Mathematical Modelling and Numerical Simulation with Applications, 2024, 4(4), 514–543

https://dergipark.org.tr/en/pub/mmnsa
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.1555670

RESEARCH PAPER

A study of fractional optimal control of overweight and obesity in a community and its impact on the diagnosis of diabetes

Erick Manuel Delgado Moya [1,*,‡], **Ranses Alfonso Rodriguez** [2,‡], **Alain Pietrus** [3,‡] **and Séverine Bernard** [3,‡]

[1]School of Technology, University of Campinas (UNICAMP), R. Paschoal Marmo, 1888 - Jd. Nova Itália, 484-332 - Limeira, SP, Brazil, [2]Department of Applied Mathematics, Florida Polytechnic University, Lakeland, FL 33805, USA, [3]Department of Mathematics and Computer Sciences, University of Antilles, LAMIA (EA 4540), BP 250, 97159, Pointe-a-Pitre, Guadeloupe, France

*Corresponding Author

‡erickdelgadomoya@gmail.com (Erick Manuel Delgado Moya); ranses.alfonso@gmail.com (Ranses Alfonso Rodriguez); alain.pietrus@univ-antilles.fr (Alain Pietrus); severine.bernard@univ-antilles.fr (Séverine Bernard)

Abstract

Obesity and diabetes are diseases that are increasing every year in the world and their control is an important problem faced by health systems. In this work, we present an optimal control problem based on a model for overweight and obesity and its impact on the diagnosis of diabetes using fractional order derivatives in the Caputo sense. The controls are defined with the objective of controlling the evolution of an individual with normal weight to overweight and that overweight leads to chronic obesity. We show the existence of optimal control using Pontryagin's maximum principle. We perform a study of the global sensitivity for the model using Sobol's index of first, second and total order using the polynomial chaos expansion (PCE) with two techniques, ordinary least squares (OLS) and least angle regression (LAR) to find the polynomial coefficients, and two sampling methods, Monte Carlo and Sobol'. With the obtained results, we find that among the parameters with the greatest influence are those we used in the definition of the control system. We have that the best results are achieved when we activate the three controls. However, when we only activate two controls, it shows better results in preventing a person with normal weight from becoming overweight by controlling weight gain due to social pressure and the evolution from overweight to obesity. All strategies significantly reduce the number of cases diagnosed with diabetes over time.

Keywords: Optimal control; diabetes; global sensitivity analysis; obesity; overweight

AMS 2020 Classification: 49J15; 26A33; 37N25; 37N35; 92D30

➤ **Received:** 25.09.2024 ➤ **Revised:** 06.12.2024 ➤ **Accepted:** 23.12.2024 ➤ **Published:** 30.12.2024

1 Introduction

The prevalence of overweight and obesity increased worldwide in recent decades. According to the World Obesity Atlas 2023, the estimated value of overweight and obesity (body mass index $[BMI] \geq 25\frac{kg}{m^2}$) in 2020 among the global population aged more than 5 years was 38%, which is projected to reach 51% in 2035. In the United States (US), the prevalence of obesity ($[BMI] \geq 30\frac{kg}{m^2}$) increased from 30.5% in 1999-2000 to 41.9% in 2017-2020. By 2030, 48.9% of US adults may have obesity, with 24.2% falling into the severely obese category ($[BMI] \geq 35\frac{kg}{m^2}$) [1–3].
More than 500 million people are living with diabetes worldwide, and it is predicted to more than double to 1.3 billion people in the next 30 years. Almost all global cases (96%) diagnosed with diabetes are type 2 diabetes (T2D). High body mass index (BMI) was the primary risk for T2D – accounting for 52.2% of T2D disability and mortality – followed by dietary risks, environmental/occupational risks, tobacco use, low physical activity, and alcohol use [4].
To determine body weight state, we use the body mass index (BMI), which is defined as [5]:

$$BMI = \frac{\text{weight}}{\text{height}^2}.$$

Then individuals are considered of normal weight when $BMI \in [18.6, 24.9]$, overweight individuals are when $BMI \in [25, 29.9]$, obese individuals are when $BMI \in [30, 40]$ and in complicated situations over 40. We know that the body mass index can be high for people with high muscle mass but we assume that these cases are not included because before calculating the BMI we do a preliminary analysis. The root cause of obesity and overweight is an energy imbalance between consumed and expended calories.
Over the last few years, the study of mathematical models of prey-predator and those of general structures such as SIR (susceptible-infected-recovered), SIS (susceptible-infected-susceptible), SVIR (susceptible-vaccinated-infected-recovered) has continued its development and elements such as delay and non-local dispersion have been incorporated [6–9]. We also note a growing use of fractional derivatives in the modeling of epidemics such as HIV [10], COVID [11], Cancer [12], Cholera [13], Influenza [14], and Hepatitis B [15].
Recently, the use of mathematical models to study the behavior of overweight and obesity has increased [16–22]. Ejima et al. [16] presented a mathematical model that studies the genetic and nongenetic effects leading to obesity and among the results they obtained that homozygous individuals are more susceptible to the risk of social contagion and the risk of spontaneous weight gain. Kim and So-Yeun Kim [17] proposed a mathematical model with the inclusion of psychological and social factors for the study of obesity. Paudel [18] presented a model for the dynamics of obesity with a SIR structure and analyzed the effect of social network on the spread of obesity. Al-Tuwairqi and Matbouli [19] proposed two mathematical models to study the impact of fast food on the increase of obesity and the role of physical activity. Pietrus et al. [20] developed a mathematical model to study the impact of media on the dynamics of obesity in a population. Moya et al. [21] presented a model with ordinary differential equations that studies overweight, obesity and the impact on the diagnosis of diabetes in a population, taking into account the negative impact that an overweight or obese individuals can have on an individual of normal weight, and also the impact of social pressure on the increase of overweight cases. Based on article [21], Moya et al. [22] incorporated into the model the positive effect of the interaction between a normal weight individual and overweight or obese individuals can have and transformed the model for fractional order equations in the Caputo sense, taking advantage of the memory effect. This model is used in the formulation of the control problem addressed in this paper taking into

account the results obtained in [22].
The aim of this work is to propose and solve an optimal control problem focused on reducing overweight and obesity in a community and its impact on diabetes, taking advantage of a model that uses fractional order derivatives in the Caputo sense.
This paper is organized as follows: In Section 2, we present the definitions used in the paper. In Section 3, we introduce the model with fractional order derivatives in the Caputo sense. Section 4 contains the definition of controls, formulation of the control problem and demonstration of its basic properties. Section 5 is devoted to the study of the global sensitivity analysis of the model and the numerical simulations. We finish the paper with some conclusions in Section 6.

2 Theorical background

The following definitions are used to formulate and study the fractional order derivative model. We assume that $\alpha \in \mathbb{R}^+$, $b > 0$, $f \in AC^n[a,b]$ (absolutely continuous) and $n = [\alpha]$ (integer part of α). We define the left-sided and right-sided fractional integral Riemann-Louville for $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\alpha > 0$ as:

$$
\begin{aligned}
{}_a\mathbb{I}_t^\alpha f(t) &:= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \qquad \text{(Left)} \\
{}_t\mathbb{I}_b^\alpha f(t) &:= \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)ds}{(s-t)^{1-\alpha}}, \qquad \text{(Right)}
\end{aligned}
$$

where Γ is Gamma function, and we define $\mathbb{I}_t^\alpha f(t) = {}_0\mathbb{I}_t^\alpha f(t)$.
The left-sided and right-sided Riemann–Liouville fractional derivatives are defined as [23, 24]:

$$
\begin{aligned}
{}_a\mathbf{D}_t^\alpha f(t) &= \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds\right), \qquad \text{(Left)} \\
{}_t\mathbf{D}_b^\alpha f(t) &= \frac{d^n}{dt^n}\left(\frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f(s)ds\right), \qquad \text{(Right)}
\end{aligned}
$$

and we denote $\mathbf{D}_t^\alpha f(t) = {}_0\mathbf{D}_t^\alpha f(t)$.
The left-sided and right-sided fractional derivatives proposed by Caputo are given by [23, 24]:

$$
\begin{aligned}
{}_a^c\mathbb{D}_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-1-\alpha} f^{(n)}(s)ds, \qquad \text{(Left)} \\
{}_t^c\mathbb{D}_b^\alpha f(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-1-\alpha} f^{(n)}(s)ds, \qquad \text{(Right)}
\end{aligned}
$$

and we define ${}^c\mathbb{D}_t^\alpha f(t) = {}_0^c\mathbb{D}_t^\alpha f(t)$.
In the order fractional derivatives, we find the memory effect which is an important factor in epidemic modeling [25–27].
In recent decades, works have been presented where models with fractional orders are used and compared with real data and it has been obtained that they can capture real behaviors, see [28–31].
Now, we will present a general formulation of the fractional order optimal control problem (FOCP) and obtain the necessary conditions for the optimality of the FOCP: Find the optimal control $u(t)$ that minimizes the functional J defined as:

$$
J(u) = \int_0^b f(t,x,u)dt, \tag{1}
$$

subject to the controlled model

$$ {}^{c}\mathbb{D}_{t}^{\alpha}\, x(t) = g(t, x, u), \tag{2} $$

with initial condition

$$ x(0) = x_I, \tag{3} $$

where $x(t)$ and $u(t)$ are the state and control variables respectively, f and g are differential functions and $0 < \alpha \leq 1$.

Theorem 1 *If $f(x, u)$ is a minimizer of (1) that satisfies the constraint (2) and the initial condition (3), then there is a function $\lambda \in \mathbb{C}^1[0, b]$ such that the triplet (x, u, λ) satisfies:*

i. the co-state and state systems, respectively

$$ {}^{c}\mathbb{D}_{t}^{\alpha}\, x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), u(t), \lambda(t)), $$

$$ {}_{t}^{c}\mathbb{D}_{b}^{\alpha}\, \lambda(t) = \frac{\partial H}{\partial x}(t, x(t), u(t), \lambda(t)), $$

ii. the stationary condition

$$ \frac{\partial H}{\partial u}(t, x(t), u(t), \lambda(t)) = 0, $$

iii. and the condition of transversality

$$ {}_{t}\mathbb{I}_{b}^{1-\alpha}\, \lambda\big|_{t=b} = \lambda(b) = 0, $$

where the Hamiltonian H is defined by

$$ H(t, x, u, \lambda) = f(t, x, u) + \lambda(t) \cdot g(t, x, u). $$

Lemma 1 *The following equations are equivalent:*

$$ {}_{t}^{c}\mathbb{D}_{b}^{\alpha}\, \lambda(t) = \frac{\partial H}{\partial x}(t, x(t), u(t), \lambda(t)), $$
$$ {}^{c}\mathbb{D}_{t}^{\alpha}\, \lambda(b-t) = \frac{\partial H}{\partial x}(b-t, x(b-t), u(b-t), \lambda(b-t)), $$

where $\alpha \in (0, 1]$.

The proof of Theorem 1 and Lemma 1 can be found in [24] and its applications in [24, 32–34].

3 Model construction

In this section, we present the model that we will use in the definition of the optimal control problem. The current model is based on those with ordinary differential equations presented in [20] and [21] and the current version with fractional equations in Caputo's sense found in [22],

where we increase the positive and negative impact of the interactions between overweight, obese and normal weight individuals.

Based on the body mass index, we define the model compartments as: normal weight individuals, S, overweight individuals, O_w, obese individuals, O_b and diabetic individuals, D. We assume that for the diagnosis of overweight and obesity in atypical cases (where the physical examination shows that the patient has other factors that alter the BMI), medical examinations are necessary because there are cases where the body mass index may be increased by muscle, inflammations in the body and other factors.

We define the negative impact rate as

$$\lambda_O = (\alpha^*)^\alpha \frac{(O_w + \epsilon O_b)}{N},$$

and positive impact rate as

$$\lambda_S = \frac{(\beta^*)^\alpha S}{N},$$

where α^* and β^* are the effective contact rate and N is the total population ($N = S + O_w + O_b + D$). The rates M_S and M_D represent the entry of individuals with normal weight and diabetes respectively.

We assume that the impact of overweight and obese on a person with normal weight is different, for this, we use the modification parameter ϵ such that $\epsilon > 1$.

The rate α_1 characterizes cases diagnosed with diabetes that are not directly associated with weight gain including genetic, racial, hereditary and other factors. Parameters α_2 and α_3 are the rates of diabetes diagnoses associated with overweight and obesity, respectively. The rate of death associated with diabetes is denoted by μ_D.

The rate δ characterize individuals who are overweight but improve to normal weight and η is for obese individuals who become overweight. These two parameters are not related to the interaction with a normal weight individual. Individuals who increase in body weight from overweight to obese are defined by the rate γ.

The rate β_1 represents the social pressure that causes an individual with normal weight to become obese. This rate is characterized by stress, lack of time for healthy eating and physical exercise, sedentary lifestyle, etc.

The mortality rate from natural causes in the population is defined as μ and we assume that it is the same from any compartment. We define d as the mortality rate associated with overweigth and obesity.

Table 1 shows the definition of the parameters and their reference values and Figure 1 presents the dynamics diagram of the model.

