Black Sea Journal of Engineering and Science
doi: 10.34248/bsengineering.1553162

Open Access Journal
e-ISSN: 2619 – 8991

Research Article
Volume 7 - Issue 6: 1310-1314 / November 2024

ON THE GENERALIZED WEIGHTED STATISTICAL CONVERGENCE

Çiğdem BEKTAŞ[1], Erdal BAYRAM[2]*

[1]Fırat University, Faculty of Science, Department of Mathematics, 23119, Elazığ, Türkiye
[2]Tekirdağ Namık Kemal University, Faculty of Arts and Sciences, Department of Mathematics, 59030, Tekirdağ, Türkiye

Abstract: Statistical convergence and summability represent a significant generalization of traditional convergence for sequences of real or complex values, allowing for a broader interpretation of convergence phenomena. This concept has been extensively examined by numerous researchers using various mathematical tools and applied to different mathematical structures over time, revealing its relevance across multiple disciplines. In the present study, a generalized definition of the concepts of statistical convergence and summability, termed $(\Delta_v^m)_u$ −generalized weighted statistical convergence and $(\Delta_v^m)_u$ −generalized weighted by $[\overline{N}_t]$ −summability for real sequences, is introduced using the weighted density and generalized difference operator. Based on this definition, several fundamental properties and inclusion results, obtained by differentiating the components used in the definitions, are provided.

Keywords: Generalized difference sequence, Weighted density, Weighted statistical convergence, Weighted summability

***Corresponding author:** Tekirdağ Namık Kemal University, Faculty of Arts and Sciences, Department of Mathematics, 59030, Tekirdağ, Türkiye
E mail: ebayram@nku.edu.tr (E. BAYRAM)
Çiğdem BEKTAŞ https://orcid.org/0000-0003-0397-3193
Erdal BAYRAM https://orcid.org/0000-0001-8488-359X

Received: September 20, 2024
Accepted: October 28, 2024
Published: November 15, 2024

Cite as: Bektaş Ç, Bayram E. 2024. On the generalized weighted statistical convergence. BSJ Eng Sci, 7(6): 1310-1314.

1. Introduction

The mathematical structure defines different types of convergence. Convergence types defined on the same structure can also be compared. One of the most recently studied types of convergence is statistical convergence. This form of convergence was initially proposed by Fast (1951) and Steinhaus (1951), and subsequently developed by Schoenberg (1959) to extend the standard topological convergence of real sequences. Salat (1980), Fridy (1985) and Connor (1988) gave remarkable properties of the statistical convergence of real sequences. Afterward, it has been discussed and generalized from many different perspectives or in many different mathematical structures . For instance, we refer the reader for weighted statistical convergence or statistical convergence by using the difference operator (Et and Nuray, 2001; Güngör and Et, 2003; Karakaya and Chishti, 2009; Mursaleen et al. 2012; Belen and Mohiuddine, 2013; Kadak, 2016; Ghosal, 2016; Braha et al., 2021; Et et al., 2021; Kandemir et al., 2023).

In this research, we investigate a generalization of statistical convergence by incorporating the difference operator outlined in (Et and Esi, 2000) along with a sequence of multipliers. Our motivation in this investigation is grounded in the findings of previous studies, specifically studies (Ghossal, 2016; Kandemir et al., 2023).

2. Materials and Methods

The basic tool used in statistical convergence is the concept of asymptotic (natural) density and it is defined for a set $A \subseteq \mathbb{N}^+$ as $\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{a \leq n : a \in A\}|$. Here, the vertical bars show how many elements are included in the enclosed set. Moreover, $\delta(A) = 0$ for the finite set A, $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$ and $\delta(A) \leq \delta(B)$ whenever $A \subseteq B$.

Definition 2.1. A real sequence $x = (x_n)$ is called statistical converges to γ if for every $\varepsilon > 0$ the set $\{k \leq n : |x_k - \gamma| \geq \varepsilon\}$ has a natural density of zero, i.e.

