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RE-VISIT $\mathcal{I}^*$-SEQUENTIAL TOPOLOGY

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Abstract. In this paper, $\mathcal{I}^*$-sequential topology is defined on a topological space (X, τ) by considering any ideal $\mathcal{I}$ which is a family of subset of natural numbers $\mathbb{N}$. It has been proven that $\mathcal{I}^*$-sequential topology is finer than $\mathcal{I}$-sequential topology. In connection with this fact, the notions $\mathcal{I}^*$-continuity and $\mathcal{I}^*$-sequential continuity are shown to be coincided. Additionally, $\mathcal{I}^*$-sequential compactness and related notions are defined and investigated.

1. Introduction and Preliminaries

Examining convergence of sequences is one of the main and famous problem in mathematical analysis. Especially, taking into consider different type convergence methods has led to a better understanding of the geometric and algebraic structure of the studied space. Statistical convergence, which is the most interesting method in terms of how it is defined, was introduced by Fast [6] and Steinhouse [23] in the year 1951, independently. Over the years, many studies on statistical convergence have been conducted and many application in different field of mathematics like, summability theory [21], number theory [5], trigonometric series [26], optimization and approximation theory [8] and etc. were given.

Recall the notion of statistical convergence in a topological space. For any subset A in $\mathbb{N}$, the asymptotic density of A is defined by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : k \leq n\}|$$

when the limit exists.

A sequence $\tilde{x} = (x_n)$ in the topological space (X, τ) is said to be statistically convergent to a point $x \in X$ if

$$\delta(\{n \in \mathbb{N}, x_n \notin U\}) = 0,$$

holds for any neighborhood U of x. It is denoted by $st - \lim_{x\to\infty} x_n = x$.

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A subset $F \subseteq X$ is called sequentially closed if for each sequence $\tilde{x} = (x_n)$ in F with $x_n \to x \in X$ then $x \in F$ holds. A space (X, τ) is called sequential topological space if each sequentially closed subset of X is closed.

A sequence $\tilde{x} = (x_n) \subset X$ is said to be eventually in an open subset U of X, if there exists $n_0(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n > n_0$. A subset $G \subseteq X$ is said to be sequentially open if $X - G$ is sequentially closed. Then, it is obvious that, a subset $U \subseteq X$ is sequentially open if and only if for each sequence $\tilde{x} = (x_n)$ converging to a point x in U, then $\tilde{x} = (x_n)$ is eventually in U.

After that in 2000, P. Kostyrko, et al. in [12] introduced the notion of ideal convergence which is completely different classical convergence but only its particular case coincides with classical and statistical convergence. Because of the flexibility of the ideal concept, several results in different spaces were given in [7, 9, 10, 11, 14, 18, 19, 20, 24]. Between the years 2012-2019, authors of the papers [2, 3, 4, 13, 16, 25] extended the notion of $\mathcal{I}$-convergence of a sequence to any topological space and proved several properties of this concept in a topological space. And very recently, the idea of $\mathcal{I}$-convergent is generalized and $\mathcal{I}^*$-convergent is defined.

Definition 1. [12] *Let S be a set and $\mathcal{I}$ be a sub family of $P(S)$. $\mathcal{I}$ is called an ideal on S if (i) For all $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$ hold.*

The ideal $\mathcal{I}$ is called an admissible ideal if $\{s\} \in \mathcal{I}$ holds for all $s \in S$; and it is called proper ideal if $S \notin \mathcal{I}$. A proper ideal is called maximal ideal if it is maximal element ordered by inclusion in the set of all proper ideals defined on S.

An ideal $\mathcal{I}$ is called non trivial if $\mathcal{I} \neq \phi$ and $S \notin \mathcal{I}$.

Example 1. *$\mathcal{I}_{Fin} := \{A \subset \mathbb{N} : A \text{ is finite set}\}$ and $\mathcal{I}_\delta := \{A \subset \mathbb{N} : \delta(A) = 0\}$ are admissible and proper ideal on the set of natural numbers.*

Example 2. [11] *Let $\mathbb{N} = \bigcup_{i=1}^{\infty} \Delta_i$ be a decomposition of $\mathbb{N}$ such that for all $i \in \mathbb{N}$ the set Δ_i are infinite subsets of $\mathbb{N}$ and $\Delta_i \cap \Delta j = \phi$ holds for all $i \neq j$. Let*

$$\mathcal{I} := \{B \subset \mathbb{N} : B \text{ intersect at most finite number of } \Delta_j' s\}.$$

Then, $\mathcal{I}$ is an admissible and nontrivial ideal.

Definition 2. *Let $\mathcal{I}$ be an ideal and $K \subset S$ be any set. The set K is said*

(i) $\mathcal{I}$-thin if $K \in \mathcal{I}$,

(ii) $\mathcal{I}$-non thin if $K \notin \mathcal{I}$,

(iii) relatively $\mathcal{I}$-non thin if there exist $A \in \mathcal{I}$ such that $A \in K$.

The set of $\mathcal{I}$-thin, $\mathcal{I}$-non-thin and relatively $\mathcal{I}$-non-thin sets are denoted by $\mathcal{I}_t$, $\mathcal{I}_{nt}$ and $\mathcal{I}_{rnt}$, respectively.

The dual notion of ideal is called filter and defined as follows:

Definition 3. [19] *A family $\mathscr{F} \subseteq \mathscr{P}(S)$ is said to be filter if (i) $A \cap B \in \mathscr{F}$ for all $A, B \in \mathscr{F}$, and (ii) If $A \in \mathscr{F} \wedge A \subseteq B$, then $B \in \mathscr{F}$ hold.*

A filter $\mathscr{F}$ is called proper if $\phi \notin \mathscr{F}$. For every non-trivial ideal $\mathcal{I}$ defines a filter associated by $\mathcal{I}$ as $\mathscr{F}(\mathcal{I}) := \{A \subseteq S : S - A \in \mathcal{I}\}$ on the set S.

