{2.2}$$

where $\phi(x) > 0$ and $a_k(x)$, $k = 0, 1, 2, \cdots, n-1, n$ are functions of x.

In equation (2.2), when $\phi(x) = 1$, this equation is called homogeneous multiplicative linear differential equation, otherwise it is called non-homogeneous multiplicative linear differential equation.

3. Main Results

In spectral theory, the Lagrange identity is a foundational result that connects a differential operator with its formal adjoint. Through integration, this leads directly to Green's formula. Green's formula helps determine when an operator is self-adjoint by examining the boundary terms. Here, self-adjointness is crucial in spectral theory for several reasons such as all eigenvalues are real, eigenfunctions corresponding to different eigenvalues are orthogonal, and the spectrum is bounded below.

3.1. **Non-Homogeneous Multiplicative Boundary Value Problems.** In this section, non-homogeneous multiplicative boundary value problems are discussed. The multiplicative counterparts of concepts such as the adjoint operator, self-adjoint operator, Lagrange identity and Green's formula are defined for this problem.

Consider the non-homogeneous multiplicative boundary value problem

$$T[y](x) = \{y^{**}\}^{a_0(x)}\{y^*\}^{a_1(x)}y^{a_2(x)} = h(x), \tag{3.1}$$

$$\begin{aligned} B_1[y] &= y(a)^{a_{11}}y^*(a)^{a_{12}}y(b)^{b_{11}}y^*(b)^{b_{12}} = 1, \\ B_2[y] &= y(a)^{a_{21}}y^*(a)^{a_{22}}y(b)^{b_{21}}y^*(b)^{b_{22}} = 1 \end{aligned} \tag{3.2}$$

where the functions $a_i := a_i\,(x)$ for $i = 0, 1, 2$ on the exponentials are continuous real valued functions on the interval $[a, b]$; a_{ij}, b_{ij}, for $i, j = 1, 2$ are real constants and $a_0\,(x) \neq 0$ for all $x \in [a, b]$.

Definition 3.1. (Multiplicative Formal Adjoint Operator) Let the operator T be the multiplicative differential operator defined by the left-hand side of equation (3.1). In other words, let it be

$$T[y] = \{y^{**}\}^{a_0}\{y^*\}^{a_1}y^{a_2}. \tag{3.3}$$

Then, the multiplicative differential operator $\widetilde{T}$ defined in the form

$$\widetilde{T}\,[y] = (y^{a_0})^{**}((y^{a_1})^*)^{-1}y^{a_2} \tag{3.4}$$

is called *the multiplicative formal adjoint operator* of the operator T.

Example 3.2. The multiplicative formal adjoint operator of the multiplicative differential operator $T[y] = \{y^{**}\}\{y^*\}^6y^{10}$ is of the form

$$\widetilde{T}\,[y] = \{y^{**}\}\{y^*\}^{-6}y^{10}$$

for $a_0 = 1, a_1 = 6, a_2 = 10$ according to formula (3.4).

Example 3.3. The multiplicative formal adjoint operator of the multiplicative differential operator $T[y] = \{y^{**}\}^{2x^2}\{y^*\}^{7x}y^{-2}$ is of the form

$$\begin{aligned} \widetilde{T}\,[y] &= (y^{2x^2})^{**}((y^{7x})^*)^{-1}y^{-2} \\ &= [(y^*)^{2x^2}y^{4x}]^*[(y^*)^{7x}y^7]^{-1}y^{-2} \\ &= (y^{**})^{2x^2}(y^*)^xy^{-5} \end{aligned}$$

for $a_0 = 2x^2, a_1 = 7x, a_2 = -2$ according to formula (3.4).

The relationship between the multiplicative differential operator T and its multiplicative formal adjoint operator $\widetilde{T}$ are defined as the multiplicative Lagrange identity.

Theorem 3.4. *Suppose that the multiplicative differential operator T is the operator defined by equation (3.1), and the multiplicative differential operator $\widetilde{T}$ is the* *multiplicative formal adjoint of T. Therefore, the multiplicative Lagrange identity is of the form*

$$\left\{\{T[u]\odot v\}\ominus\left\{u\odot\widetilde{T}\,[v]\right\}\right\}=\frac{d^*}{dx}[P\,(u,v)], \tag{3.5}$$

where,

$$P\,(u,v)=\frac{\{u^*\}^{\ln v^{a_0}}\{u\}^{\ln v^{a_1}}}{\{u\}^{\ln\,(v^{a_0})^*}}. \tag{3.6}$$

Proof. From Lemma 2.5, formulas (3.3) and (3.4), the proof can be easily shown similarly to that in [14]. □