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - \gamma| \geq \varepsilon\}| = 0.$$

Therefore, the element γ is referred to as the statistical limit of (x_n) and is represented as $st - \lim_{n\to\infty} x_n = \gamma$. S represents all statistically convergent sequences (Fast, 1951).

In order to extend the notion of statistical convergence, researchers have explored various generalizations of the concept of asymptotic density. One such generalization is referred to as weighted density. Let (q_n) be a sequence in $\mathbb{R}^+ \cup \{0\}$ such that $\liminf q_n > 0$ and $\lim Q_n = \infty$, where $Q_n = q_1 + q_2 + \ldots + q_n$ for all $n \in \mathbb{N}$. Accordingly, the weighted density is defined for a set $A \subseteq \mathbb{N}^+$ as

$$\delta_w(A) = \lim_{n\to\infty} \frac{1}{Q_n} |\{a \leq Q_n : a \in A\}|.$$

Similarly, $\delta_w(A) = 0$ for the finite set A, $\delta_w(\mathbb{N} \setminus A) = 1 -$

$\delta_w(A)$ and $\delta_w(A) \leq \delta_w(B)$ whenever $A \subseteq B$. The concept of weighted statistical convergence, first proposed by Karakaya and Chisti (2009) and subsequently refined by Mursaleen et al. (2012), is defined through the use of weighted density as follows.

Definition 2.2. A real sequence $x = (x_n)$ is called weighted statistical converges to γ if, for every $\varepsilon > 0$, $\delta_w(\{k \leq Q_n: q_k|x_k - \gamma| \geq \varepsilon\}) = 0$ holds.

Hence, the element γ is called weighted statistical limit of (x_n) and it is denoted by $S_{\overline{N}} - \lim x = \gamma$. The set of all weighted statistically convergent sequences is denoted by $S_{\overline{N}}$ (Mursaleen et al., 2012).

Also, as a generalization, Ghosal (2016) introduced the notion of weighted statistical convergence of order $\alpha \in (0,1]$.

A sequence space is defined as a linear subspace of the space $\mathbb{R}^{\mathbb{N}}$, which is denoted w. The classical sequence spaces ℓ_∞, c, c_0, ℓ_p with $1 < p < \infty$ are all bounded, convergent, null and p-absolutely summable sequences, respectively. One method for creating new sequence spaces is through the use of the difference operator, defined as $\Delta x_k = x_k - x_{k+1}$. The difference sequence spaces $\lambda(\Delta) = \{(x_k) \in w: \Delta(x_k) \in \lambda\}$ where λ is any of the classical sequence spaces. This concept was first introduced by Kızmaz (1981). Following this, quite a lot of work was done with some generalizations by using the difference operator in some way. One such example is the different sequence spaces of a positive integer order m, $\lambda(\Delta^m) = \{(x_k) \in w: \Delta^m(x_k) \in \lambda\}$ for $m \in \mathbb{N}$ where $\Delta^0 x_k = x_k$ and $\Delta^m x_k = \Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}$ (Et and Çolak, 1995). Clearly, $\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}$ holds. Subsequently, Et and Esi (2000) were expanded in the following manner:

$$\lambda(\Delta_v^m) = \{x = (x_k): (\Delta_v^m x_k) \in \lambda\}$$

where $(\Delta_v^m x_k) = (\Delta_v^{m-1}x_k - \Delta_v^{m-1}x_{k+1})$ for any fixed sequence of nonzero complex numbers $v = (v_k)$ such that

$$\Delta_v^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k+i}x_{k+i}.$$

There is considerable literature on difference sequence spaces by choosing different classical sequence spaces of λ for either case where α is integer or fractional. For instance, we refer the reader to the difference operator (Et and Çolak, 1995; Et and Esi, 2000; Bektas and Çolak, 2005; Barlak 2020).