The fractional derivative operator in the Caputo sense, ${}^c\mathbb{D}_t^\alpha$ has a α dimension, then on the right-hand side of the model all parameters will have power dimension α except the modification parameters [35]. The model that studies the behavior of overweight and obesity and its impact on the diagnosis of diabetes in a population using the fractional derivatives in the Caputo sense,

found in [22], is:

$$\begin{aligned}
{}^{c}\mathbb{D}_t^{\alpha} S &= M_S^{\alpha} + (\delta^{\alpha} + \lambda_S) O_w - (\mu^{\alpha} + \alpha_1^{\alpha} + \beta_1^{\alpha} + \lambda_O) S, \\
{}^{c}\mathbb{D}_t^{\alpha} O_w &= (\lambda_O + \beta_1^{\alpha}) S + \eta^{\alpha} O_b - (\lambda_S + \gamma^{\alpha} + \mu^{\alpha} + d^{\alpha} + \delta^{\alpha} + \alpha_2^{\alpha}) O_w, \\
{}^{c}\mathbb{D}_t^{\alpha} O_b &= \gamma^{\alpha} O_w - (\eta^{\alpha} + \mu^{\alpha} + t_H d^{\alpha} + \alpha_3^{\alpha}) O_b, \\
{}^{c}\mathbb{D}_t^{\alpha} D &= M_D^{\alpha} + \alpha_1^{\alpha} S + \alpha_2^{\alpha} O_w + \alpha_3^{\alpha} O_b - (\mu_d^{\alpha} + \mu^{\alpha} + t_D d^{\alpha}) D,
\end{aligned} \tag{4}$$

with initial conditions

$$S(0) = S_0 \geq 0,\ O_w(0) = O_{w0} \geq 0,\ O_b(0) = O_{b0} \geq 0,\ D(0) = D_0 \geq 0 \text{ and } \alpha \in (0, 1].$$

Table 1. Parameters description of model (4)

Parameter	Description	Value	Reference
M_S	Recruitment rates for normal weight	667.685	[36]
M_D	Recruitment rates for diabetic individuals	4.1	[21, 22]
α^*	Effective contact rates (negative impact)	2	[21, 22]
β^*	Effective contact rates (positive impact)	0.2	[21, 22]
d	death rate associated with weight gain	0.07	[21, 22]
μ	Natural death rate	$\frac{1}{70.5}$	[21, 22, 37]
μ_d	Diabetes death rate	0.013	[22]
η	Rate of weight reduction from obese to overweight	0.1	[20]
γ	Rate of weight gain from overweight to obese	0.0015	[20]
δ	Rate of weight reduction from overweight to normal weight	0.002	[20]
α_1	Rate of diagnosis of diabetes not associated with body weight	0.1	[21, 22]
α_2	Rate of diabetes diagnosis in overweight individuals	0.35	[21, 22, 38, 39]
α_3	Rate of diagnosis of diabetes in obese individuals	0.4	[21, 22, 38, 39]
β_1	Rate of weight gain associated with social factors	0.25	[21, 22]
ϵ, t_D, t_H	Modification parameters	1.02, 1.03	Assumed

For model (4), we proved the existence and non-negativity of the solution and found the biologically feasible region where the model makes biological sense and we calculated and studied the basic reproduction number in [22].

4 Definition of the controls and its policy

The objective of the controls is to reduce the number of overweight and obese individuals in the population and as a consequence avoid new diagnoses of diabetes. The controls are divided into three categories, the first control regulates negative relationships, the second the effect of social pressure and the third the evolution from overweight to obese. Controls are defined as:

- u_1: Control that regulates the negative impact of the influence that overweight and obese individuals have on individuals with normal weight in order to prevent an individual with normal weight from changing their lifestyle and becoming overweight.
- u_2: Social pressure leads individuals to give up a healthy life, whether due to overwork, sedentary lifestyle, or poor diet. This control focuses on regulating the impact of social pressure on weight gain in individuals with normal weight. This would include performing other activities, mainly physical exercise, walking in wooded areas and encouraging the individual to change their diet by continually avoiding fast food, time management, Visit to nutritionists and nutritionists, etc.

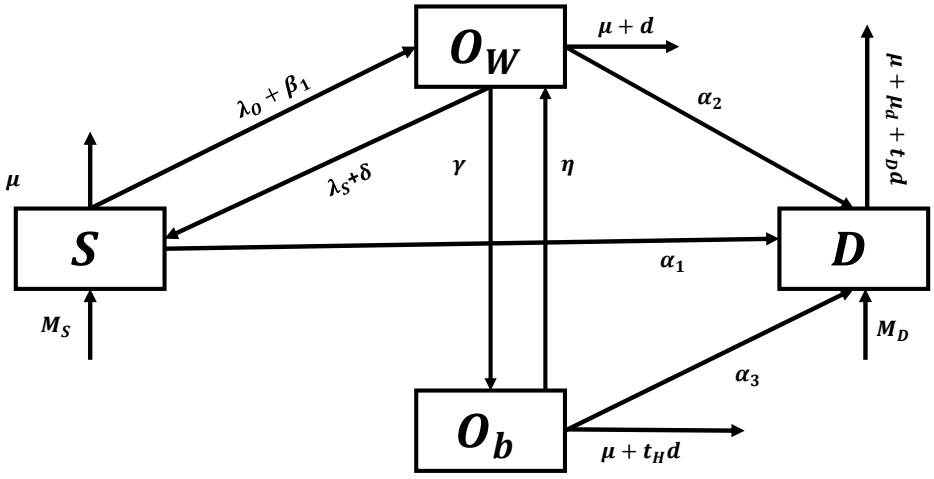


Figure 1. Flow chart of Model (4)

- u_3: The objective of this control is focused on preventing the disease from developing and leading to obesity. It consists of encouraging a gradual change in lifestyle, doing physical exercise, visiting a nutritionist to follow a healthy diet and eating habits and doing activities, avoiding stress and controlling schedules, etc.

All controls have economic and socio-psychological factors associated with them. For example gyms and many physical activities, nutritional consultations, maintaining a diet, etc. may have an economic cost and already the interest to practice this healthy lifestyle is related to psychological-social factors. We assume that the difficulty and cost of the u_3 control is higher than the others because an individual who is already overweight and obese needs more economic and psychological-social elements to improve his lifestyle and overcome his disease[1] [40].

Optimal control problem formulation

Now, we present the optimal control problem with fractional derivatives in the Caputo sense. The problem is theoretically focused on the control of new cases of overweight and obesity.
The functional objective to be minimized is:

$$J(u_1, u_2, u_3) = \int_{t_0}^{t_f} \left(O_w(t) + O_b(t) + \frac{1}{2}(B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t)) \right) dt.$$

The coefficients B_1, B_2 and B_3 represent the weight constants associated with the implementation of the controls over a finite time horizon $[t_0, t_f]$ (t_0 is the initial and t_f the final moment of time) with $B_i > 0, i = 1, 2, 3$ $B_1 \leq B_2 < B_3$. The cost involved in the application of control in compartments S, O_w and O_b are given by $\int_{t_0}^{t_f} \frac{B_1 u_1^2(t)}{2} dt$, $\int_{t_0}^{t_f} \frac{B_2 u_2^2(t)}{2} dt$ and $\int_{t_0}^{t_f} \frac{B_3 u_3^2(t)}{2} dt$.

1 Obesity is a chronic, complex disease defined by excessive fat deposits that can impair health. Obesity can increase the risk of type 2 diabetes, heart disease and certain types of cancer

Then, your goal is to find the optimal controls u_1^*, u_2^*, u_3^* that satisfy:

$$J(u_1^*, u_2^*, u_3^*) = \min_{U_{ad}} J(u_1, u_2, u_3), \tag{5}$$

where $U_{ad} = \{(u_1, u_2, u_3) : \text{ Lebesgue measurable }, 0 \leq u_i \leq 1, i = 1, 2, 3, \forall t \in [t_0, t_f]\}$. The constraints of the optimal control problem is model (4) with the incorporation of the controls as follows:

$$\begin{aligned}
{}^c\mathbb{D}_t^\alpha S &= f_1^\alpha = M_S^\alpha + (\delta^\alpha + \lambda_S)O_w - (\mu^\alpha + \alpha_1^\alpha + (1-u_2)\beta_1^\alpha + (1-u_1)\lambda_O)S, \\
{}^c\mathbb{D}_t^\alpha O_w &= f_2^\alpha = ((1-u_1)\lambda_O + (1-u_2)\beta_1^\alpha)S + \eta^\alpha O_b - (\lambda_S + (1-u_3)\gamma^\alpha + \mu^\alpha + d^\alpha + \delta^\alpha + \alpha_2^\alpha)O_w, \\
{}^c\mathbb{D}_t^\alpha O_b &= f_3^\alpha = (1-u_3)\gamma^\alpha O_w - (\eta^\alpha + \mu^\alpha + t_H d^\alpha + \alpha_3^\alpha)O_b, \\
{}^c\mathbb{D}_t^\alpha D &= f_4^\alpha = M_D^\alpha + \alpha_1^\alpha S + \alpha_2^\alpha O_w + \alpha_3^\alpha O_b - (\mu_d^\alpha + \mu^\alpha + t_D d^\alpha)D,
\end{aligned} \tag{6}$$

with initial conditions

$$S(0) = S_0 \geq 0, O_w(0) = O_{w0} \geq 0, O_b(0) = O_{b0} \geq 0, D(0) = D_0 \geq 0 \text{ and } \alpha \in (0, 1].$$

For the given optimal control problem the Hamiltonian is defined as:

$$H^\alpha(t) = O_w(t) + O_b(t) + \frac{1}{2}(B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t)) + \sum_{n=1}^{4} \lambda_n f_n^\alpha,$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the adjoint variables.
We have the following important theorem:

Theorem 2 *If u_1^*, u_2^* and u_3^* are the controls associated to (5), S^*, O_w^*, O_b^* and D^* are corresponding optimal paths them, there are co-state variables λ_n, $n = 1, .., 4$, such that the control system (6) and, the following conditions are satisfied:*

- *the co-state equations using the equivalence of Lemma 1:*

$$\begin{aligned}
{}^c\mathbb{D}_t^\alpha \lambda_1(t') &= (\lambda_2 - \lambda_1)[(1-u_1)\lambda_O + (1-u_2)\beta_1^\alpha] - \lambda_1(\mu^\alpha + \alpha_1^\alpha), \\
{}^c\mathbb{D}_t^\alpha \lambda_2(t') &= 1 + (1-u_2)(\lambda_3 - \lambda_2)\gamma^\alpha + (\lambda_1 - \lambda_2)(\lambda_S + \delta^\alpha) - \lambda_2(d^\alpha + \mu^\alpha + \alpha_2^\alpha), \\
{}^c\mathbb{D}_t^\alpha \lambda_3(t') &= 1 + (\lambda_3 - \lambda_1)\eta^\alpha - \lambda_3(\mu^\alpha + t_H d^\alpha + \alpha_3^\alpha), \\
{}^c\mathbb{D}_t^\alpha \lambda_4(t') &= -\lambda_4(\mu_d^\alpha + \mu^\alpha + t_D d^\alpha),
\end{aligned} \tag{7}$$

  with $t' = t_f - t$,
- *with transversality conditions:*

$$\lambda_n(t_f) = 0, \qquad n = 1, ..., 4, \tag{8}$$

- *and the optimality conditions:*

$$H^\alpha(S^*, O_w^*, O_b^*, D^*, \lambda_n, u_k^*) = \min_{0 \leq u_k \leq 1} H^\alpha(S^*, O_w^*, O_b^*, D^*, \lambda_n, u_k^*), \tag{9}$$

where $n = 1, ..., 4$ *and* $k = 1, 2, 3$. *Futhermore, the control functions* $u_k^*, k = 1, 2, 3$ *are given by*

$$\begin{aligned} u_1^* &= \min\{\max\{0, \frac{\lambda_O(\lambda_2 - \lambda_1)S}{B_1}\}, 1\}, \\ u_2^* &= \min\{\max\{0, \frac{\beta_1^\alpha(\lambda_2 - \lambda_1)S}{B_2}\}, 1\}, \\ u_3^* &= \min\{\max\{0, \frac{(\lambda_3 - \lambda_2)\gamma^\alpha O_w}{B_3}\}, 1\}, \end{aligned}$$

with the stationary conditions $\dfrac{\partial H^\alpha}{\partial u_k} = 0, k = 1, 2, 3.$