Theorem 3.5. *(Multiplicative Green's Formula) The multiplicative differential operator T is the operator defined by equation (3.1), and the multiplicative differential operator $\widetilde{T}$ is the multiplicative formal adjoint of T. Therefore, the multiplicative Green's formula is of the form*

$$\int_a^b \left\{\left(\{T\,[u]\odot v\}\ominus\left\{u\odot\widetilde{T}\,[v]\right\}\right)(x)\right\}^{dx}=P\,(u,v)\,(x)|_a^b\,, \tag{3.7}$$

where, $P\,(u,v)\,(x)|_a^b=\frac{P\,(u,v)(b)}{P\,(u,v)(a)}$.

Proof. The proof is easily obtained by multiplicative integrating equality (3.5) from a to b similarly to that in [14]. □

Corollary 3.6. From Lemma 2.5, the multiplicative Green's formula can be also expressed in the form

$$\langle T\,[u]\,,v\rangle_*=P\,(u,v)\,(x)|_a^b\left\langle u,\widetilde{T}\,[v]\right\rangle_*. \tag{3.8}$$

Example 3.7. Suppose that T is the multiplicative differential operator defined by equation (3.1), and $\widetilde{T}$ is the multiplicative formal adjoint of T. Therefore,

$$\begin{aligned}\int_a^b \{T\,[u]\odot v\}^{dx}&=\int_a^b\left\{(T\,[u])^{\ln v}\right\}^{dx}=\int_a^b\left\{(\{u^{**}\}^{a_0}\{u^*\}^{a_1}u^{a_2})^{\ln v}\right\}^{dx}\\&=\int_a^b\left\{\{u^{**}\}^{\ln v^{a_0}}\right\}^{dx}\int_a^b\left\{\{u^*\}^{\ln v^{a_1}}\right\}^{dx}\int_a^b\left\{\{u\}^{\ln v^{a_2}}\right\}^{dx}.\end{aligned} \tag{3.9}$$

If multiplicative integration by parts method is applied twice to the first integral and once to the second integral on the right of (3.9), the following integrals are obtained

$$\int_a^b\left\{\{u^{**}\}^{\ln v^{a_0}}\right\}^{dx}=\left.\frac{\{u^*\}^{\ln v^{a_0}}}{\{u\}^{\ln(v^{a_0})^*}}\right|_a^b\int_a^b\left\{\{u\}^{(\ln v^{a_0})^{**}}\right\}^{dx},$$

$$\int_a^b\left\{\{u^*\}^{\ln v^{a_1}}\right\}^{dx}=\{u\}^{\ln v^{a_1}}\Big|_a^b\int_a^b\left\{\{u\}^{-\ln\,(v^{a_1})^*}\right\}^{dx}.$$

If these expressions are substituted into (3.9),

$$\int_a^b \{T[u]\odot v\}^{dx} = \frac{\{u^*\}^{\ln v^{a_0}}\{u\}^{\ln v^{a_1}}}{\{u\}^{\ln (v^{a_0})^*}}\Bigg|_a^b \int_a^b \left\{u^{\ln\left[(v^{a_0})^{**}((v^{a_1})^*)^{-1}v^{a_2}\right]}\right\}^{dx}$$

$$= P\,(u,v)|_a^b \int_a^b u^{\ln \widetilde{T}\,[v]}\,dx = P\,(u,v)|_a^b \int_a^b \left\{u\odot \widetilde{T}\;[v]\right\}^{dx}$$

is found. Consequently, equality (3.8) is obtained.

Now, let T and $\widetilde{T}$ be two multiplicative differential operators defined on $C^{2,*}[a,b]$. Here, $C^{2,*}[a,b]$ is the space of functions defined on the interval $[a,b]$ whose multiplicative derivatives up to the second order are continuous.

Suppose that $D(T), D\left(\widetilde{T}\right)\subset C^{2,*}[a,b]$ and $D(T)$ be the set of functions that belong to $C^{2,*}[a,b]$ and satisfy the boundary conditions given in (3.2). This is the domain of the multiplicative differential operator T. The domain $D\left(\widetilde{T}\right)$ of the multiplicative formal adjoint $\widetilde{T}$ is the set of all functions v for which

$$\langle L[u],v\rangle_* = \left\langle u,\widetilde{T}\;[v]\right\rangle_* \tag{3.10}$$

holds for all $u\in D(T)$.

If the multiplicative Green's formula (3.7) is considered along with (3.10), it is seen that the domain of $\widetilde{T}$ consists of the functions v for which

$$P\,(u,v)(x)|_a^b = 1 \tag{3.11}$$

holds for all $u\in D(L)$.