Among the studies conducted for the purpose of generalizing of statistical convergence, there are studies utilizing the difference operator. Recently, Kandemir et al. (2023) introduced the concept of Δ^m-weighted statistical convergence and Δ^m-weighted $(\overline{N}, Q_n)-$summability. A real sequence $x = (x_n)$ is referred Δ^m-weighted statistical convergent of order $\alpha \in (0,1]$ (or $S_{\overline{N_t}}^{\alpha}(\Delta^m)-$convergent) to γ if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{Q_n^\alpha} |\{k \leq Q_n: q_k|\Delta^m x_k - \gamma| \geq \varepsilon\}| = 0.$$

Hence, it is denoted by $S_{\overline{N_t}} - \lim x = \gamma$. The notation $S_{\overline{N_t}}^{\alpha}(\Delta^m)$ represents the set of all Δ^m-weighted statistically convergent sequences of order α. In this context, the forms of statistical convergence presented in studies Karakaya and Christi (2009) and Mursaleen et al. (2012) are obtained through a specific choice of values for m and α (see Remark 2.1, Kandemir et al. (2023)).

3. Results and Discussion

This section introduces the concepts of $(\Delta_v^m)_u$-generalized weighted statistical convergence and $(\Delta_v^m)_u$-generalized weighted $(\overline{N}, q_n)$-summability and establish the relations between them. In the results and proofs provided, u and t will be defined as follows and will not be redefined each time for the sake of simplicity:

1. U refers to the set of all real sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}^+$.
2. $q = (q_n)$ is a sequence in $\mathbb{R}^+ \cup \{0\}$ such that $\liminf q_n > 0$ and $\lim_{n\to\infty} Q_n = \infty$ where $Q_n = t_1 + t_2 + \ldots + q_n$ for all $n \in \mathbb{N}$.

Definition 3.1. A sequence $x = (x_n)$ *is said to be* $(\Delta_v^m)_u -$generalized weighted statistically convergent (or $S_{\overline{N_t}}(\Delta_v^m)_u$-convergent) to γ if for every $\varepsilon > 0$

$$\delta_w(\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{Q_n} |\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}| = 0.$$

In this way, we express $S_{\overline{N_t}}(\Delta_v^m)_u$-$\lim x = \gamma$ or $x_k \to \gamma(S_{\overline{N_t}}(\Delta_v^m)_u)$. We use the notation $S_{\overline{N_t}}(\Delta_v^m)_u$ to represent the set of all $(\Delta_v^m)_u$-generalized weighted statistically convergent sequences.

Definition 3.2. Assume that $r > 0$ is a real number and $q_1 > 0$. A sequence $x = (x_n)$ is $(\Delta_v^m)_u -$generalized weighted $[\overline{N_t}]-$summable (or $[\overline{N_t}, (\Delta_v^m)_u]_r -$ summable) to γ provided that

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=0}^n q_k |u_k \Delta_v^m x_k - \gamma|^r = 0.$$

In this way, we express $[\overline{N_t}, (\Delta_v^m)_u]_r - \lim x = \gamma$ or $x_n \to \gamma[\overline{N_t}, (\Delta_v^m)_u]_r$. We denote the set of all $(\Delta_v^m)_u -$generalized weighted $[\overline{N_t}]-$ summable sequences by $[\overline{N_t}, (\Delta_v^m)_u]_r$.

Remark 3.3. In consideration of Definitions 2.1 and 2.2, it is evident the following situations are observed:

1. $S_{\overline{N_t}}(\Delta_v^m)_u = S$ if $m = 0$ and $u_k = v_k = q_k = 1$ for all $k \in \mathbb{N}$ (Fast, 1951).
2. $S_{\overline{N_t}}(\Delta_v^m)_u = S_{\overline{N}}(\Delta_v)$ and $[\overline{N_t}, (\Delta_v^m)_u]_r = [\overline{N}, q_n]_r$ if $m = 0$ and $u_k = v_k = 1$ for all $k \in \mathbb{N}$ (Karakaya, 2009; Mursaleen et all. (2012)).
3. $S_{\overline{N_t}}(\Delta_v^m)_u = S_{\overline{N_t}}(\Delta^m)$ and $[\overline{N_t}, (\Delta_v^m)_u]_r = [\overline{N_t}](\Delta^m,_r)$ if $\alpha = 1$ and $u_k = v_k = 1$ for all $k \in \mathbb{N}$ (Kandemir et al., 2023).