Remark 1. *Let $\mathcal{I}$ be an ideal and $K \subset S$ be a set. Then, $K \in \mathcal{I}_{rnt}$ if and only if there exists a set $B \in \mathscr{F}(\mathcal{I})$ such that $K \subset B$.*

Proof. It can be obtained from definition. So, it is omitted here. □

Remark 2. *If we consider $\mathcal{I} = Fin$, then $\mathcal{I}_t = \mathcal{I}$ and $\mathcal{I}_{nt} = \mathcal{I}_{rnt} = \mathcal{F}(\mathcal{I})$ holds.*

Remark 3. *If $\mathcal{I}$ is an admissible ideal, then $\mathcal{I}_{nt} \subset \mathcal{I}_{rnt}$.*

Proof. Let $\mathcal{I}$ be an admissible ideal and $A \subset \mathbb{N}$ be a an $\mathcal{I}$ non-thin subset. Hence, the set A is not finite set because of ideal $\mathcal{I}$ is admissible. From the set theory, it is well known that A contains a finite subset B which is belongs to $\mathcal{I}$. This implies that A is in $\mathcal{I}_{rnt}$. □

Lemma 1. *Let $\mathcal{I}$ be an ideal and A be a relatively $\mathcal{I}$ non-thin sub set of $\mathbb{N}$. Then, there exists a maximal set $B \in \mathcal{I}$ such that $B \subset A$ holds.*

Proof. Denote the set

$$\mathcal{A}^* = \{B \in \mathcal{I} : B \subset A\}.$$

$\mathcal{A}^*$ is partial order family with respect to inclusion. If we consider complete order sub family $\mathcal{A}$ of $\mathcal{A}^*$, then

$$\bigcup\{B : B \in \mathcal{A}\}$$

is the upper bound of $\mathcal{A}$. Then, Zorn's Lemma says that $\mathcal{A}^*$ has a maximal element. So, proof is ended. □

Thorough the paper, we are going to consider $S = \mathbb{N}$ set of natural numbers, $\mathcal{I}$ is an arbitrary ideal and (X, τ) is a topological space. Unless otherwise stated this triple X, τ and $\mathcal{I}$ will be displayed in $(X, \tau, \mathcal{I})$ format.

Definition 4. [25] *A sequence $\tilde{x} = (x_n)$ in a topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-convergent to a point $x \in X$, if $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ holds for any neighborhood U of x and it is denoted by $\mathcal{I} - lim x_n = x$.*

Remark 4. *If we consider $\mathcal{I}_\delta$ or $\mathcal{I}_{Fin}$, then ideal convergence is coincide with statistical or classical convergence, respectively.*

If $\mathcal{I}$ is an admissible ideal, then classical convergence implies $\mathcal{I}$-convergence. The converse statement is not true if X has at least two point, in generally. Let x and y be two different elements of X and $A \in \mathcal{I}$ be any set and consider a sequence $\tilde{x} = (x_n) \subset X$ with $x_n = x$ when $n \in A$ and $x_n = y$ when $n \notin A$. It is clear that the sequence $\tilde{x}$ is $\mathcal{I}$ convergent to y but not usual convergent.

Furthermore, the set of ideal convergent sequences and the set of convergent sequences are not comparable with respect to set inclusion for non-admissible ideal. To see this let us consider non-admissible ideal $\mathcal{I} = \mathcal{P}(2\mathbb{N})$. The real valued sequence $(x_n) = (\frac{1}{n})$ convergent to 0 in $\mathbb{R}$ with usual topology τ_e. Let $\varepsilon > 0$ be an arbitrary real number such that there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon < \frac{1}{n_0 - 1}$ holds. Then, following inclusion $\{1, 2, ..., n_0\} \subset \{n : |\frac{1}{n} - 0| > \varepsilon\}$ is satisfied. Since the set $\{1, 2, ..., n_0\} \notin I$, then $\{n : |\frac{1}{n} - 0| > \varepsilon\} \notin I$ holds. This implies that the sequence $(x_n) = (\frac{1}{n})$ is not $\mathcal{I}$ convergent to zero.

Similarly, if we consider a sequence (x_n) as follows:

$$x_n = \begin{cases} 0, & n = 2^{2k}, k \in K, \\ 1, & otherwise. \end{cases}$$

It is clear that this sequence ideal convergent to 1 but it is not convergent to any point in $\mathbb{R}$.

Definition 5. [1] *Let $\mathcal{I}$ be an ideal and X be a topological space. Then,*
(i) For a subset $A \subseteq X$, $\mathcal{I}$-closure of A is defined by

$$\bar{A}^{\mathcal{I}} := \{x \in X : \exists\ (x_n) \subset A : x_n \stackrel{\mathcal{I}}{\to} x\}.$$

(ii) A subset $F \subseteq X$ is said to be $\mathcal{I}$-closed if $\bar{F}^{\mathcal{I}} = F$ holds.
(iii) A subset $A \subseteq X$ is said to be $\mathcal{I}$-open if $X - A$ is $\mathcal{I}$-closed.

It is clear that $\bar{\phi}^{\mathcal{I}} = \phi$ and $A \subseteq \bar{A}^{\mathcal{I}}$ hold. Also, it can be easily seen that any open subset of topological space $(X, \tau, \mathcal{I})$ is also $\mathcal{I}$-open.