Proof The existence of the optimal control (u_1^*, u_2^*, u_3^*) and associated to the optimal solution (S^*, O_w^*, O_b^*, D^*) comes from the convexity of the integrand of the functional (5) with respect to the $u_k, k = 1, 2, 3$ controls and the Lipschitz properties of the state system with respect to the state variables (S, O_w, O_b, D). According to Pontryagin's maximum principle, if $u_k \in U_{ad}, k = 1, 2, 3$ is an optimal control associated to (5)-(6) with the initial conditions and t_f fixed, there exists an absolutely continuous non-trivial map $\lambda : [t_0, t_f] \longrightarrow \mathbb{R}^4$, $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$ called adjoint vector such that

$${}_t^c\mathbb{D}_{t_f}^\alpha \lambda_n(t) = \frac{\partial H^\alpha}{\partial x_l}, \quad n = 1, ..., 4,$$

and using the equivalence of Lemma 1, we have:

$${}^c\mathbb{D}_{t_f}^\alpha \lambda_n(t') = \frac{\partial H^\alpha}{\partial x_l}, \quad n = 1, ..., 4,$$

with $x_l = (S, O_w, O_b, D)$, $t' = t_f - t$ and H^α is the Hamiltonian defined in (7). Moreover, the transversality conditions $\lambda_n(t_f) = 0$, $n = 1, ..., 4$ are satisfied.
Optimality is when the equations $\dfrac{\partial H^\alpha}{\partial u_k} = 0$, at $u_k^*, k = 1, 2, 3$. Then,

$$\frac{\partial H^\alpha}{\partial u_1} = B_1 u_1 - \lambda_O(\lambda_1 - \lambda_2)S = 0,$$

if and only if

$$u_1^* = \frac{\lambda_O(\lambda_2 - \lambda_1)S}{B_1},$$

to be taken on the set $\{t : 0 \leq u_1^* \leq 1\}$.
Analogously, we have for u_2 that

$$\frac{\partial H^\alpha}{\partial u_2} = B_2 u_2 - \beta_1^\alpha(\lambda_1 - \lambda_2)S = 0,$$

if and only if

$$u_2^* = \frac{\beta_1^\alpha(\lambda_2 - \lambda_1)S}{B_2},$$

to be taken on the set $\{t : 0 \leq u_2^* \leq 1\}$.
For u_3, we have:

$$\frac{\partial H^\alpha}{\partial u_3} = B_3 u_3 - \gamma_1^\alpha(\lambda_2 - \lambda_3)S = 0,$$

if and only if

$$u_3^* = \frac{\gamma_1^\alpha(\lambda_3 - \lambda_2)S}{B_3},$$

to be taken on the set $\{t : 0 \leq u_3^* \leq 1\}$.
Since the optimality conditions are only satisfied in the interior of the control set. An important element regarding the behavior of the controls is that the second derivative concerning $u_k, k = 1, 2, 3$, are:

$$\frac{\partial^2 H^\alpha}{\partial u_1^2} = B_1 > 0, \quad \frac{\partial^2 H^\alpha}{\partial u_2^2} = B_2 > 0, \quad \frac{\partial^2 H^\alpha}{\partial u_3^2} = B_3 > 0.$$

Applications of this theorem to control problems in epidemiology can be found in [24, 32–34]

5 Numerical simulations

Method and data

The method used to solve numerically fractional order system of nonlinear differential equations (4) can be found [22, 37, 41–43]. The algorithm has the structure of a PECE (Predict-Evaluate-Correct-Evaluate) method and combines a fractional order algorithm with a classical method. The method chosen is the Adams-Bashforth-Moulton method for both integrators. The key to deriving the method in the fractional variant is to use the trapezoidal quadrature product formula. This algorithm is independent of the $\alpha-$ value and behaves very similar to the classical Adams-Bashforth-Moulton method. The stability properties are unchanged in the fractional version compared to the classical algorithm.
To solve the control problem primitively, with PECE and using the initial conditions of the state variables and an estimate of the controls in the time interval $[0, t_f]$, we obtain the values of the state variables. We solve the system (7) and the traversal conditions (8) with PECE and the values of the adjoint variables $\lambda_i, i = 1, ..., 4$ are obtained. The controls are updated by a convex combination of the previous control and the value calculated in (9). This procedure is repeated iteratively until the stop condition that is when the values of the controls of the previous iteration are very close to those of the current iteration. The programming was performed in Matlab software. The parameter values used in the computational simulations are in Table 1 and the initial conditions are: $S_0 = 874.1400$, $O_{w0} = 1.2000$, $O_{b0} = 1.5000$ and $D_0 = 100.0000$ on the scale of 10000 individuals.

Global sensitivity analysis

Global sensitivity analysis (GAS) can provide information on the dependence of the model output on each of its input parameters [44, 45, 51]. An advantage of GSA methods is that they explore screening or variance decomposition to cover the limitations of local analysis. In our work, we use the Sobol' indices to perform the GAS [46]. The use of Sobol' indices for the study of biological and epidemiological models is very widespread and can be found in references such as [46–48].

The system is analyzed from a probabilistic perspective where the model input is a random vector $\mathbb{X}$, with a joint probability density function $f_{\mathbb{X}}$ with support $I_{\mathbb{X}}$. This version of the model can then be rewritten as:

$$Y = M(\mathbb{X}),$$

which has a probability density function $f_{\mathbb{Y}}$ that is unknown prior to uncertainty propagation [49]. Assuming that the quantity (or quantities if there are more than one) of interest is a scalar value and that the random input parameters are composed of independent and identically distributed uniform parameters (iid) $X_i, i = 1, ..., n$, scaled to have support $[0, 1]^m$, then the Hoeffding-Sobol decomposition is given by

$$Y = M_0 + \sum_{i=1}^{n} M_i(X_i) + \sum_{1 \le i \le j \le n} M_{ij}(X_i, X_j) + ... + M_{1,...,m}(X_i, ..., X_n),$$

where

$$\begin{aligned} M_0 &= \mathbb{E}[Y], \\ M_i(X_i) &= \mathbb{E}[Y|X_i] - M_0, \\ M_{ij}(X_i, X_j) &= \mathbb{E}[Y|X_i, X_j] - M_0 - M_i - M_j, \end{aligned}$$

$\mathbb{E}$ is the expected value, M_0 is the mean value and the terms of increasing order are conditional expectations defined in a recursive way, that characterize a unique orthogonal decomposition of the model response [46, 50].
We can now decompose the total variance of the response as follows

$$Var(Y) = \sum_{\mathbf{u}} Var(M_{\mathbf{u}}(\mathbb{X}_{\mathbf{u}})), \qquad \text{for} \quad \emptyset \neq \mathbf{u} \subset \{1, ..., n\},$$

where $Var(M_{\mathbf{u}}(\mathbb{X}_{\mathbf{u}}))$ expresses the conditional variance for the subvector $\mathbb{X}_{\mathbf{u}}$, containing the variables which indices are indicated by the subset $\mathbf{u}$ [50]. Thus, the Sobol index associated to the subset $\mathbf{u}$ is defined as the ratio between the contribution given by the interaction between the components of $\mathbf{u}$ to the variance of the model and the total variance is described as [51]:

$$S_{\mathbf{u}} = \frac{Var(M_{\mathbf{u}}(\mathbb{X}_{\mathbf{u}}))}{Var(Y)}.$$

As result of this equation, we have that for $\mathbf{u} \subset \{1, ..., n\}, \mathbf{u} \neq \emptyset$:

$$\sum_{\mathbf{u}} S_{\mathbf{u}} = \sum_{i=1}^{n} S_i + \sum_{1 \le i < j \le n} S_{ij} + ...S_{1,...,m} = 1,$$

that is, by construction the sum of all the Sobol' indices must be equal to the unit [46].
The terms

$$S_i = \frac{Var(M_i(\mathbb{X}_i))}{Var(Y)}, \qquad i = 1, ..., n,$$

are defined as the first-order Sobol' indices, and denote the individual effect of the varible $\mathbb{X}_i$ for the total model variate. The terms

$$S_{ij} = \frac{Var(M_{ij}(\mathbb{X}_{ij}))}{Var(Y)}, \qquad 1 \leq i \leq j \leq n,$$

are defined as the second-order indices and denote the effect of interaction between the varibles $\mathbb{X}_i$ and $\mathbb{X}_j$. We can construct the Sobol' indices of all order until the m-th order indices, $S_{1,\ldots,m}$, but in this work we study until the second-order indices.
To measure the total contribution of the i-th random variable X_i on the total variance, either by its single effect or by its interaction with others, and we use the total Sobol' indices defined by

$$S_i^{\top} = \sum_{i \in \mathbf{u} \subset \{1,\ldots,n\}} S_{\mathbf{u}}, \qquad i = 1, \ldots, n. \tag{10}$$

To calculate the total Sobol' indices (10), we need the underlying variances and use the alternative of surrogate models based on Polynomial Chaos Expansion (PCE) [52–55]. The polynomial chaos expansion of the computational model response is a sum of orthogonal polynomials weighted by coefficients to be determined which reads as:

$$Y = M(\mathbb{X}) = \sum_{\alpha \in \mathbb{N}^M} y_\alpha \Psi_\alpha(\mathbb{X}), \tag{11}$$

where the $\Psi_\alpha(\mathbb{X})$ are multivariate polynomials orthonormal with respect to $f_{\mathbb{X}}$, $\alpha \in \mathbb{N}^{\mathbf{M}}$ is a multi-index that identifies the components of the multivariate polynomials $\Psi_\alpha(\mathbb{X})$, and the y_α are the corresponding coefficients (coordinates) [52]. The sum in Eq. (11) needs to be truncated to a finite sum, introducing the truncated polynomial expansion of chaos:

$$Y \approx M^{PC}(\mathbb{X}) = \sum_{\alpha \in A} y_\alpha \Psi_\alpha(\mathbb{X}),$$

where $A \subset \mathbb{N}^{\mathbf{M}}$ is the set of selected multi-indices of multivariate polynomials.
Note the quality of the PCE is directly dependent on the number of terms you have in the expansion. The family of orthonormal polynomials to be used are chosen according to the input distribution of the model, where the aim is to minimize the number of terms needed in the expansion to build a good computational representation of the model. For example, for uniform we use Legendre's orthonormal polynomials, for Gaussian we use Hermite's, for Gamma we use Laguerre's and for Beta we use Jacobi's [48, 49, 56].
Given the orthogonal polynomials that are used, we define the total degree truncation scheme which corresponds to all polynomials in the $\mathbf{M}$ input variables of total degree less than or equal to p:

$$A^{\mathbf{M},p} = \{\alpha \in \mathbb{N}^{\mathbf{M}} : |\alpha| \leq p\} \qquad \text{card} A^{\mathbf{M},p} = P = \binom{\mathbf{M}+p}{p}.$$

Note that the total-degree basis grows exponentially with the degree p.
There are several methods for calculating the coefficients of the polynomial expansion of chaos for a given base. The two principal strategies to calculate the polynomial chaos coefficients non-

intrusively are projection and regression.
To estimate the surrogate error estimation, we use with the Leave-One-Out (ϵ_{LOO}) cross-validation error [48, 55], calculated by

$$\epsilon_{LOO} = \frac{\sum_{i=1}^{N_S}(M(\mathbf{x}^{(i)}) - M^{PC\setminus i}(\mathbf{x}^{(i)}))^2}{\sum_{i=1}^{N_S}((\mathbf{x}^{(i)}) - \overline{y})^2}.$$

It consists in building N_s metamodels $M^{PC\setminus i}$ each one created on a reduced experimental design $\chi \setminus \mathbf{x}^{(i)} = \{\mathbf{x}^{(j)}, j = 1, ..., N_S, j \neq i\}$ (where $\chi = \{\mathbf{x}^{(i)}, i = 1, ..., N_S, \}$ called the experimental design), and comparing its prediction on the excluded point $\mathbf{x}^{(i)}$ with the real value $y^{(i)}$ and $\overline{y}$ is the sample mean of the experimental design response [52].
Several methods exist to calculate the coefficients y_α of the polynomial chaos expansion for a given basis.
In our work, we use regression and two different techniques to estimate the coefficients.
A different approach to estimate the coefficients in Eq. (12) is to set up a least-squares minimization problem. The infinite series in Eq. (11) can be written as a sum of its truncated version Eq. (12) and a residual:

$$Y = M(\mathbb{X}) = \sum_{\alpha=0}^{P} y_\alpha \Psi_\alpha(\mathbb{X}) + \epsilon_P = y^\top \Psi(\mathbb{X}) + \epsilon_P,$$

where $P = card A^{\mathbf{M},p}$, ϵ_P is the truncation error, $y = \{y_0, ..., y_P\}^\top$ is a vector containing the coefficients and $\Psi(\mathbb{X}) = \{\psi_0(\mathbb{X}), ..., \psi_P(\mathbb{X})\}^\top$, is the vector that assembles the values of all the orthonormal polynomials in $\mathbb{X}$ and compute the model response for that samples

$$\begin{aligned} y^{(1)} &= M(\mathbf{x}^{(1)}), \\ y^{(2)} &= M(\mathbf{x}^{(2)}), \\ &\vdots \\ y^{(N_S)} &= M(\mathbf{x}^{(N_S)}). \end{aligned}$$

The classic least squares regression problem is defined as:

$$y^\top \Psi(\mathbb{X}) \approx M(\mathbf{x}),$$

and its general solution can be expressed as:

$$\hat{y} = \arg\min \mathbb{E}\left[(y^\top \Psi(\mathbb{X}) - M(\mathbf{x}))^2\right]. \tag{12}$$

Now, we focused on simulating with the following methods:
Method I: Ordinary Least-Squares (OLS). A direct approach to solving Eq. (12) is given by Ordinary Least-Squares (OLS). Given a sample $\chi = \{\mathbf{x}^{(1)}, ..., \mathbf{x}^{(N)}\}$ of size N of $\mathbb{X}$ (the experimental design) and the corresponding model responses $\mathbf{Y} = \{\mathbf{y}^{(1)}, ..., \mathbf{y}^{(N)}\}^\top$, the ordinary least-squares solution of Eq. (12) reads:

$$\hat{y} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{Y},$$

where

$$A_{ij} = \Psi(\mathbf{x}^{(i)}), \quad i = 1, ..., n, \quad j = 0, ..., P-1.$$

The main advantage of the least squares minimization method over the projection method is that an arbitrary number of points can be used to calculate the coefficients, as long as they are a representative sample of the random input vector. This method theoretically, its error, methodology and example can be found in [48, 55, 57, 58].