Definition 3.8. (Multiplicative Adjoint Operator) *The multiplicative adjoint operator* of T is an operator $\widetilde{T}$ with domain $D\left(\widetilde{T}\right)$.

Example 3.9. The multiplicative adjoint operator of $T[y]=\{y^{**}\}\{y^*\}^6 y^{10}$ with a domain

$$D(T)=\left\{y : y\in C^{2,*}[0,\pi]; y(0)=y(\pi)=1\right\}$$

is calculated as follows:

From Example 3.2, the multiplicative formal adjoint operator of T is $\widetilde{T}[y] = \{y^{**}\}\{y^*\}^{-6}y^{10}$. To complete the solution, we need to determine the domain $D\left(\widetilde{T}\right)$ of the operator $\widetilde{T}$.

Let $v\in C^{2,*}[0,\pi]$. Also,

$$P(u,v)(x)|_0^\pi = \frac{\{u^*\}^{\ln v}\{u\}^{\ln v^6}}{\{u\}^{\ln v^*}}(x)\Bigg|_0^\pi = 1$$

must be satisfied for all $u\in D(T)$ and $v\in D\left(\widetilde{T}\right)$.

Considering the domain of the operator T, where $u(0)=u(\pi)=1$, then equality can be easily written as

$$\frac{\{u^*(\pi)\}^{\ln v(\pi)}}{\{u^*(0)\}^{\ln v(0)}} = 1. \tag{3.12}$$

The goal is to find a condition for the function v. Therefore, in order for equality (3.12) to hold for all $u \in D(T)$, it must be that $v(0) = v(\pi) = 1$.

As a result, since v is an arbitrary variable, the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{y : y \in C^{2,*}[0,\pi]\,; y(0) = y(\pi) = 1\right\}.$$

Definition 3.10. (Multiplicative Self-Adjoint Operator) Let $T = \widetilde{T}$ and $D(T) = D\left(\widetilde{T}\right)$, then the operator T is called *the multiplicative self-adjoint operator*.

Definition 3.11. (Multiplicative Adjoint Boundary Value Problem) Let $B[u] = 1$ denote the boundary conditions for the operator T, and $\widetilde{B}\,[u] = 1$ indicate the boundary conditions for the adjoint operator $\widetilde{T}$. Then, the boundary value problem

$$\widetilde{T}\,[u] = 1; \widetilde{B}\,[u] = 1$$

is called *the multiplicative adjoint* of the boundary value problem

$$T[u] = 1; B[u] = 1.$$

Example 3.12. It can be seen from Example 3.2 that the boundary value problem

$$\{y^{**}\}\{y^{*}\}^{-6}y^{10} = 1,$$
$$y(0) = y(\pi) = 1$$

is the multiplicative adjoint of the boundary value problem

$$\{y^{**}\}\{y^{*}\}^{6}y^{10} = 1,$$
$$y(0) = y(\pi) = 1.$$

Example 3.13. The multiplicative adjoint of the boundary value problem

$$y^{**}y = 1,$$
$$y^{*}(0)y^{*}(\pi) = 1,\ \ y(0) = 1$$

is calculated as follows:

Here, the multiplicative adjoint of the operator $T[y] = y^{**}y$ is given by $\widetilde{T}\,[y] = y^{**}y$.

The domain of T is the form

$$D(T) = \left\{u : u \in C^{2,*}[0,\pi]\,; u(0) = u(\pi) = 1, u^{*}(0) = 1\right\}.$$

The domain of the multiplicative adjoint operator $\widetilde{T}$ consists of the functions v that satisfy the equality

$$P\,(u,v)(x)|_0^\pi = \left.\frac{\{u^{*}\}^{\ln v}}{\{u\}^{\ln v^{*}}}\right|_0^\pi = 1$$

for all $u \in D(T)$.

Considering the domain of the operator T, where $u^{*}(\pi) = u^{*}(0)^{-1}$ and $u(0) = 1$, then the equality

$$\frac{\{u^{*}(0)\}^{-\ln v(0)v(\pi)}}{\{u(\pi)\}^{\ln v^{*}(\pi)}} = 1 \tag{3.13}$$

is obtained. Therefore, in order for equality (3.13) to hold for all $u \in D(T)$, it must be that

$$v(0)\,v(\pi) = 1,\ \ v^{*}(\pi) = 1.$$

This implies that the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{v : v \in C^{2,*}[0,\pi];\ \ v^{*}(\pi) = 1, v(0)\,v(\pi) = 1\right\}.$$

