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On (k,3)-arcs derived by Ceva configurations in PG(2,5)

Elif Altıntaş Kahriman[a,*], Ayşe Bayar[b]

[a]*Haliç University, Faculty of Engineering, Software Engineering, İstanbul, 34000, Turkey,*
ORCID: 0000-0002-3454-0326
[b]*Eskişehir Osmangazi University, Faculty of Science, Department of Mathematics and Computer Sciences, Eskişehir, 26000, Turkey,*
ORCID: 0000-0002-2210-5423

Abstract

In this study, we investigate complete (k,2)-arcs and (k,3)-arcs derived from a Ceva configuration in the projective plane of order five by implementing an algorithm in C#. Our results indicate the existence of a complete (6,2)-arc that has no points in common with the (7,3)-arc formed by the Ceva configuration. Furthermore, we identify eight different complete (10,3)-arcs that include a Ceva configuration. Additionally, we explore cyclic order Ceva configurations, denoted as C_1, C_2, C_3, and C_4, all of which have a common center. The vertices of each configuration C_i are on the sides of the preceding configuration C_{i-1}, with i ranging from 2 to 4. We determine different thirty-two complete *(10,3)-arcs* and different two complete *(6,2)-arcs* by constructing cyclic order Ceva configurations C_1, C_2, C_3, C_4 with a common center in PG(2,5).

Keywords: projective plane; (k,3)-arc; Ceva configuration

1. Introduction

In projective geometry, arcs are very important and have many uses in combinatorics and related domains. A k-arc is defined as a set K of k points (where k ≥ 3) that are not all located on the same line in a finite projective plane π (which need not be Desarguesian). A $(p+2)$ −arc is referred to as a hyperoval. The highest value of k can be achieved only when p is even, while $k \geq p+2$ if the plane π has an order of p. Ovals are commonly referred to in the literature, with Hirschfeld being a notable source [1]. Research on arcs in projective planes is extensive, particularly regarding full (k,2)-arcs that create complete quadrangles, leading to Fano planes in the projective plane, as analyzed in [2, 3]. The identification and classification of Fano subplanes in a projective plane of order nine, related to parts of a left nearfield of order nine, are described in [4].

Fano configurations in 5-dimensional projective spaces over GF(2) are discovered in [5]. In the projective planes of order nine and twenty-five, the simplest Cartesian Group techniques for classifying (k,3)-arcs are outlined in references [6, 7]. The research by Altıntaş investigates (k,2)-arcs in the projective plane of order five, coordinated by elements from GF(5), using an arc-finding algorithm developed in C# [8]. Complete (k,n)−arcs, n=2,3,4 related to Desargues configuration are determined in [9].

One of the conclusions that eluded Euclid is Ceva's Theorem, which will be stated and examined from now on. It appears nowhere in his thirteen volumes of the Elements. Giovanny Ceva established the theory that bears his name circa 1678. The outcome was published in his work De Lineis Rectis, also known as The Straight Line. In addition, he addressed various additional geometric results and rediscovered Menelaus' Theorem. According to the Ceva Theorem, if we draw segments from a triangle's vertices to the interior points of its opposite sides, the segments will only be concurrent that is, share a common point if and only if the ratios taken cyclically that the cevians on the opposite sides determine multiply out to 1 [10].

In the recent Benyi-Curgus generalization of the theorems of Ceva and Menelaus [11], both the collinearity of points and the concurrence of straightlines specified by six points on the edges of a triangle are characterized. Both Menelaus's and Ceva's theorems have intriguing applications in projective and Euclidean geometries.

Theorems of Ceva and Menelaus are well-known conclusions. However, these theorems characterize a projective property through an affine property: concurrency in Ceva's theorem and collinearity in Menelaus' theorem. Benitez aims to get over this, therefore. In particular, the cross ratio a projective quantity is used in the study to characterize the concurrence of the cevians [12]. One can express Menelaus' theorem in a projective form by using the dual of this latter characterization. Nicolae discusses the Ceva-Menelaus transformation of a line into four curves. This proved to be a parabola, a hyperbola, or a bean ellipse. The triangle's three straight lines are tangent to each of the conics. Furthermore, in the study the harmonic transform for a ceviana has no envelope since it is a beam of straight lines flowing through a point is discovered [13].