Method II: Least Angle Regression (LARS). A complementary strategy to favor high-dimensional sparsity is to directly modify the least-squares minimization problem (12) by adding a penalty term of the form $\lambda \parallel y \parallel_1$, i.e., solving:

$$\hat{y} = \arg\min \mathbb{E}\left[(y^\top \Psi(\mathbb{X}) - M(\mathbb{X}))^2\right] + \lambda \parallel y \parallel_1,$$

where the regularization term $\parallel \hat{y} \parallel_1 = \sum_{\alpha \in A} |y_\alpha|$ forces the minimization to favour low-rank solutions. The LARS algorithm used, in theoretical form, code and examples can be found in [59–63].

In our method, we took into account a stopping criterion that consists of stopping adding regressors after the error estimate is above its minimum value for at least 10% of the maximum number of possible iterations [64–66].

For each method we will experiment with different sampling techniques (sampling methods) [67–73]. In particular, we will use Monte Carlo (MC) and Sobol' sequence sampling (Sobol') [70, 71].

The code was implemented in MATLAB R2024A and we used the UQLab library [55, 74, 75] as an implementation aid and based on the code presented in [48]. We used the time period years and the maximum time period is 15 years.

The sensitivity study was carried out for the case ($\alpha = 1.0$) using the methods that are in the reference [48, 52, 53] and we can observe the solution of the model for fractional orders studied ($\alpha = 0.5, 0.7, 0.9, 1.0$) the reported values change but the asymptotic behavior of the results does not.

We use the independent uniform random variables for each of the parameters as the probabilistic input model, with upper and lower bounds of 1.5% dispersion around the mean, and consider the fixed nominal values presented in Table 1. To build the PCE surrogate model 1000 experimental design samples were taken with a maximum polynomial degree is in $[10, 13]$ interval. For all the cases studied, the leave-one-out error (ϵ_{LOO}) is in the range of $[8.615006e-07, 5.658380e-06]$ and the quality of the surrogate approximation showed an reasonable to the purpose.

Table 2 shows the first order Sobol' indices and Table 3 shows the total Sobol' indices for the OLS and LARS methods with the MC and Sobol' sampling techniques. Table 4 presents the second order Sobol' indices for the OLS and LARS methods with MC and Sobol' sampling techniques. Figure 2a and Figure 3a show the first order and total order Sobol' indices for the OLS method and the different sampling techniques. Figure 2b and Figure 3b present the first order and total order Sobol indices for the LARS method and Figure 4a and Figure 4b show the second order Sobol' indices for OLS and LARS, respectively.

The parameter that has the most influence is the rate of effective contact associated with the impact that an overweight and obese person has on a person with normal weight (α^*). With the different techniques studied and sampling methods to obtain the Sobol' indices, the first and total order Sobol' indices average was 0.503448 and 0.502248, respectively.

In decreasing order, the parameter δ follows, that is the rate that leads an overweight person to a

normal weight. This parameter is important because its increase defines control of overweight in the community. The average value of the first-order Sobol' index is 0.171327 and for the total index we have 0.171188.

The parameter β_1 at the third position with an average value of the Sobol' index of first-order of 0.162849 and in the total-order of 0.163707. We can observe that the two ways for a normal weight person to become overweight that we studied with the model are in the first and third order of influence.

The other parameters with consecutive influence are the death rate associated with overweight and obesity with an average first order Sobol' index of 0.034274 and total 0.00342985, the parameter γ which is associated with people who manage to overcome obesity and reach the state of overweight with average values of 0.025539 and 0.02535 of first and total order index.

Finally we have M_S which is the recruitment rate which measures the entry of new individuals to the dynamics and has an average value of 0.024462 and 0.024749, for the first and total order, respectively.

The other parameters of the dynamics were studied but their first and total order Sobol' indices are below the threshold value 10^{-4}.

It is important to mention that the influence of the parameters for the first order and for the total order of the Sobol' indices coincide. Another important factor is that the parameters associated with weight gain from a normal weight state (α^*, β_1) and weight loss from obesity to overweight and from overweight to normal weight (δ, γ) are among the most influential parameters and have an obvious importance in the control of overweight and obesity in the community.

Using as a reference the Sobol's index of second-order with the different techniques and sampling methods we have that the relationship between parameters that has more influence is between M_S and α^* that are the recruitment rate and the effective contact rate of the influence of obese and overweight are an individual with normal weight.

Continuing in decreasing order, the relationship that continues is between α^* and β_1 which represent the pathways for an individual to gain weight studied in the model, that is through the influence of the obese and overweight and through social pressure.

Parameter α^* is also related to α_2 which is the rate of diabetes diagnosis in overweight individuals.

Next, we have the relationship between the recruitment rate M_S and β_1 and α_2 which are the weight gain of the individual to become overweight and the diagnosis of diabetes in overweight individuals, so we can say that with the recruitment rate the social pressure can lead more people to become overweight and with it a higher diagnosis of diabetes.

With the use of MC sampling method there appears a relationship which is not there when we use Sobol' sampling which is the relationship between β_1 and α_2 which both are related to M_S and in the opposite case there appears the relationship between α^* and d which represents the relationship between the effective contact rate of the impact of an obese and overweight on a normal weight individual with the death associated with obesity and overweight. The use of different PCE and sampling techniques showed us relationships that are important to keep in mind that would appear with one methodology and not with the other, see Table 4 and Figure 4.

All parameters participating in the relationships characterized by second-order Sobol' indices are among the seven most influential in the first and total order Sobol' indices.

The information provided by the global stability contributes both independently and jointly between parameters helping to make decisions for parameter estimation, in the qualitative analysis of the model and in the construction of the computational simulations and we offer different techniques that contribute to obtain this information between the model and the parameters.

Table 2. Comparison of first-order Sobol' indices for the peak duration using OLS and LARS for different sampling methods

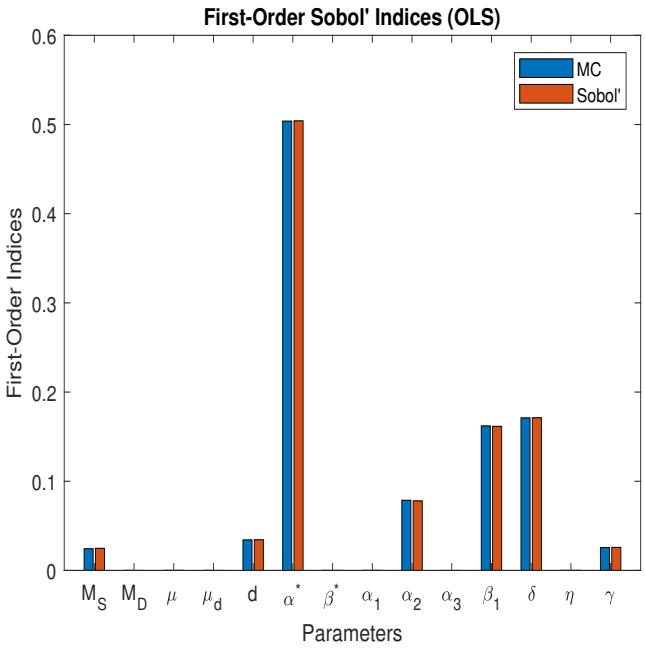
	OLS		LARS	
Parameters	MC	Sobol'	MC	Sobol'
M_S	0.024408	0.024807	0.024408	0.024287
M_D	0.000004	0.000000	0.000000	0.000000
μ	0.000036	0.000014	0.000032	0.000007
μ_d	0.000001	0.000000	0.000000	0.000000
d	0.034268	0.034460	0.034492	0.033900
α^*	0.503782	0.504101	0.502158	0.503751
β^*	0.000004	0.000003	0.000000	0.000000
α_1	0.000001	0.000005	0.000000	0.000000
α_2	0.078682	0.078010	0.078719	0.077754
α_3	0.000000	0.000000	0.000000	0.000000
β_1	0.162075	0.161534	0.163478	0.163431
δ	0.171080	0.171233	0.171233	0.171762
η	0.000010	0.000002	0.000001	0.000000
γ	0.025712	0.025855	0.025479	0.025110

Table 3. Comparison of total-order Sobol' indices for the peak duration using OLS and LARS for different sampling methods

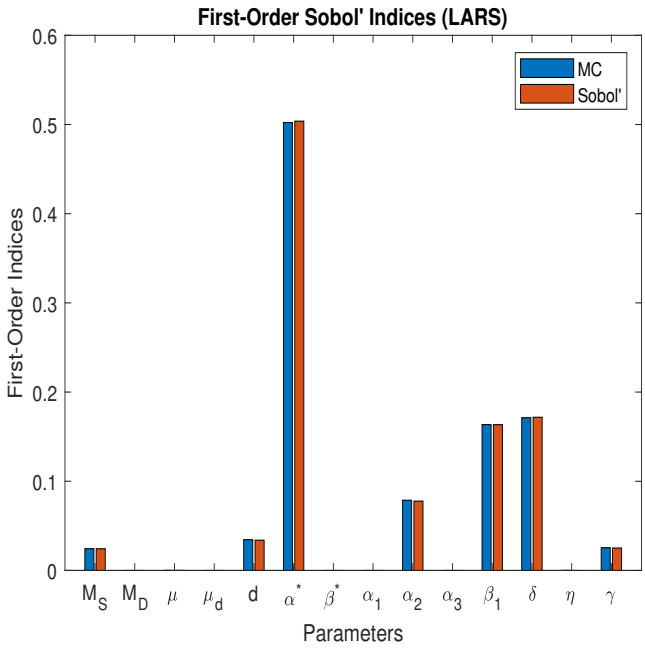
	OLS		LARS	
Parameters	MC	Sobol'	MC	Sobol'
M_S	0.025017	0.025862	0.024073	0.024045
M_D	0.000010	0.000013	0.000016	0.000013
μ	0.000027	0.000020	0.000040	0.000027
μ_d	0.000004	0.000006	0.000003	0.000004
d	0.034043	0.034105	0.034076	0.034970
α^*	0.502332	0.503263	0.503175	0.500023
β^*	0.000019	0.000006	0.000008	0.000016
α_1	0.000024	0.000008	0.000013	0.000012
α_2	0.078201	0.078304	0.078119	0.078095
α_3	0.000027	0.000009	0.000006	0.000009
β_1	0.161226	0.161132	0.164832	0.167639
δ	0.173204	0.171339	0.170200	0.170009
η	0.000028	0.000013	0.000023	0.000028
γ	0.025642	0.025932	0.025416	0.025150

Table 4. Second-order Sobol' indices. The table shows the influence of the relationships between model parameters using the OLS and LARS techniques with the MC and Sobol' sampling techniques

	OLS		LARS		
Relationship	MC	Sobol	MC	Sobol	Average
$M_S - \alpha^*$	0.003929	0.003999	0.003816	0.003874	0.003904
$\beta_1 - \alpha^*$	0.003724	0.003601	0.003578	0.003641	0.003636
$\alpha^* - \alpha_2$	0.001699	0.001737	0.001586	0.001784	0.001701
$M_S - \beta_1$	0.001686	0.001828	0.001570	0.001829	0.001782
$M_S - \alpha_2$	0.000883	0.0.00074	0.000885	0.000915	0.000856
$\beta_1 - \alpha_2$	0.000730	0	0.000750	0	0.000370
$\alpha^* - d$	0	0.000807	0	0.000735	0.000385

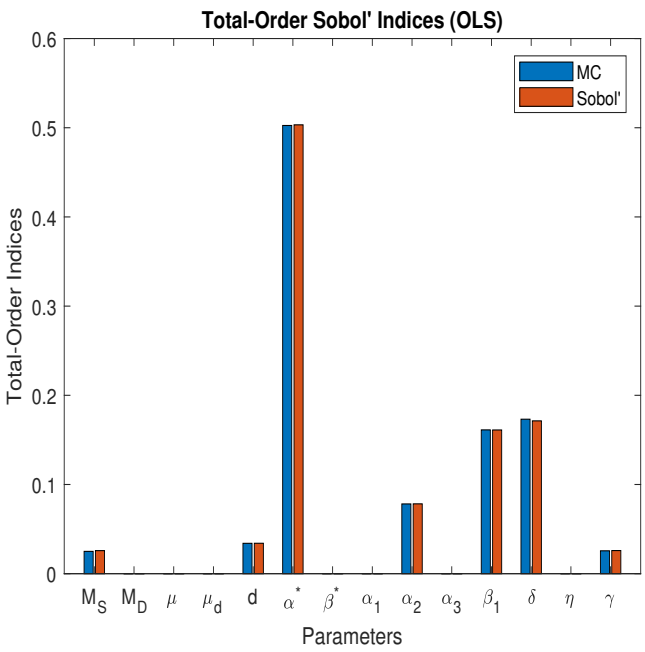


(a) First-order Sobol' indices using the OLS method and the MC and Sobol' sampling techniques