In the paper [22], $\mathcal{I}$-closure and $\mathcal{I}^*$-closure of a set A was defined by using $\mathcal{I}$ non-thin sequences. Let us recall it: A sequence $\tilde{x} = (x_n)_{n\in M}$ is called $\mathcal{I}$-thin if $M \in \mathcal{I}$, otherwise it is called $\mathcal{I}$-non-thin. Then, $\mathcal{I}$-closure of a set A is

$$\bar{A}^{\mathcal{I}} := \{x \in X : \exists\ (x_n)_{n\in M} \subset A : (x_n)_{n\in M} \stackrel{\mathcal{I}_M}{\to} x\}$$

where $\mathcal{I}_M := \{M \cap A : A \in \mathcal{I}\}$.

It is clear that $\mathcal{I}_M$ is an ideal $\mathcal{I}_M \subset \mathcal{I}$ for any subset $M \subset \mathbb{N}$.

Remark 5. *It is clear that $\mathcal{I}_M$ is an (admissible) ideal for any (admissible) ideal and $\mathcal{I}_M \subset \mathcal{I}$ holds for any subset $M \subset \mathbb{N}$.*

Theorem 1. *Let (X, τ) be a topological space, $\mathcal{I}$ be an ideal and $M \notin \mathcal{I}$. Then, $(x_n)_{n\in M} \stackrel{\mathcal{I}_M}{\to} x$ if and only if $(x_n)_{n\in M} \stackrel{\mathcal{I}}{\to} x$*

Proof. From the definitions, proof can be obtained easily. So it is omitted here. □

2. Further properties of $\mathcal{I}^*$- sequential topological space

Through the paper, we consider any ideal unless said otherwise. Let's remember the definition of $\mathcal{I}^*$-convergence of sequences for any ideal $\mathcal{I}$.

Definition 6. [13] *Let $(X, \tau, \mathcal{I})$ be a topological space. A sequence $\tilde{x} = (x_n)$ in X is said to be $\mathcal{I}^*$-convergent to a point $x \in X$ if there exist a set $M \in \mathscr{F}(\mathcal{I})$ where*

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$$

such that for any neighborhood U of x, there exists $N(U) \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m_k > N(U)$.

If X has an algebraic structure, then the Definition 6 can be reformulated in the following form as called decomposition theorem:

Theorem 2. *A sequence $\tilde{x} = (x_n)$ in $(X, \tau, \mathcal{I})$ is $\mathcal{I}^*$-convergent to $x \in X$ if and only if it can be written as $x_n = t_n + s_n$ for all $n \in \mathbb{N}$ such that $\tilde{t} = (t_n) \subset X$ is a $\mathcal{I}_{Fin}$-convergent to x and $\tilde{s} = (s_n) \subset X$ is non zero only in a set of $\mathcal{I}$.*

Proof. Assume that $x_n := t_n + s_n$ is satisfied for all $n \in \mathbb{N}$ where $t_n \to x(\mathcal{I}_{Fin})$ and (s_n) is non zero only in a set from ideal $\mathcal{I}$. Since $t_n \to x(\mathcal{I}_{Fin})$, then for any neighborhood U of x

$$\{n \in \mathbb{N} :\ t_n \notin U\} \in \mathcal{I}_{Fin}$$

holds. Let $M := \mathbb{N} - \{n \in \mathbb{N} :\ t_n \notin U\}$. Then, $s_n = 0$ for all $n \in M$. So, $x_n = t_n$ and this implies that for any neighborhood U of x $x_n \in U$ holds for all $n \in M$. Hence, $x_n \stackrel{\mathcal{I}^*}{\to} x$.

Conversely, let $x_n \xrightarrow{\mathcal{I}^*} x$. Then, there exists $M \in F(\mathcal{I})$ such that $(x_n)_{n\in M}$ convergent to x. Take into consider sequences $\tilde{t} = (t_n)$ and $\tilde{s} = (s_n)$ as follow

$$t_n := \begin{cases} x_n, & n \in M, \\ x, & n \notin M, \end{cases} \text{ and } s_n := \begin{cases} 0, & n \in M, \\ x_n - x, & n \notin M. \end{cases}$$

It is clear that $t_n \to x(\mathcal{I}_{Fin})$ and (s_n) is nonzero only on a set from the ideal $\mathcal{I}$ and $x_n = t_n + s_n$ holds for all $n \in \mathbb{N}$. □

In [13], it was pointed out that $\mathcal{I}^*$-convergence implies that $\mathcal{I}$-convergence. In the following example, we will show that the converse statement is not true, in generally.

Example 3. *Let $(\mathbb{R}, \tau_e)$ be an Euclidean topological space and let $B_n(0) := (-\frac{1}{2n}, \frac{1}{2n})$ for $n \in \mathbb{N}$ be a monotonically decreasing open base at zero. Define a real valued sequence $\tilde{x} = (x_n)$ such that*

$$x_n \in B_n(0) - B_{n+1}(0)$$

where $x_n = \frac{2n+1}{4n^2+4n}$. It is clear that $x_n \to 0$, $n \to \infty$.

Consider the ideal given in Example 2 and let us note that any Δ_i is a member of $\mathcal{I}$.

Let $\tilde{y} = (y_n)$ be a sequence defined by $y_n = x_j$ if $n \in \Delta j$. Let U be any open set containing zero. Choose a positive integer m such that $B_n(0) \subset U$ holds for all $n > m$. Then,

$$\{n : y_n \notin U\} \subset \Delta_1 \cup \Delta_2 \cup \Delta_3 ... \cup \Delta_m \in \mathcal{I}$$

implies that $y_n \xrightarrow{\mathcal{I}} 0$ satisfies.

Now, suppose that $y_n \xrightarrow{\mathcal{I}^*} 0$ holds. Hence, there exists a set

$$M := \mathbb{N} - H = \{m_1 < m_2 < ... < m_k ...\} \in \mathscr{F}(\mathcal{I})$$

where $H \in \mathcal{I}$ such that for any neighborhood U of zero there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ for all $m_k > N$.