Menelaus and Ceva theorems in projective planes P_2(F), where F is the field of characteristic not equal to two, were given by Kelly B. Funk [14]. Menelaus and Ceva 6-figures were first introduced in Moufang projective planes by Kaya and Çiftçi in [15]. Menelaus and Ceva's 6-figures in fibered geometry were examined in [16] with multiple degrees of membership of the points and lines of the basic geometry. Intuitionistic fuzzy projective Menelaus and Ceva's conditions in the intuitionistic fuzzy projective plane with base plane that is projective plane are given by Akça et al. [17].

The primary goal of this work is to examine (k,3)-arcs in PG(2,5) that are associated with the Ceva configuration. In Section 2, we present specific terms that are important for our research. The projective plane of order five over GF(5) is constructed in Section 3 together with its lines, points, and incidence relation. A Ceva configuration is then determined in this projective plane. In Section 4, we introduce our algorithm and method to find all the results that related to (k,n)− arcs derived by a Ceva configuration in PG(2,5). It is demonstrated that eight distinct complete (10,3)-arcs exist, each containing a Ceva configuration and a complete (6,2)-arc formed by the remaining points, utilizing the algorithm implemented in C#. Also, cylic order Ceva configurations C_1, C_2, C_3, C_4 derived from the Ceva configuration C_1 are defined in PG(2,5). We give the examples related to the complete (10,3)-arcs and complete (6,2)-arcs derived by the cyclic order Ceva configurations C_1, C_2, C_3, C_4 with the common center. In the last section, we give our results.

2. Preliminaries

This section offers a review of important definitions and theorems concerning projective planes, along with an outline of certain properties of arcs in these planes.

Definition 1. *A projective plane (N, D, ∘) consists of a set N of points, and a set D of subsets of N, called lines, such that every pair of points is contained in exactly one line, every two different lines intersect in exactly one point, and there exist four points, no three of which are collinear.*

Definition 2. *The vector space V(n+1,q) is (n+1)-dimensional and consists of vectors with coordinates from the finite field GF(q). The projective space PG(n,q) is defined as the collection of points, each corresponding to a line that passes through the origin in V(n+1,q). Specifically, this means that each point in PG(n,q) can be represented as P(x), where x is any non-zero vector in V(n+1,q). If K is the finite field GF(q), also denoted as F_q, then the n-dimensional projective plane is referred to as PG(n,K) or PG(n,q). In this context, q represents the order of PG(n,q). The number of points in this projective plane can be determined using the formula*

$$\theta(\mathrm{n}) = \frac{q^{n+1}-1}{q-1}.$$

$(x_1, x_2, \dots, x_n)$ represents a point in N, where $x_1, x_2, \dots, x_n$ are not all zero, and $(\lambda x_1, \lambda x_2, \dots, \lambda x_n) \equiv (x_1, x_2, \dots, x_n)$, $\lambda \in K \backslash \{0\}$. Similarly, the notation [a_1,a_2,...,a_n] denotes any line in D, where $a_1, a_2, \dots, a_n$ are not all zero. The relationship $[\mu a_1, \dots, \mu a_n] \equiv [a_1, \dots, a_n]$ holds for $\mu \in K \backslash \{0\}$. The projective plane P_2K is characterized as a point-line geometry (N, D, ∘) defined by K. The incidence relation is given by ∘: $(x_1, \dots, x_n) \circ [a_1, \dots, a_n]$ if and only if $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = 0$.

Let *p* denote a prime number and *r* a positive integer. The projective plane of order $q = p^r$ over the finite field $K = GF(p^r)$, where p^r represents the number of elements, is expressed as $P_2K = PG(2, p^r)$ [18].

Definition 4. *Let P be a projective plane. In P, a 6-figure is a sequence of 6 distinct points* $(A_1A_2A_3, A_1'A_2'A_3')$ *such that* $A_1A_2A_3$ *is a triangle, and* $A_1' \in A_2A_3$, $A_2' \in A_3A_1$, $A_3' \in A_1A_2$. $A_1, A_2, A_3, A_1', A_2', A_3'$ *are called vertices of this 6-figure. If the lines* A_1A_1', A_2A_2', A_3A_3' *are concurrent,* $(A_1A_2A_3, A_1'A_2'A_3')$ *is called a Ceva 6-figure* [15].