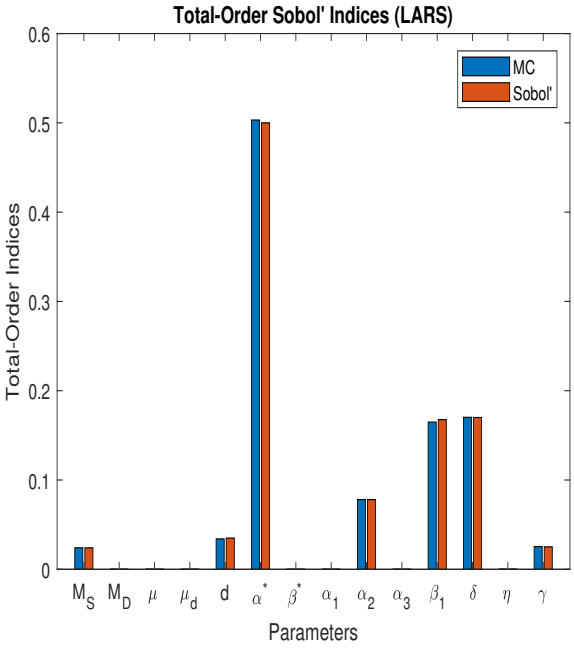


(b) First-order Sobol' indices using the LARS method and the MC and Sobol' sampling techniques

Figure 2. First-order Sobol' indices using the OLS and LARS methods and the MC and Sobol' sampling techniques



(a) Total-order Sobol' indices using the OLS method and the MC and Sobol' sampling techniques

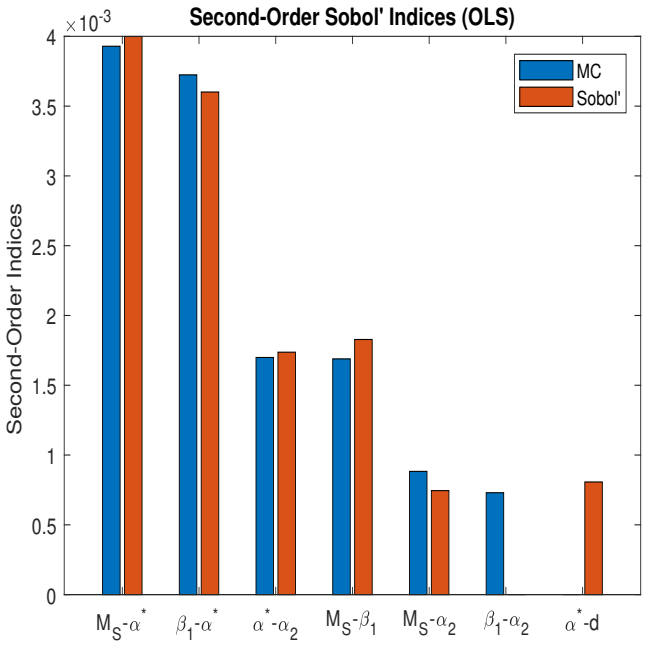


(b) Total-order Sobol' indices using the LARS method and the MC and Sobol' sampling techniques

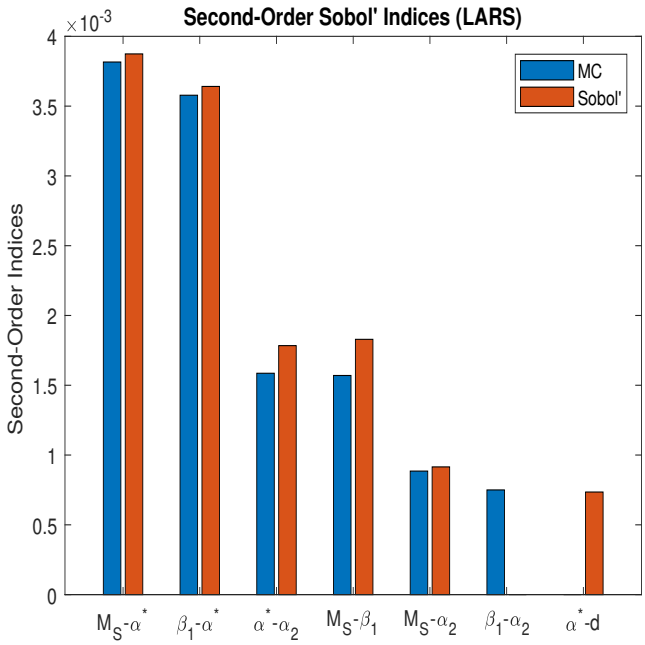
Figure 3. Total-order Sobol' indices using the OLS and LARS methods and the MC and Sobol' sampling techniques

Numerical simulations

For the simulations of the optimal control problem the time period is annual for a total time of 15 years and we assume the values for the weight constants associated with the implementation of the controls are $B_1 = 150$, $B_2 = 300$ and $B_3 = 550$.



(a) Second-order Sobol' indices using OLS and with sampling using MC and Sobol'



(b) Second-order Sobol' indices using LARS and with sampling using MC and Sobol'

Figure 4. Sobol's second order indices using OLS and LARS for PCE using MC and Sobol' as sampling techniques

We constructed three possible strategies:

- Strategy I: $u_1 = 0, u_2 \neq 0, u_3 \neq 0$. This strategy focuses on controlling weight gain due to social pressure and the progression from overweight to obese.
- Strategy II: $u_3 = 0, u_1 \neq 0, u_2 \neq 0$. In this strategy, we control the two ways taken into account in the construction of the model for an individual with normal weight to evolve into overweight.
- Strategy III: $u_i \neq 0, i = 1, 2, 3$. All controls are active; we control the ways in which an individual with normal weight becomes overweight and the evolution from overweight to obese.

The control strategies are based on the results of simulations and the study of the parameters presented in [22]. Due to one of the objectives of the model, which is the impact of social pressure on weight gain, and is presented in [21] and [22], the control over social pressure that causes an individual with normal weight to become obese will always be active.
Now, let us study the impact of strategies for different fractional orders and compartments.
For the behavior of individuals with normal weight without application of controls, we can observe that with the increase of fractional orders the number of individuals with normal weight is reduced. For all control strategies and fractional orders, the number of cases with normal weight increases with respect to the model without control, which implies that fewer people are gaining weight and are able to maintain an adequate weight. The best results are achieved when all the controls are activated followed by strategy II when only the controls are active in the negative interactions of obese and overweight with overweight people and the control of weight gain by social pressure on individuals with normal weight, see Table 5 and Figure 5a, Figure 5b, Figure 5c and Figure 5d.
In the case of overweight individuals, there is a growth and then it stabilizes for the different fractional orders and the higher the fractional order the higher the number of overweight individuals reported. For all the control strategies studied the number of individual overweights is reduced but for order $\alpha = 0.5$ the asymptotic behavior is analogous to the model without control
For constructed strategy and different fractional orders, we observed a decrease in the asymptotic

behavior which is relevant to strategy II and III. The strategy that showed the best results was strategy III because it significantly reduced over time the number of overweight individuals followed by strategy II. For higher fractional orders the impact of the controls is more significant, see Table 6 and Figure 6a, Figure 6b, Figure 6c and Figure 6d. In the obese compartment, we have that for all fractional orders we initially have a decrease, then a growth,h and then stabilization. In this case, the higher fractional orders the higher the number of obese reported at the end of the study for the model without controls. The maximum and minimum values for the different fractional orders and control strategies can be found in Table 7. In this case, the impact of the strategies is behaviorally more effective for lower fractional orders and strategies III and II show the best results respectively. We can observe that with strategies III and II for fractional orders $\alpha = 0.5$ and $\alpha = 0.7$, not only control growth during the study (happens for all orders) but also cause a significant decrease in the number of obese reported, see Figure 7a, Figure 7b, Figure 7c and Figure 7d. The diabetes compartment is directly related to the behaviors associated with body weight. In this case, for all fractional orders, we have a growth behavior in the number of diabetic individuals and it is maintained when applying the different control strategies. We can conclude that the higher the fractional order, the higher the number of reported diabetics. All strategies significantly reduce the number of reported diabetics over time, but the strategy with the best results was Strategy III followed by Strategy II, see Table 8 and Figure 8a, Figure 8b, Figure 8c and Figure 8d. The strategy that showed the best results was Strategy III, where all controls are active, followed by Strategy II. All strategies are associated with the seven parameters with the highest Sobol' indices (specifically α^*, β_1 and γ). Strategy II, which showed the best results when controls were not applied, is associated with parameters α^* and β_1, which are in the three parameters with the highest Sobol' index of the first and total order, and they are also related by the Sobol' index of the second order, which does not show any significant relationship with parameter γ or any other parameter of the model. We can conclude from the construction of the model, the control strategies, and the values used in the simulations that if we manage to prevent an individual from gaining weight, either by interaction with overweight and/or obese people, and reduce social pressure (Strategy II), we manage to reduce overweight and obesity in the community and, as a consequence, diabetes.

Table 5. Maximum and minimum values of the simulations of Model (4) of the normal weight individual compartment without strategies and with the different control strategies ($\times 10000$).

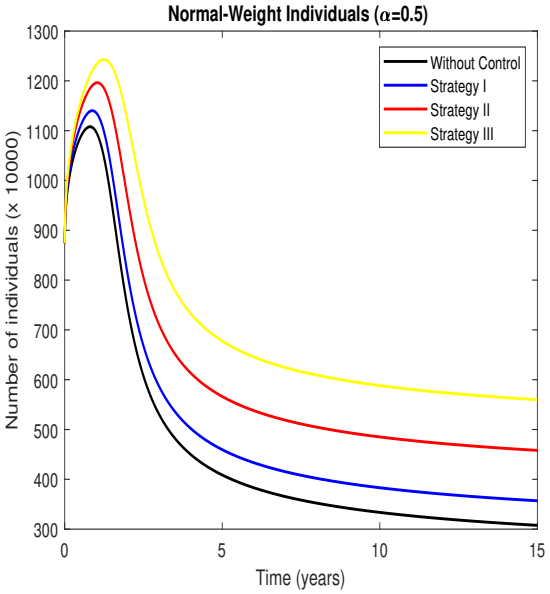
	Without Controls		Strategy I		Strategy II		Strategy III	
α	Min	Max	Min	Max	Min	Max	Min	Max
0.5	307.6150	1.1079e+03	356.9388	1.1401e+03	458.1019	1.1903e+03	560.4600	1.2250e+03
0.7	252.7330	1.8687e+03	301.7025	1.2094e+03	402.0024	1.2793e+03	503.0536	1.3312e+03
0.9	225.1468	1.2461e+03	274.0151	1.2974e+03	373.9049	1.3843e+03	474.0821	1.4214e+03
1.0	218.2975	1.2918e+03	266.4800	1.3494e+03	362.5733	1.4464e+03	462.3616	1.4859e+03

Table 6. Maximum and minimum values of the simulations of Model (4) of the overweight individual compartment without strategies and with the different control strategies ($\times 10000$).

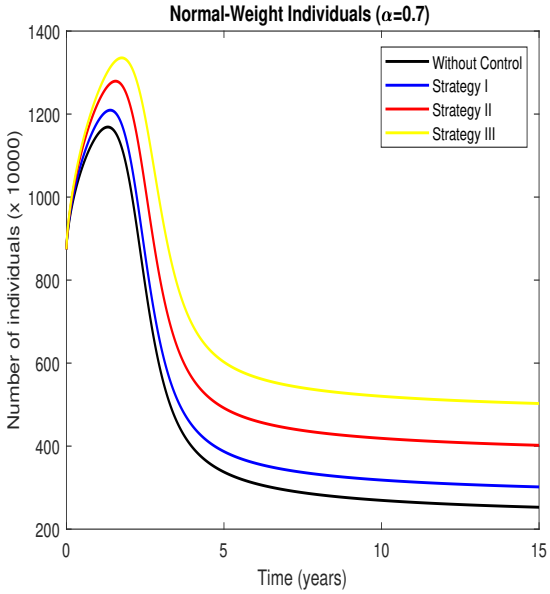
	Without Controls		Strategy I		Strategy II		Strategy III	
α	Min	Max	Min	Max	Min	Max	Min	Max
0.5	1.2000	1.1050e+03	1.2000	875.8010	1.2000	601.5214	1.2000	465.5587
0.7	1.2000	1.2355e+03	1.2000	980.0019	1.2000	635.4550	1.2000	563.9612
0.9	1.2000	1.3707e+03	1.2000	1.0280e+03	1.2000	652.7550	1.2000	574.7420
1.0	1.2000	1.3857e+03	1.2000	1.0455e+03	1.2000	681.5733	1.2000	581.6812

Table 7. Maximum and minimum values of the simulations of Model (4) of the obese individual compartment without strategies and with the different control strategies ($\times 10000$).