Consequently, since u and v are arbitrary variables, the multiplicative adjoint of given multiplicative boundary value problem takes the form

$$y^{**}y = 1,$$
$$y(0)y(\pi) = 1,\ \ y^{*}(\pi) = 1.$$

Example 3.14. The multiplicative adjoint of the boundary value problem

$$\{y^{**}\}^{2x^2}\{y^{*}\}^{7x}y^{-2} = 1,$$
$$y(1) = y(\pi) = 1$$

is calculated as follows:

From Example 3.3, the multiplicative adjoint of the operator

$$T[y] = \{y^{**}\}^{2x^2}\{y^{*}\}^{7x}y^{-2}$$

is given by

$$\widetilde{T}[y] = (y^{**})^{2x^2}(y^{*})^{x}y^{-5}.$$

The domain of T is the form

$$D(T) = \left\{u : u \in C^{2,*}[1,\pi]; u(1) = u(\pi) = 1\right\}.$$

The domain of the multiplicative adjoint operator $\widetilde{T}$ consists of the functions v that satisfy the equality

$$P(u,v)(x)|_{1}^{\pi} = \left.\frac{\{u^{*}\}^{\ln v^{2x^2}}\{u\}^{\ln v^{7x}}}{\{u\}^{\ln\left(\{v^{*}\}^{2x^2}v^{4x}\right)}}(x)\right|_{1}^{\pi} = 1.$$

Considering the domain of the operator T, where $u(1) = u(\pi) = 1$, then the equality

$$\left\{\frac{\{u^{*}(\pi)\}^{2\pi^2\ln v(\pi)}}{\{u^{*}(1)\}^{2\ln v(1)}}\right\} = 1 \tag{3.14}$$

is obtained. Therefore, in order for equality (3.14) to hold for all $u \in D(T)$, it must be that

$$v(1) = 1,\ \ v(\pi) = 1.$$

This implies that the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{v : v \in C^{2,*}[0,\pi]; v(1) = 1, v(\pi) = 1\right\}.$$

Consequently, since u and v are arbitrary variables, the multiplicative adjoint of given multiplicative boundary value problem takes the form

$$\{y^{**}\}^{2x^2}\{y^{*}\}^{x}y^{-5} = 1,$$
$$y(0) = y(\pi) = 1.$$

3.2. **Multiplicative Fredholm Alternative.** In this section, a criterion will be provided for determining whether a solution exists for a non-homogeneous multiplicative boundary value problem.

Theorem 3.15. *(Multiplicative Fredholm Alternative) Let T and B denote a multiplicative operator and a set of boundary conditions, respectively. Then, the necessary condition for the non-homogeneous multiplicative boundary value problem*

$$T\left[y\right]\left(x\right)=h\left(x\right),\ x\in\left(a,b\right),\ h\left(x\right)>0$$
$$B\left[y\right]=1$$

to have a solution is that

$$\int\limits_a^b\left\{h\left(x\right)\odot z\left(x\right)\right\}^{dx}=1 \tag{3.15}$$

for every solution z of the homogeneous multiplicative adjoint boundary value problem

$$\widetilde{T}\left[z\right]\left(x\right)=1,\ x\in\left(a,b\right)$$
$$\widetilde{B}\left[z\right]=1.$$

Proof. Let us consider in turn the following non-homogeneous multiplicative boundary value problem and its multiplicative adjoint boundary value problem

$$T\left[y\right]=h,\ B\left[y\right]=1, \tag{3.16}$$

$$\widetilde{T}\left[y\right]=1,\ \widetilde{B}\left[y\right]=1, \tag{3.17}$$

respectively.

If u is a solution to the multiplicative boundary value problem (3.16), then equality

$$\left\langle T\left[u\right],v\right\rangle_*=\left\langle h,v\right\rangle_*$$

is satisfied for any function v.

Similarly, if v is a solution to the multiplicative adjoint boundary value problem (3.17), then

$$\left\langle u,\widetilde{T}\left[v\right]\right\rangle_*=\left\langle u,1\right\rangle_*=1$$

is satisfied for any function u.

According to the last two equalities, the multiplicative Green's formula (3.8) is taken into account and

$$\left\langle T\left[u\right],v\right\rangle_*=\left\langle T\left[u\right],v\right\rangle_*$$

or

$$\left\langle h,v\right\rangle_*=1 \tag{3.18}$$

is obtained. That is, equality (3.15) holds.

As a result, since u and v are arbitrary variables, the proof is completed. □

Corollary 3.16. A necessary condition for the non-homogeneous multiplicative boundary value problem (3.16) to have a solution is that equation (3.18) holds for all solutions v of the homogeneous multiplicative adjoint boundary value problem (3.17). That is, the function h must be multiplicatively orthogonal to all solutions of the homogeneous multiplicative adjoint boundary value problem (3.17) within the problem's domain.