Let $l \in \mathbb{N}$ be a fixed number and assume that

$$H \subset \Delta_1 \cup \Delta_2 \cup \Delta_3 ... \cup \Delta_l$$

then $\Delta_i \subset \mathbb{N} - H$ holds for all $i > l+1$. Therefore, for each $i > l+1$, there is infinitely many k's such that $y_{m_k} = x_i$. But, the limit $lim y_{n_k}$ doesn't exists because of $x_i \neq x_j$ for all $i \neq j$.

Theorem 3. *Let (X, τ) be a topological space, and $\mathcal{I}$ be a finite ideal. Then, $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are coincided.*

Proof. We already know that if $x_n \xrightarrow{\mathcal{I}^*} x$ then $x_n \xrightarrow{\mathcal{I}} x$ for any ideal. Let a sequence $x_n \xrightarrow{\mathcal{I}} x$, then for any neighborhood U of x, we have $A := \{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

Consider $M = \mathbb{N} - A \in \mathscr{F}(\mathcal{I})$ and arrange M as

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\}.$$

Since the set A is finite, then there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m > N$. Therefore, $x_n \xrightarrow{\mathcal{I}^*} x$, hods. □

Theorem 4. *Let $(X, \tau, \mathcal{I})$ be a topological space. If every sub-sequence (x_{n_k}) of $(x_n) \subseteq X$ is $\mathcal{I}^*$- convergent to a point $x_0 \in X$, then (x_n) is $\mathcal{I}^*$- convergent to x_0.*

Proof. Let us assume that $(x_{n)}$ is not $\mathcal{I}^*$-convergent to point x_0. Then, for all $M \in \mathscr{F}(\mathcal{I})$ and for all $N \in \mathbb{N}$ there exists $n_k > N$ such that $x_{n_k} \notin U$, where U is any neighborhood of x_0. If we take $N = 1$, then there exists the sub-sequence $(x_{n_k}) \notin U$, for all $n_k > 1$. This means that there exists a sub-sequence of (x_n) which is not converging to the point of x_0 which is contradiction. □

Now, let's see with the following example that the converse of Theorem 4 is not true, in generally.

Example 4. *Let $(\mathbb{R}, \tau_e)$ be a topological space, $\mathcal{I}$ be any ideal and $K \in \mathscr{F}(\mathcal{I})$ be an arbitrary set. Define a sequence as*

$$y_n = \begin{cases} 2^n, & n \notin K, \\ \frac{1}{n}, & n \in K. \end{cases}$$

The sequence (y_n) is $\mathcal{I}^*$-convergent to zero but its subsequence $(y_{n_k}) = (2^{n_k})$ for $n_k \notin K$ is not $\mathcal{I}^*$-convergent.

Lemma 2. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{J}$ and $\tilde{x} = (x_n)$ be a sequence in a topological space (X, τ). Then, $x_n \xrightarrow{\mathcal{I}^*} x$ implies $x_n \xrightarrow{\mathcal{J}^*} x$.*

Proof. Let $(x_n) \xrightarrow{\mathcal{I}^*} x$ holds. That is, there exists $M \in \mathscr{F}(\mathcal{I})$ as

$$M = \{m_1 < m_2 < \dots, < m_k < \dots\}$$

such that for any neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_{m_k} \in U$ holds for all $m_k > N$ holds. Since $\mathbb{N} - M \in \mathcal{I}$, then from the assumption $\mathbb{N} - M \in \mathcal{J}$ is satisfied. So, $(x_n) \xrightarrow{J^*} x$. □

It is stated in (Lemma 2 in [1]) that every subsequence of $\mathcal{I}$-convergent sequence in a topological space (X, τ) is also $\mathcal{I}$-convergent. Moreover, Example 4 shows that this statement is not true for $\mathcal{I}^*$-convergence. Because of this reason, when defining $\mathcal{I}^*$-closure of a set A, the sequence itself will be considered instead of its subsequences.

Definition 7. *Let $(X, \tau, \mathcal{I})$ be a topological space. Then,*
(i) $\mathcal{I}^*$-Closure of a set A is defined by

$$\overline{A}^{\mathcal{I}^*} := \{x \in X : \exists (x_n) \subset A \text{ such that } x_n \xrightarrow{\mathcal{I}^*} x\}$$

(ii) A subset $F \subseteq X$ is said to be $\mathcal{I}^*$-closed if $\overline{F}^{\mathcal{I}^*} = F$ holds.
(iii) A subset $U \subseteq X$ is said to be $\mathcal{I}^*$-open if $X - U$ is $\mathcal{I}^*$-closed.

Remark 6. *It is clear that $\overline{\phi}^{\mathcal{I}^*} = \phi$ and $A \subset \overline{A}^{\mathcal{I}^*}$ are true for any $A \subseteq X$.*

Theorem 5. *Let (X, τ) be a topological space and $\mathcal{I}$ is an admissible ideal. Then, every $\mathcal{I}$-open subset is $\mathcal{I}^*$- open.*

Proof. Let U be an $\mathcal{I}$-open subset of X. Then, $X - U$ is $\mathcal{I}$-closed such that $X - U = \overline{X - U}^{\mathcal{I}}$ holds. To prove $X - U = \overline{X - U}^{\mathcal{I}^*}$ it is sufficient to show that $\overline{X - U}^{\mathcal{I}^*} \subset X - U$ holds. Let $x \in \overline{X - U}^{\mathcal{I}^*}$ be an arbitrary element. Then, there exists a sequence $\tilde{x} = (x_n) \subset X - U$ such that $x_n \xrightarrow{\mathcal{I}^*} x$ holds. Therefore, Theorem 1 gives that $x_n \xrightarrow{\mathcal{I}} x$ holds. This implies that $x \in \overline{X - U}^{\mathcal{I}} = X - U$. Hence, the proof is completed. □

Corollary 1. *Let (X,τ) be a topological space and $\mathcal{I}$ be a finite ideal. Then, $A \subset X$ is $\mathcal{I}$-open if and only if A is $\mathcal{I}^*$-open.*

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideal such that $\mathcal{I} \subset \mathcal{J}$ and X be a topological space. If $U \subset X$ is $\mathcal{J}^*$- open then it is $\mathcal{I}^*$-open.*

Proof. Let U be $\mathcal{J}^*$-open then $X - U$ is $\mathcal{J}^*$-closed and $X - U = \overline{X - U}^{\mathcal{J}^*}$ holds.