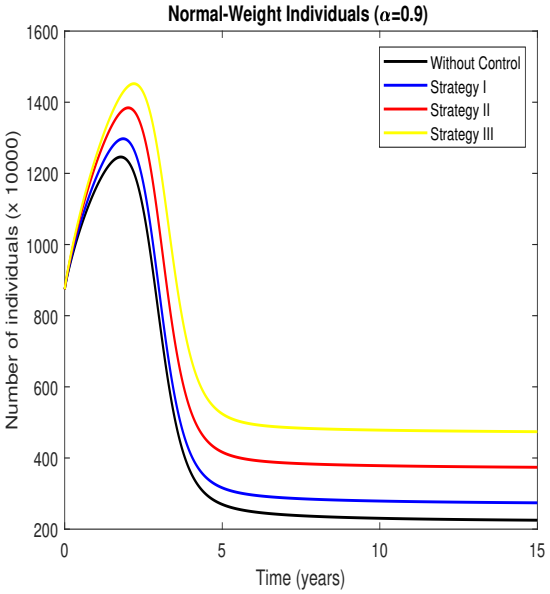
	Without Controls		Strategy I		Strategy II		Strategy III	
α	Min	Max	Min	Max	Min	Max	Min	Max
0.5	1.0100	1.5000	0.4176	1.5000	0.2177	1.5000	0.1828	1.5000
0.7	1.2544	1.5000	0.3746	1.5000	0.1554	1.5000	0.1128	1.5000
0.9	1.2953	1.5000	0.3644	1.5000	0.0929	1.5000	0.0807	1.5000
1.0	1.3245	1.5000	0.3595	1.5000	0.0806	1.5000	0.0642	1.5000



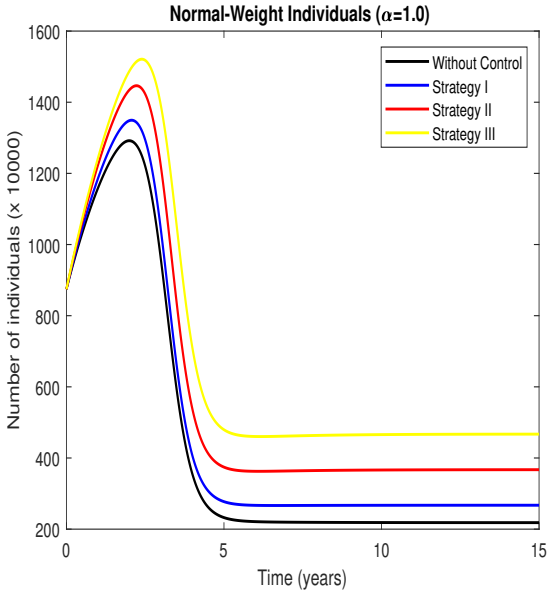
(a) Behavior of normal weight individuals over time with and without controls for $\alpha = 0.5$



(b) Behavior of normal weight individuals over time with and without controls for $\alpha = 0.7$



(c) Behavior of normal weight individuals over time with and without controls for $\alpha = 0.9$

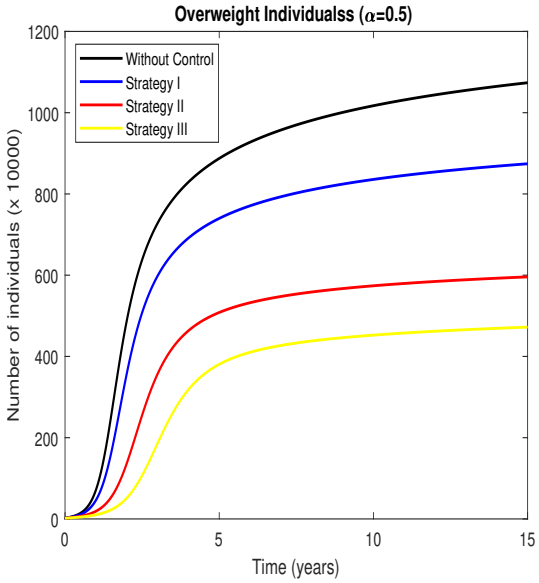


(d) Behavior of normal weight individuals over time with and without controls for $\alpha = 1.0$

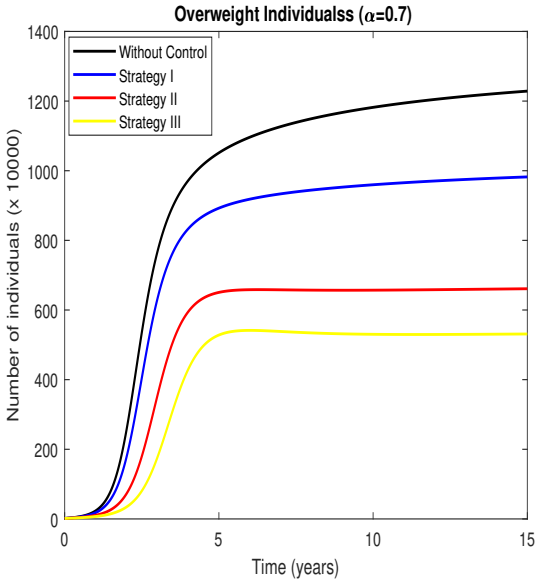
Figure 5. Behavior of normal weight individuals over time with and without controls for different fractional orders

Table 8. Maximum and minimum values of the simulations of Model (4) of the diabetic individual compartment without strategies and with the different control strategies ($\times 10000$).

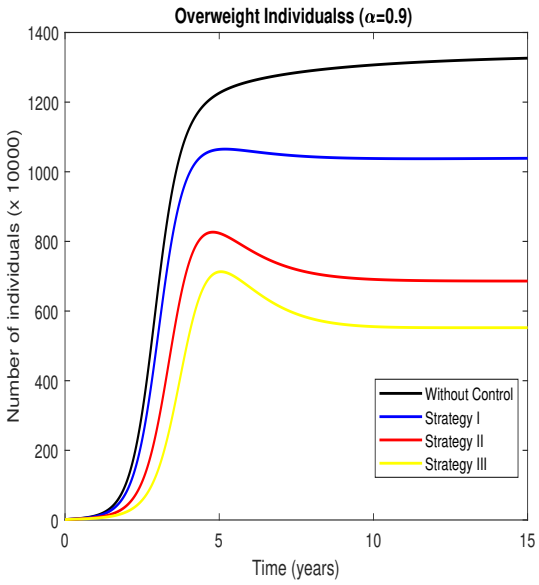
	Without Controls		Strategy I		Strategy II		Strategy III	
α	Min	Max	Min	Max	Min	Max	Min	Max
0.5	100.0000	1.5063e+03	100.0000	681.5702	100.0000	195.1652	100.0000	175.6160
0.7	100.0000	2.6223e+03	100.0000	1.2649e+03	100.0000	366.8341	100.0000	199.8900
0.9	100.0000	4.2701e+03	100.0000	2.2035e+03	100.0000	653.1200	100.0000	296.6318
1.0	100.0000	5.3356e+03	100.0000	2.8479e+03	100.0000	854.9811	100.0000	385.9001



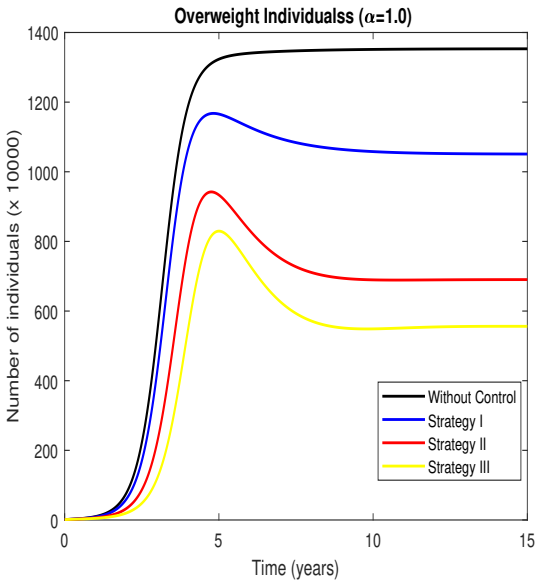
(a) Behavior of overweight individuals over time with and without controls for $\alpha = 0.5$



(b) Behavior of overweight individuals over time with and without controls for $\alpha = 0.7$

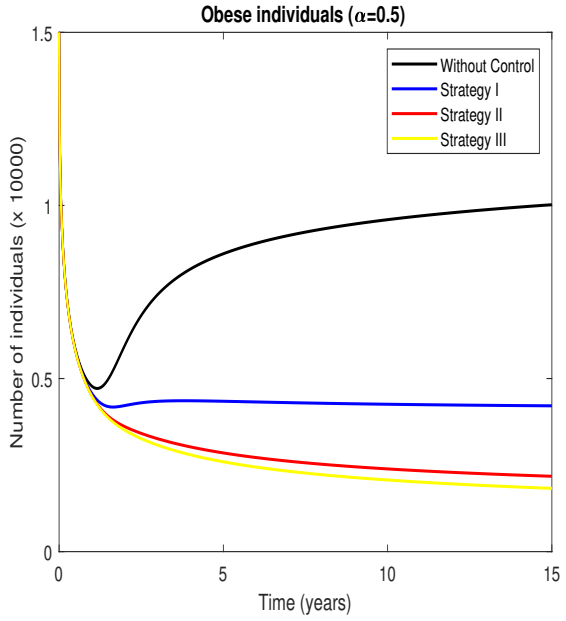


(c) Behavior of overweight individuals over time with and without controls for $\alpha = 0.9$

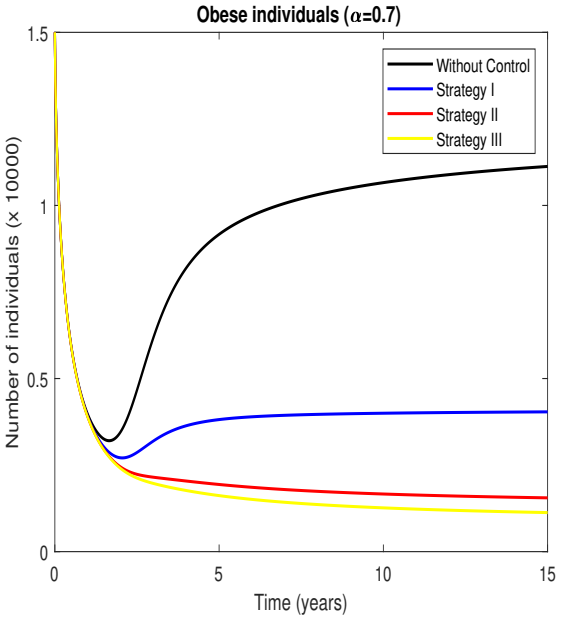


(d) Behavior of overweight individuals over time with and without controls for $\alpha = 1.0$

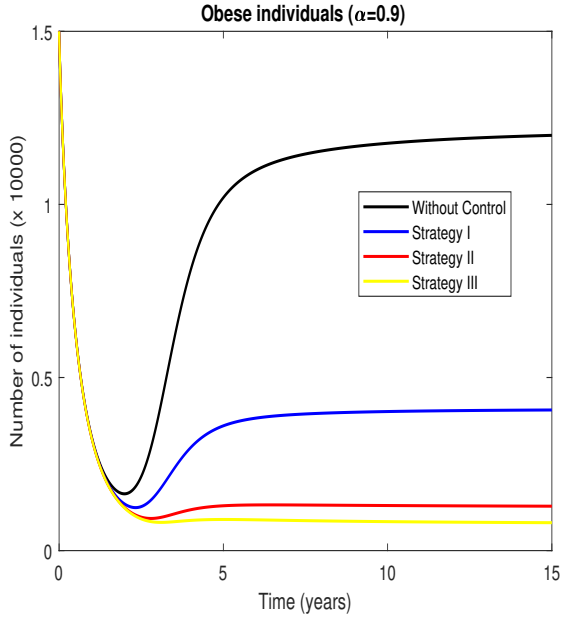
Figure 6. Behavior of overweight individuals over time with and without controls for different fractional orders



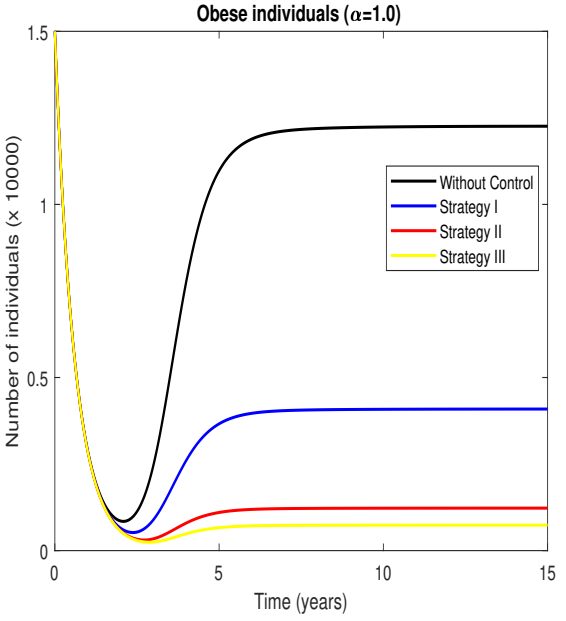
(a) Behavior of obese individuals over time with and without controls for $\alpha = 0.5$