Definition 5. *In a projective plane, a (k,n)-arc K is defined as a set of k points such that any line intersects K at exactly n points, with no line intersecting the set at more than n points, where n ≥ 2 [19].*

Definition 6. *A line l in a projective plane is defined as an μ-secant of a (k,n)-arc K if it intersects K at μ points. Let τ_i represent the total number of i-secants to K. The notations*σ_i *or* $\sigma_i(Q)$ *denote the count of i-secants to the set K that pass through a point Q, which is part of* P \ K*. A point Q is classified as an index zero point if the condition* $\sigma_n(Q) = 0$ *holds [20].*

If no (k+1,n)-arc contains a (k,n)-arc, then the (k,n)-arc is considered complete [20].

Definition 7. *The points out of a* (k, n) *−arc K in P which passes through it i-secant of K is called a point of index i* [20].

3. The Projective plane of order 5

The study considers PG(2,5), which is built over GF(5) using the irreducible polynomial $f(x) = x^3 + 2x^2 + x - 1$. The elements of GF(5) with thirty one points and thirty one lines are 0, 1, 2, 3, and 4. In the projective plane order five, every line consists of six points, and each point is associated with six lines that pass through it [21].

The projective plane of order five has a point set N defined as $\boldsymbol{N} = \{\boldsymbol{N_i} | \, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{31}\}$ where

$\boldsymbol{N_1} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{1})$, $\boldsymbol{N_2} = (\boldsymbol{1}, \boldsymbol{1}, \boldsymbol{1})$, $\boldsymbol{N_3} = (\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{2})$, $\boldsymbol{N_4} = (\boldsymbol{1}, \boldsymbol{4}, \boldsymbol{2})$, $\boldsymbol{N_5} = (\boldsymbol{1}, \boldsymbol{4}, \boldsymbol{3})$,
$\boldsymbol{N_6} = (\boldsymbol{1}, \boldsymbol{3}, \boldsymbol{4})$, $\boldsymbol{N_7} = (\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{3})$, $\boldsymbol{N_8} = (\boldsymbol{1}, \boldsymbol{3}, \boldsymbol{1})$, $\boldsymbol{N_9} = (\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{4})$, $\boldsymbol{N_{10}} = (\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{4})$,

$\boldsymbol{N_{11}} = (\mathbf{1}, \mathbf{0}, \mathbf{1})$, $\boldsymbol{N_{12}} = (\mathbf{1}, \mathbf{2}, \mathbf{1})$, $\boldsymbol{N_{13}} = (\mathbf{1}, \mathbf{2}, \mathbf{3})$, $\boldsymbol{N_{14}} = (\mathbf{1}, \mathbf{3}, \mathbf{0})$, $\boldsymbol{N_{15}} = (\mathbf{0}, \mathbf{1}, \mathbf{3})$, $\boldsymbol{N_{16}} = (\mathbf{1}, \mathbf{1}, \mathbf{3})$, $\boldsymbol{N_{17}} = (\mathbf{1}, \mathbf{3}, \mathbf{3})$, $\boldsymbol{N_{18}} = (\mathbf{1}, \mathbf{3}, \mathbf{2})$, $\boldsymbol{N_{19}} = (\mathbf{1}, \mathbf{4}, \mathbf{0})$, $\boldsymbol{N_{20}} = (\mathbf{0}, \mathbf{1}, \mathbf{4})$, $\boldsymbol{N_{21}} = (\mathbf{1}, \mathbf{1}, \mathbf{0})$, $\boldsymbol{N_{22}} = (\mathbf{0}, \mathbf{1}, \mathbf{1})$, $\boldsymbol{N_{23}} = (\mathbf{1}, \mathbf{1}, \mathbf{2})$, $\boldsymbol{N_{24}} = (\mathbf{1}, \mathbf{4}, \mathbf{4})$, $\boldsymbol{N_{25}} = (\mathbf{1}, \mathbf{0}, \mathbf{2})$, $\boldsymbol{N_{26}} = (\mathbf{1}, \mathbf{4}, \mathbf{1})$, $\boldsymbol{N_{27}} = (\mathbf{1}, \mathbf{2}, \mathbf{0})$, $\boldsymbol{N_{28}} = (\mathbf{0}, \mathbf{1}, \mathbf{2})$, $\boldsymbol{N_{29}} = (\mathbf{1}, \mathbf{1}, \mathbf{4})$, $\boldsymbol{N_{30}} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$, $\boldsymbol{N_{31}} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$.