Theorem 3.4. Assume that $x = (x_k) \in \overline{N_t}, (\Delta_v^m)_u]_r$ *and*

$[\overline{N_t}, (\Delta_v^m)_u]_r - lim x = \gamma$. *Then* x *is* $S_{\overline{N_t}}(\Delta_v^m)_u$ −statistically converges to γ, if one of the following two conditions holds:

1. $r \in (0, 1)$ and $0 \le |u_k \, \Delta_v^m \, x_k - \gamma| < 1$ for every k,
2. $r \in [1, \infty)$ and $1 \le |u_k \, \Delta_v^m \, x_k - \gamma| < \infty$ for every k.

Proof. Since the sequence $x = (x_k)$ is $[\overline{N_t}, (\Delta_v^m)_u]_r$ −summable to γ we see that

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r = 0.$$

If both (1) and (2) are satisfied, it can be seen that for every k,

$$q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r \ge q_k |u_k \, \Delta_v^m \, x_k - \gamma|$$

holds. Then we obtain, for any $\varepsilon > 0$,

$$\sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r$$
$$\ge \sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|$$
$$\ge \sum_{\substack{k=1 \\ k\in\{k \le Q_n : q_k |u_k \Delta_v^m x_k - \gamma| \ge \varepsilon\}}}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|$$
$$\ge \sum_{\substack{k=1 \\ k\in\{k \le Q_n : q_k |u_k \Delta_v^m x_k - \gamma| \ge \varepsilon\}}}^{n} \varepsilon$$
$$\ge |\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma| \ge \varepsilon\}| \, \varepsilon$$

and this implies

$$\varepsilon \delta_w(\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma| \ge \varepsilon\}) \le \frac{1}{Q_n} \sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r.$$

Taking limit as $n \to \infty$, this means $\delta_w(\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma| \ge \varepsilon\}) = 0$, so that $x \in S_{\overline{N_t}}(\Delta_v^m)_u$.

Theorem 3.5. Suppose that the sequence $x = (x_k)$, $S_{\overline{N_t}}(\Delta_v^m)_u$ −converges to γ and for every $k \in \mathbb{N}$, $q_k |u_k \, \Delta_v^m \, x_k - \gamma| \le M$ for some $M > 0$. Then (x_n) *is* $[(\overline{N}, q_n), (\Delta_v^m)_u]_r$ −summable to γ, if one of the following two conditions holds:

1. $r \in (0, 1)$ and $1 \le M < \infty$,
2. $r \in [1, \infty)$ and $0 \le M < 1$.

Proof. If x, $S_{\overline{N_t}}(\Delta_v^m)_u$ −statistically converges to γ then for every $\varepsilon > 0$ we have $\delta_w(K_{Q_n}(\varepsilon)) = 0$ where $K_{Q_n}(\varepsilon) = \{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma| \ge \varepsilon\}$. If both 1 and 2 are satisfied, it can be seen that for every $k \in \mathbb{N}$,

$$q_k |u_k \, \Delta_v^m \, x_k - \gamma| \le M$$

holds. Then, for every $\varepsilon > 0$, we obtain

$$\sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r$$
$$= \sum_{\substack{k=1 \\ k\notin\{k \le Q_n : q_k |u_k \Delta_v^m x_k - \gamma| \ge \varepsilon\}}}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r$$
$$+ \sum_{\substack{k=1 \\ k\in\{k \le Q_n : q_k |u_k \Delta_v^m x_k - \gamma| \ge \varepsilon\}}}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r$$
$$\le \varepsilon^r Q_n + M |\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma| \ge \varepsilon\}|.$$

That is, for every $\varepsilon > 0$

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_k |u_k \, \Delta_v^m \, x_k - \gamma|^r \le \varepsilon^r + M \delta_w(K_{Q_n}(\varepsilon))$$

holds. As a result, we deduce x, $[\overline{N_t}, (\Delta_v^m)_u]_r$ −summable to γ.