(b) Behavior of obese individuals over time with and without controls for $\alpha = 0.7$



(c) Behavior of obese individuals over time with and without controls for $\alpha = 0.9$

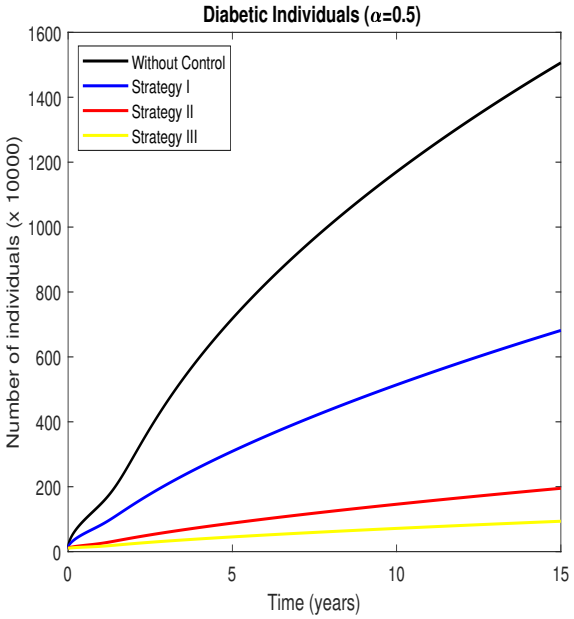


(d) Behavior of obese individuals over time with and without controls for $\alpha = 1.0$

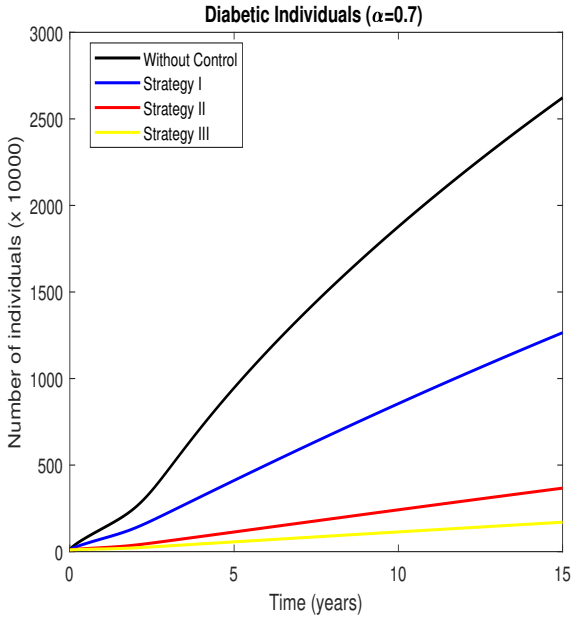
Figure 7. Behavior of obese individuals over time with and without controls for different fractional orders

6 Conclusions

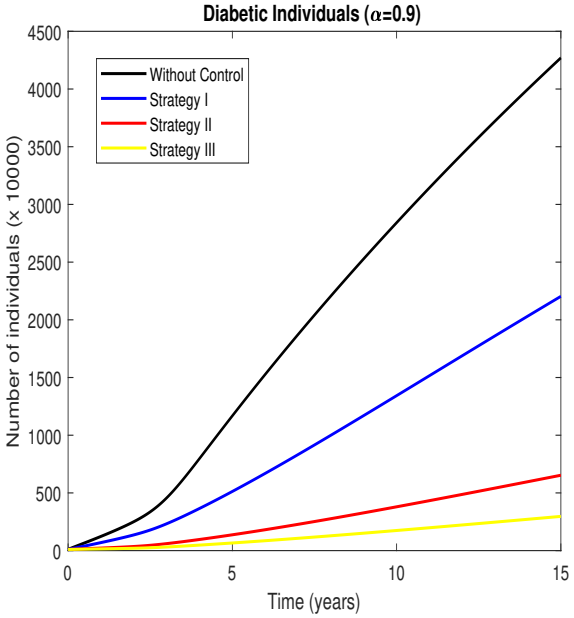
In this work, we present an optimal control problem focused on the reduction of overweight and obesity in a community and its impact on the diagnosis of diabetes. The model used in the formulation of the control problem is published in [22] and its basic properties such as the existence of solution, the biologically feasible region and a study of the basic reproduction number were demonstrated. Three controls were defined centered on individuals moving from normal and obese weight in the pathways studied in the model, by interaction with overweight and



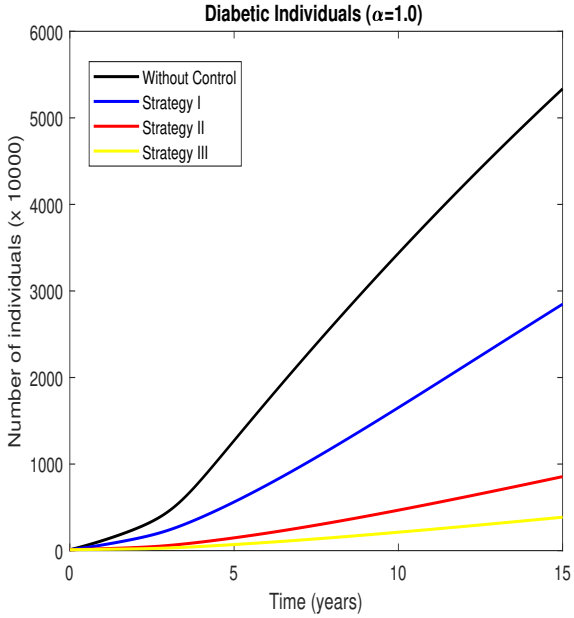
(a) Behavior of diabetic individuals over time with and without controls for $\alpha = 0.5$



(b) Behavior of diabetic individuals over time with and without controls for $\alpha = 0.7$



(c) Behavior of diabetic individuals over time with and without controls for $\alpha = 0.9$



(d) Behavior of diabetic individuals over time with and without controls for $\alpha = 1.0$

Figure 8. Behavior of diabetic individuals over time with and without controls for different fractional orders

obese individuals and by social pressure, and on overweight individuals reaching the obese state. We show the existence of optimal control using Pontryagin's maximum principle. Using data from the literature, we performed a global sensitivity study of the model and simulations of the optimal control problem. For global sensitivity analysis, we use Sobol's index in particular first, second and total order. To find the Sobol' indices, we use the exponentiation in chaos polynomials using two methodologies the OLS and LARS and also two techniques of Monte Carlo and Sobol' sampling. We also used different orders, first, second and total, and in the case of first and total we matched the order and the influential parameters. The results showed that for a threshold

value of 10^{-4}, we have seven influential parameters, in particular the most influential parameter was the effective contact rate in the negative effect that an overweight or obese individual can have on a normal weight individual. In addition, the other possible case of weight gain in a normal-weight individual is in third place and all parameters that are involved with the controls are among the most influential second Sobol' indices. Three control strategies were developed, focused on controlling social pressure and the evolution from overweight to obese, controlling all the studied ways for a normal-weight individual to become overweight and the ways to go from normal weight to overweight and from overweight to obese. All strategies were efficient not only in reducing overweight and obesity in the community but also the number of cases of diabetes. The strategy that showed the best results qualitatively and quantitatively was to control all possible ways of going from normal weight to overweight and from overweight to obese (all active controls). The second with the best results was to control only the possible cases addressed in the model of going from normal weight to overweight. With the model and the data used, we show that the most influential parameter in the model is the negative impact that an obese or overweight individual can have on an individual with normal weight and also that by controlling all possible evolutions from normal weight to overweight and from overweight to obese we reduce not only overweight and obesity in the community but also diabetes and that if we control only the ways in which an individual with normal weight evolves to overweight we would also obtain good results.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Ethical approval

The authors declare that this research complies with ethical standards. This research does not involve human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 13179.

Author's contributions

E.M.D.M.: Methodology, Conceptualization, Validation, Software, Data Curation, Writing the Original Draft. R.A.R., A.P. and S.B.: Writing - Review & Editing, Supervision. All authors have read and agreed to the published version of the manuscript

Acknowledgements

The authors would like to thank the reviewers and the editorial team of the journal.

References

[1] Jeong, S.M., Jung, J., Yang, Y.S., Kim, W., Cho, I.Y., Lee, Y.B. et al. Obesity fact sheet: Prevalence of obesity and abdominal obesity in adults, adolescents, and children in Korea from 2012 to 2021. *Journal of Obesity & Metabolic Syndrome*, 33(1), 27-35, (2024). [CrossRef]

[2] Chong, B., Jayabaskaran, J., Kong, G., Chan, Y.H., Chin, Y.H., Goh, R. et al. Trends and predictions of malnutrition and obesity in 204 countries and territories: an analysis of the Global Burden of Disease Study 2019. *eClinicalMedicine Part of The Lancet*, 57, 101850, (2023). [CrossRef]

[3] Hu, K. and Staiano, A.E. Trends in obesity prevalence among children and adolescents aged 2 to 19 years in the US from 2011 to 2020. *JAMA Pediatrics*, 176(10), 1037-1039, (2022). [CrossRef]

[4] Ong, K.L., Stafford, L.K., McLaughlin, S.A., Boyko, E.J., Vollset, S.E., Smith, A.E. et al. Global, regional, and national burden of diabetes from 1990 to 2021, with projections of prevalence to 2050: a systematic analysis for the Global Burden of Disease Study 2021. *The Lancet*, 402(10397), 203-234. [CrossRef]

[5] Nuttall, F.Q. Body mass index: obesity, BMI, and health: a critical review. *Nutrition Today*, 50(3), 117-128, (2015). [CrossRef]

[6] Bentout, S., Djilali, S. and Atangana, A. Bifurcation analysis of an age-structured prey–predator model with infection developed in prey. *Mathematical Methods in the Applied Sciences*, 45(3), 1189-1208, (2022). [CrossRef]

[7] Bentout, S. and Djilali, S. Asymptotic profiles of a nonlocal dispersal SIR epidemic model with treat-age in a heterogeneous environment. *Mathematics and Computers in Simulation*, 203, 926-956, (2023). [CrossRef]

[8] Paul, S., Mahata, A., Mukherjee, S., Das, M., Mali, P.C., Roy, B. et al. Study of fractional order SIR model with MH type treatment rate and its stability analysis. *Bulletin of Biomathematics*, 2(1), 85-113, (2024). [CrossRef]

[9] Bentout, S., Djilali, S., Kuniya, T. and Wang, J. Mathematical analysis of a vaccination epidemic model with nonlocal diffusion. *Mathematical Methods in the Applied Sciences*, 46(9), 10970-10994, (2023). [CrossRef]

[10] Nabil, H. and Hamaizia, T. A three-dimensional discrete fractional-order HIV-1 model related to cancer cells, dynamical analysis and chaos control. *Mathematical Modelling and Numerical Simulation with Applications*, 4(3), 256-279, (2024). [CrossRef]

[11] Iwa, L.L., Omame, A. and Chioma, S. A fractional-order model of COVID-19 and Malaria co-infection. *Bulletin of Biomathematics*, 2(2), 133-161, (2024). [CrossRef]

[12] Özköse, F., Şenel, M.T. and Habbireeh, R. Fractional-order mathematical modelling of cancer cells-cancer stem cells-immune system interaction with chemotherapy. *Mathematical Modelling and Numerical Simulation with Applications*, 1(2), 67-83, (2021). [CrossRef]

[13] Ahmed, I., Akgül, A., Jarad, F., Kumam, P. and Nonlaopon, K. A Caputo-Fabrizio fractional-order cholera model and its sensitivity analysis. *Mathematical Modelling and Numerical Simulation with Applications*, 3(2), 170-187, (2023). [CrossRef]

[14] Evirgen, F., Uçar, E., Uçar, S. and Özdemir, N. Modelling influenza a disease dynamics under Caputo-Fabrizio fractional derivative with distinct contact rates. *Mathematical Modelling and Numerical Simulation with Applications*, 3(1), 58-73, (2023). [CrossRef]

[15] Din, A. and Abidin M.Z. Analysis of fractional-order vaccinated Hepatitis-B epidemic model with Mittag-Leffler kernels. *Mathematical Modelling and Numerical Simulation with Applications*, 2(2), 59-72, (2022). [CrossRef]

[16] Ejima, K., Thomas, D.M. and Allison, D.B. A mathematical model for predicting obesity transmission with both genetic and nongenetic heredity. *Obesity*, 26(5), 927–933, (2018). [CrossRef]

[17] Kim, S. and Kim, S.Y. Mathematical modeling for the obesity dynamics with psychological and social factors. *East Asian Mathematical Journal*, 34(3), 317-330, (2018). [CrossRef]

[18] Paudel, L.P. Mathematical modeling on the obesity dynamics in the southeastern region and the effect of intervention. *Universal Journal of Mathematics and Applications*, 7(3), 41-52, (2019). [CrossRef]

[19] Al-Tuwairqi, S.M. and Matbouli, R.T. Modeling dynamics of fast food and obesity for evaluating the peer pressure effect and workout impact. *Advances in Difference Equations*, 2021, 59, (2021). [CrossRef]

[20] Bernard, S., Cesar T. and Pietrus, A. The impact of media coverage on obesity. *Contemporary Mathematics*, 3(1), 60-71, (2022). [CrossRef]

[21] Moya, E.D., Pietrus, A. and Bernard, S. Mathematical model for the Study of obesity in a population and its impact on the growth of diabetes. *Mathematical Modelling and Analysis*, 28(4), 611-635, (2021). [CrossRef]

[22] Moya, E.M.D., Pietrus, A., Bernard, S. and Nuiro, S.P. A mathematical model with fractional order for obesity with positive and negative interactions and its impact on the diagnosis of diabetes. *Journal of Mathematical Sciences and Modelling*, 6(3), 133-149, (2023). [CrossRef]

[23] Camargo, R.F. and Oliveira E.C. *Cálculo Fracionário*. Livraria da Fısica: São Paulo, (2015).