We must to prove $\overline{X - U}^{\mathcal{I}^*} \subset X - U$. Let $x \in \overline{X - U}^{\mathcal{I}^*}$ be an arbitrary point, then there exists a sequence $(x_n) \subset X - U$ such that (x_n) is $\mathcal{I}^*$-convergent to a point $x \in X - U$. Then by Theorem 2 the sequence (x_n), $\mathcal{J}^*$ converges to x. Hence, $x \in \overline{X - U}^{\mathcal{J}} = X - U$ this implies that $x \in X - U$ and U is $\mathcal{J}^*$-open. □

Definition 8. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, $\mathcal{I}^*$ interior of A is defined as*

$$A^{o^{\mathcal{I}^*}} := A - \overline{(X - A)}^{\mathcal{I}^*}.$$

Lemma 3. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the set A is $\mathcal{I}^*$-open if and only if $A^{o^{\mathcal{I}^*}} = A$.*

Proof. Let A be $\mathcal{I}^*$-open subset of topological space $(X,\tau,\mathcal{I})$. Then, $X - A$ is $\mathcal{I}^*$-closed and $X - A = \overline{X - A}^{\mathcal{I}^*}$ holds. This implies that

$$A^{o^{\mathcal{I}^*}} = A - \overline{(X - A)}^{\mathcal{I}^*} = A - (X - A) = A.$$

Conversely assume that $A = A^{o^{\mathcal{I}^*}}$ holds. Considering the definition, the equality $A = A - \overline{(X - A)}^{\mathcal{I}^*}$ is obtained. This implies that $A \cap \overline{(X - A)}^{\mathcal{I}^*} = \phi$ holds. Therefore, $\overline{(X - A)}^{\mathcal{I}^*} \subset X - A$. Hence, $X - A$ is $\mathcal{I}^*$- closed and the set A is $\mathcal{I}^*$-open. □

Theorem 7. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the following statements are equivalent:*

(i) A is $\mathcal{I}^*$- closed.

(ii) $A = \bigcap\{F : F \text{ is } \mathcal{I}^* - \text{ closed and } A \subset F\}$.

Proof. From the definitions it is obvious that $(i) \Rightarrow (ii)$. So, we are going to prove $(ii) \Rightarrow (i)$. To show that $\bar{A}^{\mathcal{I}^*} = A$ holds it is sufficient to prove that $\bar{A}^{\mathcal{I}^*} \subseteq A$ holds. Let $x_0 \in \bar{A}^{\mathcal{I}^*}$ is an arbitrary point, then there exists a sequence $(x_n) \subset A$ such that (x_n) is $\mathcal{I}^*$-convergent to x_0. Assume that $x_0 \notin A$. So, (ii) implies that

$$x_0 \notin \bigcap\{F : F \text{ is } \mathcal{I}^* - \text{ closed and } A \subset F\}.$$

Hence, there exists an $\mathcal{I}^*$-closed set F such that $A \subset F$ and $x_0 \notin F$. Since $(x_n) \subset A \subset F$, then $x_0 \in F$ which is a contradiction to assumption. □

Theorem 8. *Let A be a subset of topological space $(X,\tau,\mathcal{I})$. Then, the following statements are equivalent:*

(i) A is $\mathcal{I}^*$- open.

(ii) $A = \bigcup\{U : U \text{ is } \mathcal{I}^* - \text{open and } U \subset A\}$.

Proof. From the definitions $(i) \Rightarrow (ii)$ is obvious. So, we are going to prove inverse of this case. Let us consider $A = \bigcup\{U : U \text{ is } \mathcal{I}^* - \text{open and } U \subset A\}$. To prove A is $\mathcal{I}^*$- open subset of X, we must to show that $A = A^{o^{\mathcal{I}^*}}$ holds. It is known that $A^{o^{\mathcal{I}^*}}$ always subset of A. So, it is sufficient to show that $A \subset A^{o^{\mathcal{I}^*}}$ holds.

Let $x_0 \in A$ be an arbitrary point, then there is an open subset U of A such that $x_0 \in U$. Since $U \subset A$ then $x_0 \in A^{o^{\mathcal{I}^*}}$ and this implies that $A \subset A^{o^{\mathcal{I}^*}}$ holds. □

Definition 9. *Let $\mathcal{I}$ be an ideal and U be a subset of topological space X. A sequence $\tilde{x} = (x_n) \subset X$ is $\mathcal{I}^*$-eventually in U if there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ holds for all $m \in M$.*

In the following a new characterization will be given for the $\mathcal{I}^*$-open set.

Proposition 1. *Let $\mathcal{I}$ be a maximal ideal and (X, τ) be a topological space. Then, a subset $U \subseteq X$ is $\mathcal{I}^*$-open if and only if each $\mathcal{I}^*$-convergent sequence to a point $x \in U$ in X is $\mathcal{I}^*$-eventually in U.*

Proof. Let us assume that U be an $\mathcal{I}^*$ open subset of (X, τ). Consider an arbitrary sequence $\tilde{x} = (x_n) \subset X$ which is $\mathcal{I}^*$-convergent to a point $x \in U$. Since U is $\mathcal{I}^*$-open, then it is neighborhood of the point x. So, $E := \{n : x_n \notin U\} \in \mathcal{I}$ and $M(= \mathbb{N} - E) = \{n : x_n \in U\} \in \mathscr{F}(\mathcal{I})$ holds. Hence, for all $m \in M$ such that $x_m \in U$ and this implies that $\tilde{x}$ is $\mathcal{I}^*$-eventually in U.