Table 1 presents the incidence relation between the points and the lines in the projective plane *PG(2,5).* In this table, each row corresponds to a specific line, denoted as $\boldsymbol{D_i}$, where $\boldsymbol{i}$ ranges from 1 to 31. For each line $\boldsymbol{D_i}$, the table lists the points on that line.

Table 1. The points and lines based on incidence relation.

D_1	N_2	N_3	N_{17}	N_{22}	N_{24}	N_{30}
D_2	N_3	N_4	N_{18}	N_{23}	N_{25}	N_{31}
D_3	N_4	N_5	N_{19}	N_{24}	N_{26}	N_1
D_4	N_5	N_6	N_{20}	N_{25}	N_{27}	N_2
D_5	N_6	N_7	N_{21}	N_{26}	N_{28}	N_3
D_6	N_7	N_8	N_{22}	N_{27}	N_{29}	N_4
D_7	N_8	N_9	N_{23}	N_{28}	N_{30}	N_5
D_8	N_9	N_{10}	N_{24}	N_{29}	N_{31}	N_6
D_9	N_{10}	N_{11}	N_{25}	N_{30}	N_1	N_7
D_{10}	N_{11}	N_{12}	N_{26}	N_{31}	N_2	N_8
D_{11}	N_{12}	N_{13}	N_{27}	N_1	N_3	N_9
D_{12}	N_{13}	N_{14}	N_{28}	N_2	N_4	N_{10}
D_{13}	N_{14}	N_{15}	N_{29}	N_3	N_5	N_{11}
D_{14}	N_{15}	N_{16}	N_{30}	N_4	N_6	N_{12}
D_{15}	N_{16}	N_{17}	N_{31}	N_5	N_7	N_{13}
D_{16}	N_{17}	N_{18}	N_1	N_6	N_8	N_{14}
D_{17}	N_{18}	N_{19}	N_2	N_7	N_9	N_{15}
D_{18}	N_{19}	N_{20}	N_3	N_8	N_{10}	N_{16}
D_{19}	N_{20}	N_{21}	N_4	N_9	N_{11}	N_{17}
D_{20}	N_{21}	N_{22}	N_5	N_{10}	N_{12}	N_{18}
D_{21}	N_{22}	N_{23}	N_6	N_{11}	N_{13}	N_{19}
D_{22}	N_{23}	N_{24}	N_7	N_{12}	N_{14}	N_{20}
D_{23}	N_{24}	N_{25}	N_8	N_{13}	N_{15}	N_{21}
D_{24}	N_{25}	N_{26}	N_9	N_{14}	N_{16}	N_{22}
D_{25}	N_{26}	N_{27}	N_{10}	N_{15}	N_{17}	N_{23}
D_{26}	N_{27}	N_{28}	N_{11}	N_{16}	N_{18}	N_{24}
D_{27}	N_{28}	N_{29}	N_{12}	N_{17}	N_{19}	N_{25}
D_{28}	N_{29}	N_{30}	N_{13}	N_{18}	N_{20}	N_{26}
D_{29}	N_{30}	N_{31}	N_{14}	N_{19}	N_{21}	N_{27}
D_{30}	N_{31}	N_1	N_{15}	N_{20}	N_{22}	N_{28}
$\boldsymbol{D_{31}}$	N_1	N_2	N_{16}	N_{21}	N_{23}	N_{29}

4. An algorithm for constructing (k,n)-arcs related to the Ceva configuration in the projective plane of order five

In this section, we present an algorithm used to construct (k,n) −arcs, and to identify $(k,3)$ −arcs based on secant distributions in *PG(2,5).*