[24] Kheiri, H. and Jafari, M. Optimal control of a fractional-order model for the HIV/AIDS epidemic. *International Journal of Biomathematics*, 11(07), 1850086, (2018). [CrossRef]

[25] Diethelm, K. *The Analysis of Fractional Differential Equations*. Springer Berlin: Heidelberg, (2014). [CrossRef]

[26] Barros, L.C.D., Lopes, M.M., Pedro, F.S., Esmi, E., Santos, J.P.C.D. et al. The memory effect on fractional calculus: an application in the spread of COVID-19. *Computational and Applied Mathematics*, 40, 72, (2021). [CrossRef]

[27] Saeedian, M., Khalighi, M., Azimi-Tafreshi, N., Jafari, G.R. and Ausloos, M. Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model. *Physical Review E*, 95(2), 022409, (2017). [CrossRef]

[28] Baleanu, D., Ghassabzade, F.A., Nieto, J.J. and Jajarmi, A. On a new and generalized fractional model for a real cholera outbreak. *Alexandria Engineering Journal*, 61(11), 9175-9186, (2022). [CrossRef]

[29] Monteiro, N.Z. and Mazorche, S.R. Fractional derivatives applied to epidemiology. *Trends in*

Computational and Applied Mathematics, 22(2), 157-177, (2021). [CrossRef]

[30] Vellappandi, M., Kumar, P. and Govindaraj, V. Role of fractional derivatives in the mathematical modeling of the transmission of Chlamydia in the United States from 1989 to 2019. *Nonlinear Dynamics*, 111, 4915–4929, (2023). [CrossRef]

[31] Inc, M., Acay, B., Berhe, H.W., Yusuf, A., Khan, A. and Yao, S.W. Analysis of novel fractional COVID-19 model with real-life data application. *Results in Physics*, 23, 103968, (2021). [CrossRef]

[32] Wang, H., Jahanshahi, H., Wang, M.K., Bekiros, S., Liu, J. and Aly, A.A. A Caputo–Fabrizio fractional-order model of HIV/AIDS with a treatment compartment: Sensitivity analysis and optimal control strategies. *Entropy*, 23(5), 610, (2021). [CrossRef]

[33] Sweilam, N.H., Al-Mekhlafi, S.M. and Baleanu, D. Optimal control for a fractional tuberculosis infection model including the impact of diabetes and resistant strains. *Journal of Advanced Research*, 17, 125-137, (2019). [CrossRef]

[34] Baba, B.A. and Bilgehan, B. Optimal control of a fractional order model for the COVID–19 pandemic. *Chaos, Solitons & Fractals*, 144, 110678, (2021). [CrossRef]

[35] Diethelm, K. A fractional calculus based model for the simulation of an outbreak of dengue fever. *Nonlinear Dynamics*, 71, 613–619, (2013). [CrossRef]

[36] Pinto, C.M.A. and Carvalho, A.R.M. Diabetes mellitus and TB co-existence: Clinical implications from a fractional order modelling. *Applied Mathematical Modelling*, 68, 219-243, (2019). [CrossRef]

[37] Moya, E.M.D. and Rodrigues, D.S. A mathematical model for the study of latent tuberculosis under 3HP and 1HP regimens. *Mathematical Modelling and Control*, 4(4), 400-416, (2024). [CrossRef]

[38] Centers for Disease Control and Prevention, National Center for Health Statistics, *About Underlying Cause of Death, 1999-2020*. http://wonder.cdc.gov/ucd-icd10.html

[39] Centers for Disease Control and Prevention, National Diabetes Statistics Report, (2024). https://www.cdc.gov/diabetes/pdfs/data/statistics/national-diabetes-statistics-report.pdf

[40] World Health Organization (WHO), Obesity and Overweight, (2024). https://www.who.int/news-room/fact-sheets/detail/obesity-and-overweight#:~:text=Obesity%20is%20a%20chronic%20complex,the%20risk%20of%20certain%20cancers.

[41] Diethelm, K., Ford, N.J. and Freed, A.D. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29, 3-22, (2002). [CrossRef]

[42] Diethelm, K., Ford, N.J. and Freed, A.D. Detailed error analysis for a fractional Adams method. *Numerical Algorithms*, 36, 31-52, (2004). [CrossRef]

[43] Diethelm, K. and Freed, A.D. The FracPECE subroutine for the numerical solution of differential equations of fractional order, Forschung und wissenschaftliches Rechnen, Beiträge zum Heinz-Billing-Preis 1998. *Gesellschaft für wissenschaftliche Datenverarbeitung, Göttingen*, 57-71, (1999). [CrossRef]

[44] Razavi, S., Jakeman, A., Saltelli, A., Prieur, C., Iooss, B., Borgonovo, E. et al. The future of sensitivity analysis: an essential discipline for systems modeling and policy support. *Environmental Modelling & Software*, 137, 104954, (2021). [CrossRef]

[45] Boas, S.E., Jimenez, M.I.N, Merks, R.M. and Blom, J.G. A global sensitivity analysis approach

for morphogenesis models. *BMC Systems Biology*, 9, 85, (2015). [CrossRef]

[46] Sobol, I.M. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and Computers in Simulation*, 55(1-3), 271-280, (2001). [CrossRef]

[47] Janon, A., Klein, T., Lagnoux, A., Nodet, M. and Prieur, C. Asymptotic normality and efficiency of two Sobol index estimators. *ESAIM: Probability and Statistics*, 18, 342-364, (2014). [CrossRef]

[48] Tosin, M., Côrtes, A.M.A. and Cunha A. A A tutorial on sobol'global sensitivity analysis applied to biological models. In *Networks in Systems Biology: Computational Biology* (pp. 93-118). Springer: Cham, (2020). [CrossRef]

[49] Ghanem, R., Higdon, D. and Owhadi, H. *Handbook of Uncertainty Quantification*. Springer: New York, (2017). [CrossRef]

[50] Konakli, K. and Sudret B. Global sensitivity analysis using low-rank tensor approximations. *Reliability Engineering & System Safety*, 156, 64-83, (2016). [CrossRef]

[51] Smith, R.C. *Uncertainty Quantification: Theory, Implementation, and Applications*. Society for Industrial and Applied Mathematics: 3600 University City Science Center Philadelphia, United States, (2013). [CrossRef]

[52] Sudret, B. Global sensitivity analysis using polynomial chaos expansions. *Reliability Engineering & System Safety*, 93(7), 964-979, (2008). [CrossRef]

[53] Sudret, B. and Caniou, Y. Analysis of covariance (ANCOVA) using polynomial chaos expansions. In Proceedings, *11th International Conference on Structural Safety and Reliability (ICOSSAR 2013)*, pp. 3275-3281, New York, USA, (2013). [CrossRef]

[54] Sudret, B., Blatman, G. and Berveiller, M. Response surfaces based on polynomial chaos expansions. In *Construction reliability: safety, variability and sustainability* (pp. 147-168). Wiley: New York, (2013). [CrossRef]

[55] Marelli, S. and Sudret, B. UQLab user manual-polynomial chaos expansions. *Chair of risk, Safety and uncertainty quantification, ETH Zurich*, UQLab-V1.1-104, (2018). [CrossRef]

[56] Xiu, D. *Numerical Methods for Stochastic Computations: A Spectral Method Approach*. Princeton University Press: USA, (2010). [CrossRef]

[57] Tarakanov, A. and Elsheikh, A.H. Regression-based sparse polynomial chaos for uncertainty quantification of subsurface flow models. *Journal of Computational Physics*, 399, 108909, (2019). [CrossRef]

[58] Klink, D., Meyer, P. and Steyn, W. Comparison of Coefficient Calculation Techniques for NLPLS PCE Models of Antennas. *2022 16th European Conference on Antennas and Propagation (EuCAP)*, pp. 1-5, Madrid, Spain, (2022). [CrossRef]

[59] Xu, Z., Zhou, X. and Qian, Q. The global sensitivity analysis of slope stability based on the least angle regression. *Natural Hazards*, 105, 2361-2379, (2021). [CrossRef]

[60] Petley, D. Global patterns of loss of life from landslides. *Geology*, 40(10), 927-930, (2012). [CrossRef]

[61] Efron, B., Hastie, T., Johnstone, L. and Tibshirani, R. Least angle regression. *Annals of Statistics*, 32(2), 407-499, (2004). [CrossRef]

[62] Liu, C., Yang, S.X. and Deng, L. A comparative study for least angle regression on NIR spectra analysis to determine internal qualities of navel oranges. *Expert Systems with Applications*, 42(22), 8497-503, (2015). [CrossRef]

[63] Zhang, L. and Li, K. Forward and backward least angle regression for nonlinear system identification. *Automatica*, 53, 94-102, (2015). [CrossRef]

[64] Pati, Y.C, Rezaiifar, R. and Krishnaprasad, P.S. Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition. In Proceedings, *27th Asilomar Conference on Signals, Systems and Computers*, pp. 40-44, Pacific Grove, USA, (1993, November). [CrossRef]

[65] Mallat, S.G. and Zhang, Z. Matching pursuits with time-frequency dictionaries. *IEEE Transactions on Signal Processing*, 41(12), 3397-3415, (1993). [CrossRef]

[66] Baptista, R., Stolbunov, V. and Nair, P.B. Some greedy algorithms for sparse polynomial chaos expansions. *Journal of Computational Physics*, 387, 303-325, (2019). [CrossRef]

[67] Halton, J.C. and Davis, F.J. Illustration of Sampling-Based Methods for Uncertainty and Sensitivity Analysis. *Risk Analysis*, 22(3), 591-622, (2002). [CrossRef]

[68] Tarantola, S., Becker, W. and Zeitz, B. A comparison of two sampling methods for global sensitivity analysis. *Computer Physics Communications*, 183(5), 1061-1072, (2012). [CrossRef]

[69] Pharr, M., Jakob, W. and Humphreys, G. 07 - Sampling and reconstruction. In, *Physically Based Rendering (Third Edition) From Theory to Implementation* (pp. 401-504). Morgan Kaufhann: USA, (2016). [CrossRef]

[70] Renardy, M., Joslyn, L.R., Millar, J.A. and Kirschner, D.E. To Sobol or not to Sobol? The effects of sampling schemes in systems biology applications. *Mathematical Biosciences*, 337, 108593, (2021). [CrossRef]

[71] Sun, X., Croke, B., Roberts, S. and Jakeman, A. Comparing methods of randomizing Sobol' sequences for improving uncertainty of metrics in variance-based global sensitivity estimation. *Reliability Engineering & System Safety*, 210, 107499, (2021). [CrossRef]

[72] Sheikholeslami, R. and Razavi, S. Progressive Latin Hypercube Sampling: An efficient approach for robust sampling-based analysis of environmental models. *Environmental Modelling & Software*, 93, 109-126, (2017). [CrossRef]

[73] Wang, Z., Zhao, D., Heidari, A.A, Chen, Y., Chen, H. and Liang, G. Improved Latin hypercube sampling initialization-based whale optimization algorithm for COVID-19 X-ray multi-threshold image segmentation. *Scientific Reports*, 14, 13239, (2024). [CrossRef]

[74] Marelli, S., Lamas, C. Konakli, K., Mylonas, C., Wiederkehr, P. and Sudret, B. UQLab user manual–Sensitivity analysis. *Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich*, UQLab-V0.9-106, (2015). [CrossRef]

[75] Lataniotis, C., Marelli, S. and Sudret, B. UQLab user manual-The MODEL module. *Chair of Risk, Safety and Uncertainty Quantification, ETH Zürich*, UQLab-V0.9-103, (2015). [CrossRef]

Mathematical Modelling and Numerical Simulation with Applications (MMNSA)
(https://dergipark.org.tr/en/pub/mmnsa)

How to cite this article: Moya, E.M.D., Rodriguez, R.A., Pietrus, A. and Bernard, S. (2024). A study of fractional optimal control of overweight and obesity in a community and its impact on the diagnosis of diabetes. *Mathematical Modelling and Numerical Simulation with Applications*, 4(4), 514-543. https://doi.org/10.53391/mmnsa.1555670