Let us assume each $\mathcal{I}^*$-convergent sequence to a point $x_0 \in U$ is $\mathcal{I}^*$-eventually in U. That is, if $\tilde{x}$ is a sequence which is $\mathcal{I}^*$-convergent to $x_0 \in U$, then there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ holds for all $m \in M$. Now, we are going to show that U is $\mathcal{I}^*$-open. It is enough to prove $X - U$ is $\mathcal{I}^*$-closed. To do this we will focus the inclusion $\overline{X - U}^{\mathcal{I}^*} \subseteq (X - U)$ is satisfied. Let $x \in \overline{X - U}^{\mathcal{I}^*}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset (X - U)$ such that (x_n) is $\mathcal{I}^*$-convergent to x. Assume that $x \in U$. From the assumption there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ for all $m \in M$, but we have $x_n \in X - U$, for all n which is contradiction. Hence $x \in X - U$ and U is $\mathcal{I}^*$-open. □

Lemma 4. *Let $\mathcal{I}$ be an admissible ideal and (X, τ) be a topological space. If U and V are $\mathcal{I}^*$-open subsets of X, then $U \cap V$ is $\mathcal{I}^*$-open.*

Proof. Let $\tilde{x} = (x_n)$ be an $\mathcal{I}^*$-convergent sequence in X which convergent to a point $x \in U \cap V$. Since U and V are $\mathcal{I}^*$-open sets and the sequence $\tilde{x}$ is $\mathcal{I}^*$-converging to a point x in U also in V. So, by the help of Proposition 1, the sequence $\tilde{x}$ is $\mathcal{I}^*$-eventually in U and also in V. Then, there exists $M_1, M_2 \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$ for all $m \in M_1$ and $x_m \in V$ for all $m \in M_2$. If we consider $M = M_1 \cap M_2 \in \mathscr{F}(\mathcal{I})$, then $x_m \in U \cap V$ holds for all $m \in M$. This shows that $U \cap V$ is $\mathcal{I}^*$-open subset of X. □

Theorem 9. *Let $\mathcal{I}$ be a maximal ideal and (X, τ) be a topological space. A sequence $\tilde{x} = (x_n) \subset X$ is $\mathcal{I}^*$-convergent to an element $x \in X$ if and only if for any $\mathcal{I}^*$-open subset U of X with $x \in U$, there exists $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$, for all $m \in M$.*

Proof. Let $\mathcal{I}$ be a maximal ideal and $\tilde{x} = (x_n)$ be an $\mathcal{I}^*$-convergent sequence to $x \in X$. Let U be an $\mathcal{I}^*$-open subset of X with $x \in U$. Then, $\tilde{x}$ will be $\mathcal{I}^*$-eventually in U. Hence, there exists a set $M \in \mathscr{F}(\mathcal{I})$ such that $x_m \in U$, for all $m \in M$.

The converse statement is clear from the definition of $\mathcal{I}^*$-convergence. So, it is omitted here. □

Theorem 10. *($\mathcal{I}^*$-sequential topology) Let $(X, \tau, \mathcal{I})$ be a topological space. Then, the family*

$$\tau_{\mathcal{I}^*} := \{U \in P(X) : U \text{ is } \mathcal{I}^* - \text{open set}\}$$

is a topology on X.

Proof. It is obvious that X and ϕ are $\mathcal{I}^*$-open sets. By Lemma 4, we can say that finite intersection of $\mathcal{I}^*$-open sets is $\mathcal{I}^*$-open.

Let $(U_\alpha)_{\alpha\in\Lambda}$ be an arbitrary family of elements of $\tau_{\mathcal{I}^*}$. We are going to show that their union belongs to $\tau_{\mathcal{I}^*}$. Since

$$X - \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcap_{\alpha\in\Lambda}(X - U_\alpha),$$

then it is sufficient to show that $\bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ is $\mathcal{I}^*$- closed. That is,

$$\overline{\bigcap_{\alpha\in\Lambda}(X - U_\alpha)}^{\mathcal{I}^*} = \bigcap_{\alpha\in\Lambda}(X - U_\alpha).$$

Let $x \in \overline{\bigcap_{\alpha\in\Lambda}(X - U_\alpha)}^{\mathcal{I}^*}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset \bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ such that $x_n \overset{\mathcal{I}^*}{\to} x$ holds. Therefore, for all $\alpha \in \Lambda$ the sequence $(x_n) \subseteq (X - U_\alpha)$ and $x_n \overset{\mathcal{I}^*}{\to} x$. Since the set $X - U_\alpha$ is $\mathcal{I}^*$- closed for all $\alpha \in \Lambda$, then $x \in X - U_\alpha$. Hence, $x \in \bigcap_{\alpha\in\Lambda}(X - U_\alpha)$ thus $\bigcap_{\alpha\in\Lambda} X - U_\alpha$ is $\mathcal{I}^*$-closed. □

Theorem 11. *If $\mathcal{I}$ is admissible ideal and the topological space (X, τ) has no limit point, then every $\mathcal{I}^*$-open set is $\mathcal{I}$-open set.*

Proof. Let U be an $\mathcal{I}^*$-open set, i.e. $X - U$ is $\mathcal{I}^*$-closed. To prove U is $\mathcal{I}$-open, it is enough to show that $X - U$ is $\mathcal{I}$-closed set. It is clear that $X - U \subseteq \overline{X - U}^{\mathcal{I}}$ holds. Let $x \in \overline{X - U}^{\mathcal{I}}$ be an arbitrary point. Then, there exists a sequence $(x_n) \subset X - U$ such that (x_n) is $\mathcal{I}$-convergent to x.