Method: Finding $(\boldsymbol{k},\boldsymbol{n})$ −arcs

Step 1: To identify the points and lines of PG(2,5), we utilize the irreducible polynomial $f(x) = x^3 + 2x^2 + x - 1$. By applying this polynomial, we determine thirty-one points and thirty-one lines within the projective plane, as detailed in Table 1.
Step 2: Consider a Ceva Configuration in given projective plane of order five. In Ceva configuration, a center point and six other points form a (7,3)-arc denoted by C in *PG(2,5)*, which is incomplete.
Step 3: There are 6 points not on the lines spanned by the points of Ceva configuration.
Step 4: The complete $(k,3)$ −arcs are investigated by adding these six points to $(7,3)$ −*arc*.
We present the following algorithm, implemented in C#, designed to identify complete *(k,3)-arcs* within *PG(2,5):*
Steps of Algorithm

$A \leftarrow Read(Excel\ File)$
$B \leftarrow Read(Text\ File)$
$C \leftarrow A$
while s(C)>0
  $B_i \leftarrow input(b), \{b|b \in C, b \notin B, i = s(B) + 1\}$
  j=1
  while $j \leq s(B)$
    for k=(j+1) to s(B)
      $m \leftarrow the\ index\ of\ row\ on\ B_j, B_k$
      $D \leftarrow A_{mn}; \{A_{mn}|A_{mn} \neq B_j, A_{mn} \neq B_k, n = 1, \ldots, 10\}$
      Remove a from A; $\{a|a \in A, a \in D\}$
      $C \leftarrow c;\ \{c|c \in A, c \notin C\}$
    end for
    j=j+1
  end while
end while

Theorem 1. *Let C be a Ceva configuration in PG(2,5). If the given algorithm is applied to the points of Ceva configuration to find (k,3)- arcs, there is a (6,2)- arc constructing with the remaining points.*
Proof. Let C be Ceva configuration in *PG(2,5).* If we apply the algorithm to C, six points are remained in *PG(2,5).* Since all points of a projective plane lie on a pencil of lines through a single point, the points of a Ceva configuration are on a pencil of lines through a point. So, five line through any point outside C contains one point of the Ceva configuration, while one line contains two points of C. Since any three points from the remaining set are not collinear, it follows that this set of remaining points forms a (6,2)-arc.

Theorem 2. *There exist eight distinct complete (10,3)-arcs that include a Ceva configuration in PG(2,5).*
Proof. Let C represent a Ceva configuration. By utilizing the algorithm outlined in Theorem 1 on C, we can get a . (6,2*)-arc*. Each point in this (6,2*)-arc* lies on six lines, of which five intersect the Ceva configuration C at a single point, while one line intersects it at two points. Consequently, there are three lines that intersect the configuration and pass through the points of the (6,2)-arc, which qualify as 2-secant lines. Choosing one remaining point from each of

these 2-secant lines and adding them to the *(7,3)-arc* formed by the Ceva configuration results in eight distinct complete *(10,3)-arcs*.

Theorem 3. *Let C_1, be a Ceva configuration in PG(2,5). Then the cyclic order of Ceva configurations C_1, C_2, C_3, C_4 can be constructed having the same center such that the vertices of Ceva configuration C_i, are the points on the sides of the Ceva configuration C_{i-1}, i=2,3,4.*
Proof. Let C_1 be the Ceva configuration $(A_1A_2A_3, A_1'A_2'A_3')$ with the center M *in* $PG(2,5)$. Let the vertices of C_2 with the center M be A_1', A_2', and A_3'. If we define the points A_1'', A_2'', and A_3'' as the intersection points $MA_1' \cap A_2'A_3'$, $MA_2' \cap A_1'A_3'$, and $MA_3' \cap A_1'A_2'$, respectively, C_2 is obtained as the Ceva configuration $(A_1'A_2'A_3', A_1''A_2''A_3'')$ with the center M. In the same way that we obtained C_2 from C_1, we can now define a Ceva configuration C_3 as $(A_1''A_2''A_3'', A_1'''A_2'''A_3''')$ such that the vertices of C_3 are the points on the sides of C_2 where A_1''', A_2''', and A_3''' as the intersection points $MA_1'' \cap A_2''A_3''$, $MA_2'' \cap A_1''A_3''$, and $MA_3'' \cap A_1''A_2''$. Similarly, the Ceva configuration C_4 is constructed by using the side points and the center M of C_3. C_4 can be found as $(A_1'''A_2'''A_3''', A_1A_2A_3)$ where A_1'''', A_2'''', and A_3'''' are the intersection points $MA_1''' \cap A_2'''A_3'''$, $MA_2''' \cap A_1'''A_3'''$, and $MA_3''' \cap A_1'''A_2'''$, respectively. Since the vertices of C_5 are the points A_1, A_2, A_3 on the sides of C_4, C_5 configuration is the same as C_1. Thus, C_1, C_2, C_3, C_4 are cyclic order of Ceva configurations C_i, *i=1,2,3,4* in *PG(2,5).*