Theorem 3.6. $(\Delta_v^m)_u$- generalized weighted statistically convergent sequence $x = (x_k)$ has a unique $(\Delta_v^m)_u$ −limit.

Proof. Suppose that $S_{\overline{N_t}}(\Delta_v^m)_u - \lim x = \gamma_1$, $S_{\overline{N_t}}(\Delta_v^m)_u - \lim x = \gamma_2$ and $\gamma_1 \ne \gamma_2$ hold. Choose $\varepsilon, \delta \in \mathbb{R}^+$ such that $\varepsilon = \frac{1}{2}|\gamma_1 - \gamma_2|$ and $\liminf q_k > \delta > 0$. Hence, the following inequality holds:

$$1 \le |\{k \le Q_n : q_k |\gamma_1 - \gamma_2| \ge \varepsilon\delta\}|$$
$$\le |\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_1| \ge \varepsilon\delta\}|$$
$$+ |\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_2| \ge \varepsilon\delta\}|.$$

Consequently, we have

$$1 \le \delta_w(|\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_1| \ge \varepsilon\}|) + \delta_w(|\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_2| \ge \varepsilon\}|) = 0.$$

This contradiction indicates that γ_1 must be equal to γ_2.

Theorem 3.7. $(\Delta_v^m)_u$ −generalized weighted $[\overline{N_t}]$ −summable sequence $x = (x_k)$ has a unique $[\overline{N_t}, (\Delta_v^m)_u]_r$ −limit.

Proof. The proof is identical to that presented above.

Theorem 3.8. *Let* $S_{\overline{N_t}}(\Delta_v^m)_u - lim x = \gamma_1$ and $S_{\overline{N_t}}(\Delta_v^m)_u - lim y = \gamma_2$. Then

(i) $S_{\overline{N_t}}(\Delta_v^m)_u - \lim(x + y) = \gamma_1 + \gamma_2$.
(ii) $S_{\overline{N_t}}(\Delta_v^m)_u - \lim cx = c\gamma_1$, $c \in \mathbb{R}$.

Proof. For (i), the assertion is obvious from the following inclusion:

$$\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - (\gamma_1 + \gamma_2)| \ge \varepsilon\}$$
$$\subseteq \{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_1| \ge \frac{\varepsilon}{2}\}$$
$$\cup \left\{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_2| \ge \frac{\varepsilon}{2}\right\}.$$

For (ii), the assertion is clear if $c = 0$. If $c \ne 0$, then proof can be seen from the following equality:

$$\{k \le Q_n : q_k |u_k \, \Delta_v^m \, (cx_k) - (c\gamma_1)| \ge \varepsilon\}$$
$$= \{k \le Q_n : q_k |u_k \, \Delta_v^m \, x_k - \gamma_1| \ge \frac{\varepsilon}{|c|}\}.$$

Theorem 3.9. Suppose that $q = (q_n)$ and $s = (s_n)$ be real sequences of nonnegative real numbers such that $\liminf q_n > 0$, $\liminf s_n > 0$, $\lim_{n\to\infty} Q_n = \infty$, $\lim_{n\to\infty} S_n = \infty$ where $Q_n = t_1 + t_2 + \ldots + q_n$ and $S_n = s_1 + s_2 + \ldots + s_n$ for all $n \in \mathbb{N}$. If ,for all $k \in \mathbb{N}$, $q_k \le s_k$ and $\liminf \frac{Q_n}{S_n} > 0$ hold, then the following *inclusions hold:*

$S_{\overline{N_s}}(\Delta_v^m)_u \subseteq S_{\overline{N_t}}(\Delta_v^m)_u$

and

$[\overline{N_s}, (\Delta_v^m)_u]_r \subseteq [\overline{N_t}, (\Delta_v^m)_u]_r$.