Since $\mathcal{I}$ is admissible and X has no limit point, then by [13] the sequence (x_n) will be $\mathcal{I}^*$-convergent to x. Therefore, $x \in \overline{X - U}^{\mathcal{I}^*}$. This implies that $x \in X - U$ holds. □

Corollary 2. *Under the assumption of Theorem 12, $\mathcal{I}$-sequentially and $\mathcal{I}^*$-sequentially topology are coincide.*

Definition 10. [13] *Let $\mathcal{I}$ be an ideal of $\mathbb{N}$, it is said that the ideal $\mathcal{I}$ satisfies additive property (AP) if for every countable family $(A_i)_{i\in\mathbb{N}} \subset I$, there exists a countable family $(B_i)_{i\in\mathbb{N}}$ of sets such that $A_i \triangle B_i \in F(\mathcal{I})$ for all $i \in \mathbb{N}$ and $B = \cup B_i \in I$.*

Theorem 12. *Let $\mathcal{I}$ be an admissible ideal which has the (AP)-property, and (X, τ) is first countable topological space. Then, every $\mathcal{I}^*$-open subset of X is $\mathcal{I}$-open.*

Proof. Let U be an arbitrary $\mathcal{I}^*$-open subset of X. Then, $X - U$ is $\mathcal{I}^*$-closed, so $X - U = \overline{X - U}^{\mathcal{I}^*}$ holds. To proof U is $\mathcal{I}$-open, we must show that it's complement is $\mathcal{I}$-closed. Let $x \in \overline{X - U}^{\mathcal{I}}$ be an arbitrary point. Then, there exists a sequence $x_n \subset X - U$ such that it is $\mathcal{I}$-converging to the point x. As the ideal $\mathcal{I}$ has (AP)-property and the space X is first countable by [13] it is $\mathcal{I}^*$-converging to x. So $x \in \overline{X - U}^{\mathcal{I}^*}$. So, $x \in X - U$. Hence, this fact implies that $X - U$ is $\mathcal{I}$-closed and U is $\mathcal{I}$-open. □

Corollary 3. *Under the assumption of Theorem 12, it can be say that $\mathcal{I}$-sequential and $\mathcal{I}^*$-sequential topology are coincide.*

Definition 11. *Let $(X, \tau_1, \mathcal{I})$ and $(Y, \tau_2, \mathcal{I})$ be two topological space and $f : X \to Y$ be a function. The function f is said to be (i) $\mathcal{I}^*$-continuous if $f^{-1}(U)$ is $\mathcal{I}^*$-open subset of X for every $\mathcal{I}^*$-open subset U of Y.*

(ii) sequentially $\mathcal{I}^*$-continuous if $f(x_n)$ is $\mathcal{I}^*$-convergent to $f(x)$ for each sequence (x_n) in X which (x_n) is $\mathcal{I}^*$ convergent to x.

It is well known that the definitions given above are not necessarily equivalent in classical topological spaces. In the following theorem we will show that they are equivalent notions for typologies produced with the help of ideals.

Theorem 13. *Let $(X, \tau_1, \mathcal{I})$ and $(Y, \tau_2, \mathcal{I})$ be two topological space and $f : X \to Y$ be a function. Then, f is sequentially $\mathcal{I}^*$-continuous if and only if f is $\mathcal{I}^*$-continuous function.*

Proof. Let f be a sequentially $\mathcal{I}^*$-continuous function and U be any $\mathcal{I}^*$-open set in Y. Assume that $f^{-1}(U)$ is not $\mathcal{I}^*$- open in X, equivalently $X - f^{-1}(U)$ is not $\mathcal{I}^*$- closed. We conclude from the assumption that $\overline{X - f^{-1}(U)}^{\mathcal{I}^*}$ is not subset of $X - f^{-1}(U)$. So, there exists a point $x \in \overline{X - f^{-1}(U)}^{\mathcal{I}^*}$ such that $x \notin X - f^{-1}(U)$. This means that there exists a sequence $(x_n) \subset X - f^{-1}(U)$ such that it is $\mathcal{I}^*$-converging to x and $x \in f^{-1}(U)$. Since f is sequentially continuous, then the sequence $f(x_n)$ is I^*-converging to $f(x)$. This implies that $f(x_n) \subset Y - U$ which is not in case so $f^{-1}(U)$ is $\mathcal{I}^*$-open subset of X.

Let $f : X \to Y$ be an $\mathcal{I}^*$-continuous mapping and assume that $x_n \xrightarrow{\mathcal{I}^*} x$. Then, for any neighborhood U of x, there exists $N \in \mathbb{N}$ and $M \in \mathscr{F}(\mathcal{I})$ such that $x_{m_k} \in U$ for all $m_k \in M$. Let V be any $\mathcal{I}^*$-open neighborhood of $f(x)$, then $f^{-1}(V) \subset X$ is $\mathcal{I}^*$-open and contain x. Hence, there exists $N \in \mathbb{N}$ and $M \in \mathscr{F}(\mathcal{I})$ such that $x_{m_k} \in f^{-1}(V)$. As a result of this discussion, it can easily be seen that $f(x_{m_k}) \in V$ hence $f(x_n) \xrightarrow{\mathcal{I}^*} f(x)$. □

3. Sequentially $\mathcal{I}^*$-compactness

The notion of compactness which is one of the most significant topological properties of the sets was formally introduced by M. Frechet in 1906. There are many different type of compactness introduced by mathematicians over time. Recently, using the concept of ideal the concept of $\mathcal{I}$-compactness was defined by Newcomb in [15] and studied by Rancin in the paper [17]. In this section, we will go one step further and define the concept of $\mathcal{I}^*$-sequentially compactness and examine some of its basic properties.

Let's start with the concept of boundedness in normed space which is directly related to compactness.