Corollary. Let C_1, C_2, C_3, C_4 be cyclic order of Ceva configurations C_i, i=1,2,3,4 in PG(2,5). Then there are thirty two (10,3)- arcs which is defined by C_1, C_2, C_3, and C_4.
Proof. Let C_1, C_2, C_3, C_4 be cyclic order of Ceva configurations C_i, i=1,2,3,4 in PG(2,5). Since there are eight different complete (10,3) −arcs containing a Ceva configuration in PG(2,5) from Theorem 2, thirty two different compelte (10,3)-arcs are constructed from the Ceva configurations C_1, C_2, C_3, and C_4.

Now, by taking cylic order Ceva configurations C_1, C_2, C_3, C_4 derived from the Ceva configuration $C_1 = (N_3N_2N_6, N_5N_{21}N_{24})$ in *PG(2,5)*, we give all complete (10,3)−arcs with tables and complete (6,2)−arcs for each Ceva configurations C_i, *i = 1,2,3,4*.

Example 1. Let C_1 be *(7,3)-arc* defined by the Ceva configuration $(N_3N_2N_6, N_5N_{21}N_{24})$ with the center point N_{29}. By implementing this algorithm to C_1, it is seen that $C_1 = \{N_2, N_3, N_5, N_6, N_{21}, N_{24}, N_{29}\}$ is *(7,3)-arc* but incomplete arc. In this projective plane, the points on the lines spanned by *(7,3)-arc* are deleted in PG(2,5), then N_4, N_8, N_{12}, N_{13}, N_{18}, N_{18}, N_{19} points are remained. Since six lines pass through each of these points in PG(2,5), five of them intersect the Ceva configuration in one point, and one of them intersect the Ceva configuration in two points. In Table 2, 1−secant and 2−secant lines of the Ceva configuration passing through the remaining points out of (7, 3)−arc are given in PG(2,5).

Table 2. The secant lines of the Ceva configuration.

Point	1-secant lines	2-secant lines
N_4	$D_2, D_6, D_{12}, D_{14}, D_{19}$	D_3
N_8	$D_6, D_7, D_{10}, D_{16}, D_{18}$	D_{23}
N_{12}	$D_{10}, D_{11}, D_{14}, D_{22}, D_{27}$	D_{20}
N_{13}	$D_{11}, D_{12}, D_{15}, D_{21}, D_{28}$	D_{23}
N_{18}	$D_2, D_{16}, D_{17}, D_{26}, D_{28}$	D_{20}
N_{19}	$D_{17}, D_{18}, D_{21}, D_{27}, D_{29}$	D_3

When the algorithm applied to C_1 and the remaining points, eight complete (10,3)−arcs are obtained and given in Table 3.

Table 3. The complete (10,3)-arcs containing the Ceva configuration C_1.

The incomplete (7,3)-arc C_1	The complete (10,3)-arcs
$C_1 \cup \{N_4, N_8, N_{12}\}$	$\{N_2, N_3, N_4, N_5, N_6, N_8, N_{12}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_4, N_8, N_{18}\}$	$\{N_2, N_3, N_4, N_5, N_6, N_8, N_{18}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_4, N_{12}, N_{13}\}$	$\{N_2, N_3, N_4, N_5, N_6, N_{12}, N_{13}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_4, N_{13}, N_{18}\}$	$\{N_2, N_3, N_4, N_5, N_6, N_{13}, N_{18}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_8, N_{12}, N_{19}\}$	$\{N_2, N_3, N_5, N_6, N_8, N_{12}, N_{19}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_8, N_{18}, N_{19}\}$	$\{N_2, N_3, N_5, N_6, N_8, N_{18}, N_{19}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_{12}, N_{13}, N_{19}\}$	$\{N_2, N_3, N_5, N_6, N_{12}, N_{13}, N_{19}, N_{21}, N_{24}, N_{29}\}$
$C_1 \cup \{N_{13}, N_{18}, N_{19}\}$	$\{N_2, N_3, N_5, N_6, N_{13}, N_{18}, N_{19}, N_{21}, N_{24}, N_{29}\}$