Proof. Choose $x = (x_k) \in S_{\overline{N_s}}(\Delta_v^m)_u$ such that $S_{\overline{N_s}}(\Delta_v^m)_u$-lim$x = \gamma$. For every $\varepsilon > 0$, from the selection of the sequences, it is observed that the following inequality is satisfied:

$$q_k|u_k \Delta_v^m x_k - \gamma| \leq s_k|u_k \Delta_v^m x_k - \gamma|$$

and hence

$$\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\} \subseteq \{k \leq S_n: s_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}. \quad (1)$$

Therefore

$$\frac{1}{S_n}|\{k \leq S_n: s_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}| \geq \frac{1}{S_n}|\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}| \geq \frac{Q_n}{S_n}\frac{1}{Q_n}|\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}|$$

holds. Taking limit as $n \to \infty$, since liminf$\frac{Q_n}{S_n} > 0$ holds, $x \in S_{\overline{N_s}}(\Delta_v^m)_u$ implies $x \in S_{\overline{N_t}}(\Delta_v^m)_u$. The proof for the second inclusion follows a nearly identical process.

Theorem 3.10. Assume that $q = (q_n)$ and $s = (s_n)$ be real sequences of nonnegative real numbers such that $liminf\, q_n > 0$, $liminf\, s_n > 0$, $\lim_{n\to\infty} Q_n = \infty$, $\lim_{n\to\infty} S_n = \infty$ where $Q_n = t_1 + t_2 + \ldots + q_n$ and $S_n = s_1 + s_2 + \ldots + s_n$ for all $n \in \mathbb{N}$. If ,for all $k \in \mathbb{N}$, $q_k \leq s_k$ and $limsup \frac{S_n}{Q_n} < \infty$ hold, then the following inclusions hold:

$S_{\overline{N_s}}(\Delta_v^m)_u \subseteq S_{\overline{N_t}}(\Delta_v^m)_u$

and

$[\overline{N_s}, (\Delta_v^m)_u]_r \subseteq \overline{N_t}, (\Delta_v^m)_u]_r$.

Proof. Choose $x = (x_k) \in S_{\overline{N_s}}(\Delta_v^m)_u$ such that $S_{\overline{N_s}}(\Delta_v^m)_u$-lim$x = \gamma$. According to the conditions of the theorem, the inclusion (1) above is ensured. Hence, we have

$$\frac{1}{Q_n}|\{k \leq Q_n: q_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}| \leq \frac{1}{Q_n}|\{k \leq S_n: s_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}| \leq \frac{S_n}{Q_n}\frac{1}{S_n}|\{k \leq S_n: s_k|u_k \Delta_v^m x_k - \gamma| \geq \varepsilon\}|.$$

Taking limit as $n \to \infty$, since limsup$\frac{S_n}{Q_n} < \infty$ hold, $x \in S_{\overline{N_s}}(\Delta_v^m)_u$ implies $x \in S_{\overline{N_t}}(\Delta_v^m)_u$. Again, the proof for the second inclusion is similar.

Theorem 3.11. If, for the sequences $u = (u_k) \subset \mathbb{R}^+ \cup \{0\}$ and $w = (w_k) \subset \mathbb{R}^+ \cup \{0\}$, the condition $u_k \leq w_k$ is satisfied for all $k \in \mathbb{N}$, then the following inclusions hold:

$S_{\overline{N_t}}(\Delta_v^m)_u \subseteq S_{\overline{N_t}}(\Delta_v^m)_w$

and

$[\overline{N_s}, (\Delta_v^m)_u]_r \subseteq \overline{N_t}, (\Delta_v^m)_w]_r$.

Theorem 3.12. If, for the sequences $v = (v_k) \subset \mathbb{R}^+ \cup \{0\}$ and $z = (z_k) \subset \mathbb{R}^+ \cup \{0\}$, the condition $v_k \leq z_k$ is satisfied for all $k \in \mathbb{N}$, then the following inclusions hold:

$S_{\overline{N_t}}(\Delta_v^m)_u \subseteq S_{\overline{N_t}}(\Delta_z^m)_u$

and

$[\overline{N_s}, (\Delta_v^m)_u]_r \subseteq \overline{N_t}, (\Delta_z^m)_u]_r$.