Definition 12. [20] *Let $(X, \|.\|)$ be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. A sequence $\tilde{x} = (x_n)$ in X is called (i) $\mathcal{I}$-bounded if there exist $K > 0$ such that $\{n \in \mathbb{N} : \|x_n\| > K\} \in \mathcal{I}$ holds.*

(ii) $\mathcal{I}^*$-bounded if there exists $M \in \mathscr{F}(\mathcal{I})$ such that $(x_n)_{n \in M}$ is bounded.

Remark 7. *Let $(X, \|.\|)$ be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. Then, every $\mathcal{I}$-bounded sequence is $\mathcal{I}^*$-bounded.*

Proof. Assume that $(x_n) \subset X$ is $\mathcal{I}$-bounded sequence in X. Then, there exists $K > 0$ such that $\{n : \|x_n\| > K\} \in \mathcal{I}$ holds. If we denote $M := \{m : \|x_n\| < K\}$.

Then, $M \in \mathscr{F}(\mathcal{I})$ and $\|x_n\| < K$ holds for all $n \in M$. Hence, (x_n) is $\mathcal{I}^*$- bounded sequence. □

Corollary 4. *Let X be a normed space and $\mathcal{I}$ be an ideal of $\mathbb{N}$. Then, every bounded sequence is $\mathcal{I}^*$-bounded.*

Proof. Let X be a normed space and $(x_n) \subset X$ be a bounded sequence in X. Then, the sequence (x_n) is $\mathcal{I}$-bounded which is given in [1] and by Remark 7 it is $\mathcal{I}^*$-bounded. □

Definition 13. *Let $(X, \tau, \mathcal{I})$ be a topological space. A subset $F \subset X$ is said to sequentially $\mathcal{I}^*$-compact if any sequence $(x_n) \subset F$ has an $\mathcal{I}^*$-convergent subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{I^*} x \in F$.*

Theorem 14. *Let $(X, \tau, \mathcal{I})$ be a topological space and $f : X \to \mathbb{R}$ be a sequentially $\mathcal{I}^*$-continuous function. If A is sequentially $\mathcal{I}^*$-compact subset of X, then $f(A)$ is $\mathcal{I}^*$-bounded.*

Proof. On the contrary assume that $f(A)$ is not $\mathcal{I}^*$-bounded. Then, there exists a sequence (y_n) in $f(A)$ such that it is not $\mathcal{I}^*$-bounded. That is

$$\{n \in \mathbb{N} : |y_n| < M\} \notin \mathscr{F}(\mathcal{I})$$

holds for all positive $M > 0$. Also, there exists a sequence (x_n) in A such that $f(x_n) =: y_n$ holds for all $n \in \mathbb{N}$. Since A is sequentially $\mathcal{I}^*$- compact, then there exists a subsequence (x_{n_k}) of (x_n) which is $\mathcal{I}^*$-convergent to a point x_0 in A. Moreover, f is sequentially $\mathcal{I}^*$-continuous function then $f(x_n)$ is $\mathcal{I}^*$-convergent to $f(x_0)$. So, there exists $E \in \mathscr{F}(\mathcal{I})$ where

$$E = \{m_1 < m_2 < \cdots < m_k < \cdots\}$$

such that for any neighborhood U of $f(x_0)$, there exists $N \in \mathbb{N}$ such that $f(x_{n_{m_k}}) \in U$ holds for all $m_k > N$. As a result of this analysis, it can be say that $(y_n) = f(x_{n_k})$ is $\mathcal{I}$-convergent to $f(x_0)$. Then, $\{n \in \mathbb{N} : |f(x_n)| > M\} \in \mathcal{I}$ holds for any neighborhood U of $f(x_0)$. So, we have $\{n \in \mathbb{N} : |x_n| < M\} \in \mathscr{F}(\mathcal{I})$ which is not in case so $f(A)$ is $\mathcal{I}^*$-bounded. □

Lemma 5. *Let $(X, \tau, \mathcal{I})$ and $(Y, \tau, \mathcal{I})$ be topological spaces. If X is sequentially $\mathcal{I}^*$-compact and $f : X \to Y$ is sequentially $\mathcal{I}^*$-continuous function, then $f(X)$ is sequentially $\mathcal{I}^*$-compact.*

Proof. It can be proved easily. So it is omitted. □

4. CONCLUSIONS AND SOME REMARKS

In the paper, we defined the $\mathcal{I}^*$-sequential topology on a topological space (X, τ) and proved that $\mathcal{I}^*$-sequential topology is finer then $\mathcal{I}$-sequential topology. Also, we observed that under the conditions of if the space X has no limit point and $\mathcal{I}$ be an admissible ideal then, the $\mathcal{I}$-sequentially topology and the $\mathcal{I}^*$-sequentially topology are coincide, i.e $\tau_{\mathcal{I}} = \tau_{\mathcal{I}^*}$. Also, If $\mathcal{I}$ is an admissible ideal with (AP)-property, and (X, τ) is a first countable topological space, then $\mathcal{I}$-sequentially topology and $\mathcal{I}^*$-sequentially topology are coincide, i.e $\tau_{\mathcal{I}} = \tau_{\mathcal{I}^*}$. Interestingly, it has been proven that the concepts of $\mathcal{I}^*$-continuity and $\mathcal{I}^*$-sequential continuity of a function are equivalent. As a continuation of this study, some questions can be asked:

$Q1$: Is there any topology (different from discrete topology) over X that are finer than the $\mathcal{I}^*$-sequentially topology?

$Q2$: Is there any sequential type topology between $\mathcal{I}$-sequential topology and $\mathcal{I}^*$-sequential topology on topological space X?

$Q3$: Can $\mathcal{I}$-sequential topology (or $\mathcal{I}^*$-sequential topology) be metrizable?

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