Let's start by taking the points N_5, N_{21}, N_{24} on the sides of $C_1 = (N_3N_2N_6, N_5N_{21}N_{24})$ as the vertices of a new Ceva configuration $C_2 = (N_5N_{21}N_{24}, N_1N_{10}N_{15})$ having the same center point N_{29}. If we apply the algorithm to C_2, then the remaining points are $N_7, N_{17}, N_{20}, N_{13}, N_{18}, N_{19}$. Six lines pass through each of these points. Five of them intersect C_2 in one points, and one is the remaining line, which intersect C_2 in two points. If one remaining point is chosen from each of these 2-secant lines and added to C_2, eight different complete (10,3)-*arcs* are obtained as following Table 4. And also these remaining points $N_7, N_{17}, N_{20}, N_{13}, N_{18}, N_{19}$ construct (6,2)−arcs according to Theorem 1.

Table 4. The complete (10,3)-arcs containing the Ceva configuration C_2.

The incomplete (7,3)-arc C_2	The complete (10,3)-arcs
$C_2 \cup \{N_7, N_{17}, N_{20}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{17}, N_{20}, N_{21}, N_{24}, N_{29}\}$
$C_2 \cup \{N_7, N_{17}, N_{28}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{17}, N_{21}, N_{24}, N_{28}, N_{29}\}$
$C_2 \cup \{N_7, N_{20}, N_{27}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{20}, N_{21}, N_{24}, N_{27}, N_{29}\}$
$C_2 \cup \{N_7, N_{27}, N_{28}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{21}, N_{24}, N_{27}, N_{28}, N_{29}\}$
$C_2 \cup \{N_{17}, N_{20}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{17}, N_{20}, N_{21}, N_{24}, N_{29}, N_{30}\}$
$C_2 \cup \{N_{17}, N_{28}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{17}, N_{21}, N_{24}, N_{28}, N_{29}, N_{30}\}$
$C_2 \cup \{N_{20}, N_{27}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{20}, N_{21}, N_{24}, N_{27}, N_{29}, N_{30}\}$
$C_2 \cup \{N_{27}, N_{28}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{21}, N_{24}, N_{27}, N_{28}, N_{29}, N_{30}\}$

Let C_3 and C_2 be taken instead of C_2 and C_1, respectively. Then new Ceva configuration C_3 is $(N_1N_{10}N_{15}, N_{11}N_{23}N_{31})$ with the same center point N_{29}. If we apply the algorithm to C_3, then the remaining points are $N_4, N_8, N_{12}, N_{13}, N_{18}, N_{19}$. If one remaining point is chosen from each of these 2-secant lines and added to C_3, eight different complete (10,3)-*arcs* are obtained as following Table 5.

Table 5. The complete (10,3)-arcs containing the Ceva configuration C_3.

The incomplete (7,3)-arc C_3	The complete (10,3)-arcs
$C_3 \cup \{N_7, N_{17}, N_{20}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{17}, N_{20}, N_{21}, N_{24}, N_{29}\}$
$C_3 \cup \{N_7, N_{17}, N_{28}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{17}, N_{21}, N_{24}, N_{28}, N_{29}\}$
$C_3 \cup \{N_7, N_{20}, N_{27}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{20}, N_{21}, N_{24}, N_{27}, N_{29}\}$
$C_3 \cup \{N_7, N_{27}, N_{28}\}$	$\{N_1, N_5, N_7, N_{10}, N_{15}, N_{21}, N_{24}, N_{27}, N_{28}, N_{29}\}$