Remark 3.17. If the homogeneous multiplicative adjoint boundary value problem (3.17) has only the multiplicative trivial solution, then for any continuous arbitrary function $h > 0$, the non-homogeneous multiplicative boundary value problem (3.16) has always a solution. In this case, if the boundary conditions are multiplicatively separable or periodic, the solution is also unique.

Example 3.18. Determine the conditions that the function h must satisfy for the non-homogeneous multiplicative boundary value problem

$$\begin{gathered} \{y^{**}\}\{y^{*}\}^{6}y^{10} = h\left(x\right), \\ y(0) = y(\pi) = 1 \end{gathered} \tag{3.19}$$

to have a solution.

From Example 3.2, the homogeneous multiplicative adjoint boundary value problem corresponding to (3.19) is

$$\begin{gathered} \{y^{**}\}\{y^{*}\}^{-6}y^{10} = 1, \\ y(0) = y(\pi) = 1. \end{gathered} \tag{3.20}$$

Therefore, the general solution of the problem (3.20) is computed as

$$y = e^{e^{3x}(c_1 \cos x + c_2 \sin x)}$$

with the help of techniques in [28].

Considering the boundary conditions in (3.20),

$$y\left(0\right) = e^{c_1},\ y\left(\pi\right) = e^{-c_1 e^{3\pi}} = 1$$

is obtained. Thus, $c_1 = 0$, and the solution of the problem (3.20) is

$$y = e^{ke^{3x}\sin x}$$

for any arbitrary constant $c_2 = k$.

As a result, for the non-homogeneous multiplicative boundary value problem (3.19) to have a solution, the condition

$$\int_0^{\pi} \left\{h\left(x\right) \odot e^{ke^{3x}\sin x}\right\}^{dx} = \int_0^{\pi} \left\{h(x)^{ke^{3x}\sin x}\right\}^{dx} = 1$$

should be held according to the multiplicative Fredholm alternative.

Remark 3.19. If we note that $h\left(x\right) = e^{e^{-3x}\cos x}$ in Example 3.18 then

$$\int_0^{\pi} \left\{e^{\frac{\cos 2x}{2}}\right\}^{dx} = 1$$

or

$$\int_0^{\pi} \frac{\cos 2x}{2} dx = 0$$

holds, and the condition (3.15) is satisfied. That is, for this choice of h, there is a solution of the problem (3.19). However, the solution is not unique because the solution of the problem (3.20) according to the problem (3.19) depends on a parameter k.

In fact, for $h(x) = e^{e^{-3x}\cos x}$ in Example 3.18, the solution of the problem (3.19)

$$y = e^{ke^{-3x}\sin x + \frac{x}{2}e^{-3x}\sin x}$$

with the help of techniques in [29].

Example 3.20. Determine the conditions that the function h must satisfy for the non-homogeneous multiplicative boundary value problem

$$\begin{gathered} \{y^{**}\}^{2x^2}\{y^{*}\}^{7x}y^{-2} = h(x), \\ y(1) = y(\pi) = 1 \end{gathered} \tag{3.21}$$

to have a solution.

From Example 3.3, the homogeneous multiplicative adjoint boundary value problem corresponding to (3.21) is

$$\begin{gathered} \{y^{**}\}^{2x^2}\{y^{*}\}^{x}y^{-5} = 1, \\ y(1) = y(\pi) = 1. \end{gathered} \tag{3.22}$$

The multiplicative differential equation in (3.22) is a multiplicative Cauchy-Euler equation and its general solution is

$$y = e^{c_1 x^{\frac{1-\sqrt{41}}{4}} + c_2 x^{\frac{1+\sqrt{41}}{4}}}$$

with the help of techniques in [28].

Considering the boundary conditions in (3.22), $c_1 = c_2 = 0$ and the multiplicative trivial solution $y = 1$ is obtained. According to Remark 3.17, since the problem (3.22) has only the trivial solution, for any continuous arbitrary function $h > 0$, the problem (3.21) has always a solution.

4. Conclusion

We consider non-homogeneous boundary value problems that are redefined with multiplicative calculus techniques. The concepts of adjoint operator, self-adjoint operator, Lagrange identity, Green's formula given in the multiplicative sense for this problem are expressed, and the detailed examples are provided to emphasize their importance. In conclusion, a criterion is provided for determining whether a solution exists for a non-homogeneous multiplicative boundary value problem, and also some examples are given.

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