4. Conclusion

Numerous convergence concepts have emerged after the advent of the topological convergence concept in classical analysis. Among these, statistical convergence, a generalization of topological convergence, has been the subject of intense study. Initially defined for real series, statistical convergence has been examined and generalized in numerous mathematical structures, with generalizations being made in each case. In this study, we also generalized statistical convergence and summability by using the weighted density, difference operator, and sequences of nonzero real numbers. We presented fundamental results and some relations related to existing literature. Additionally, we presented some inclusion theorems related to weighted statistical convergence and summability concepts. For further study, one can consider analogous results regarding the difference operator with fractional order instead of a positive integer m.

Author Contributions

The percentages of the authors' contributions are presented below. All authors reviewed and approved the final version of the manuscript.

	Ç.A.B.	E.B.
C	50	50
D	50	50
S	50	50
DCP	50	50
DAI	50	50
L	50	50
W	50	50
CR	50	50
SR	50	50
PM	50	50
FA	50	50

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or

humans.

References

Barlak D. 2020. Statistical convergence of order β for (λ, μ) double sequences of fuzzy numbers, 39(5): 6949-6954.

Bektaş ÇA, Çolak R. 2005. On some generalized difference sequence spaces. Thai J Math, 3(1): 83-98.

Belen C, Mohiuddine SA. 2013. Generalized weighted statistical convergence and application. Appl Math Computat, 219(18): 9821-9826.

Braha NL, Srivastava HM, Et M. 2021. Some weighted statistical convergence and associated Korovkin and Voronovskaya type theorems. J App Math Comput, 65: 429-450.

Connor JS. 1988. The statistical and strong p-Cesaro convergence of sequences. Analysis, 8: 47-63.

Et M, Çolak R. 1995. On generalized difference sequence spaces. Soochow J Math, 21(4): 377-386.

Et M, Esi A. 2000. On Köthe-Toeplitz duals of generalized difference sequence spaces. Bull Malaysian Math Sci Soc, 23: 25-32.

Et M, Nuray F. 2001. $\Delta^{m}-$Statistical Convergence. Indian J Pure Appl Math, 32(6): 961-969.

Et M, Kandemir HŞ, Çakallı H. 2021. $\Delta^{m}-$weighted statistical convergence. AIP Conf Proc, 2334: 040005.

Fast H. 1951. Sur la convergence statistique. Colloquium Mathematicum, 2: 241-244.

Fridy J. 1985. On statistical convergence, Analysis, 5: 301-313.

Ghosal S. 2016. Weighted statistical convergence of order α and its applications. J Egyptian Math Soc, 24(1): 60-67.

Güngör M, Et M. 2003. $\Delta^{r}-$strongly almost summable sequences defined by Orlicz functions. Indian J Pure Appl Math, 34(8): 1141-1151.

Kadak U. 2016. On weighted statistical convergence based on (p,q)-integers and related approximation theorems for functions of two variables. J Math Analy Appl, 443(2): 752-764.

Kandemir HŞ, Et M, Çakallı H. 2023. Weighted statistical convergence of order α. Facta Univ Series: Math Info, 38(2): 317-327.

Karakaya V, Chishti TA. 2009. Weighted statistical convergence. Iranian J Sci Technol Transact A: Sci, 33(33): 219-223.

Kızmaz K. 1981. On certain sequence spaces. Canadian Math Bull, 24(2): 169-176.

Mursaleen M, Karakaya V, Ertürk M, Gürsoy F. 2012. Weighted statistical convergence and its application to Korovkin type approximation theorem. Appl Math Comput, 218(18): 9132-9137.

Salat T. 1980. On statistically convergent sequences of real numbers. MathSlovaca, 30: 139-150.

Schoenberg IJ. 959. The integrability of certain functions and related summability methods. Amer Math Monthly, 66: 361-375.

Steinhaus H. 1951. Sur la convergence ordinaire et la convergence asymptotique. Colloquium Math, 2: 73-74.