$C_3 \cup \{N_{17}, N_{20}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{17}, N_{20}, N_{21}, N_{24}, N_{29}, N_{30}\}$
$C_3 \cup \{N_{17}, N_{28}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{17}, N_{21}, N_{24}, N_{28}, N_{29}, N_{30}\}$
$C_3 \cup \{N_{20}, N_{27}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{20}, N_{21}, N_{24}, N_{27}, N_{29}, N_{30}\}$
$C_3 \cup \{N_{27}, N_{28}, N_{30}\}$	$\{N_1, N_5, N_{10}, N_{15}, N_{21}, N_{24}, N_{27}, N_{28}, N_{29}, N_{30}\}$

Now, let C_4 and C_3 be taken instead of C_3 and C_2, respectively. Then new Ceva configuration C_4 is $(N_{11}N_{23}N_{31}, N_3N_2N_6)$ with the same center point N_{29}. If we apply the algorithm to C_4, then the remaining points are $N_7, N_{17}, N_{20}, N_{27}, N_{28}, N_{30}$. If one remaining point is chosen from each of these 2−secant lines and added to C_4, eight different complete (10,3)−arcs are obtained as following Table 6.

Table 6. The complete (10,3)-arcs containing the Ceva configuration C_4.

The completion of (7,3)-arc C_4	The complete (10,3)-arcs
$C_4 \cup \{N_7, N_{17}, N_{20}\}$	$\{N_2, N_3, N_6, N_7, N_{11}, N_{17}, N_{20}, N_{23}, N_{29}, N_{31}\}$
$C_4 \cup \{N_7, N_{17}, N_{27}\}$	$\{N_2, N_3, N_6, N_7, N_{11}, N_{17}, N_{23}, N_{27}, N_{29}, N_{31}\}$
$C_4 \cup \{N_7, N_{20}, N_{30}\}$	$\{N_2, N_3, N_6, N_7, N_{11}, N_{20}, N_{23}, N_{29}, N_{30}, N_{31}\}$
$C_4 \cup \{N_7, N_{27}, N_{30}\}$	$\{N_2, N_3, N_6, N_7, N_{11}, N_{23}, N_{27}, N_{28}, N_{29}, N_{31}\}$
$C_4 \cup \{N_{17}, N_{20}, N_{28}\}$	$\{N_2, N_3, N_6, N_{11}, N_{17}, N_{20}, N_{23}, N_{28}, N_{29}, N_{31}\}$
$C_4 \cup \{N_{17}, N_{27}, N_{28}\}$	$\{N_2, N_3, N_6, N_{11}, N_{17}, N_{23}, N_{27}, N_{28}, N_{29}, N_{31}\}$
$C_4 \cup \{N_{20}, N_{28}, N_{30}\}$	$\{N_2, N_3, N_6, N_{11}, N_{20}, N_{23}, N_{28}, N_{29}, N_{30}, N_{31}\}$
$C_4 \cup \{N_{27}, N_{28}, N_{30}\}$	$\{N_2, N_3, N_6, N_{11}, N_{23}, N_{27}, N_{28}, N_{29}, N_{30}, N_{31}\}$

5. Conclusion

In this work, it is determined that (k,2)-arcs and (k,3)-arcs obtained from a Ceva configuration in PG(2,5) by giving the algorithm implemented in C#. The following conclusions are found in PG(2,5):

1. There is a complete *(6,2)-arc* that does not contain any common points with the *(7,3)*-arc determined by a Ceva configuration.
2. There are eight different complete *(10,3)-arcs* containing a Ceva configuration.
3. There are cyclic order Ceva configurations C_1, C_2, C_3, C_4 having the same center such that the vertices of the Ceva configuration C_i, are the points on the sides of the Ceva configuration C_{i-1}, *i=2,3,4*.
4. Related to the cyclic order Ceva configurations C_1, C_2, C_3, C_4, there are two different complete *(6,2)-arcs* and thirty-two different complete *(10,3)-arcs*.

These findings show that there is a significant relationship between arcs and Ceva configurations in the projective planes.

Author Contributions

Ayşe Bayar led the conceptualization and development of the theoretical framework and provided insights during the conceptualization phase, and supervised the entire paper, ensuring the accuracy of the theoretical framework, in addition to editing the manuscript for clarity. Elif Altıntaş Kahriman performed the (k,n)-arcs analysis, contributed to the methodology by applying the algorithm, wrote the majority of the original manuscript. All authors commented on the final form of the manuscript.

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