Communications in Advanced Mathematical Sciences
ISSN (Online) 2651-4001
Vol. 8, No. 1, 1-10, 2025
https://doi.org/10.33434/cams.1559808
Research Article

The Mathematical Dynamics of the Caputo Fractional Order Social Media Addiction Design

Bahar Karaman[1]*, Emrah Karaman[2]

Abstract
The paper presents the mathematical dynamics and numerical simulations for a fractional-order social media addiction (FSMA) model. This addiction structure is replaced by involving the Caputo fractional (CF) derivative to get the FSMA model. In this study, our main goal is to understand how the fractional derivative impresses the dynamics of the model. Thus, the theoretical properties are first examined. Afterward, the stability properties of the mentioned model are discussed. Besides, the fractional backward differentiation formula (FBDF) displays numerical simulations of the model. Observing both theoretical and numerical results, the two equilibrium points' stability is not impacted by the order of fractional derivatives. However, each solution converges more quickly to its stationary state for higher values of the fractional-order derivative. Finally, we would like to say that the acquired numerical results are compatible with our theoretical outcomes.

Keywords: Fractional backward differentiation formula(FBDF), Social media addiction, Stability analysis
2020 AMS: 65L05, 34A08, 34A12, 34A34,34D23

[1] *Department of Mathematics, Eskişehir Technical University, Eskişehir, Türkiye, bahar_korkmaz@eskisehir.edu.tr, 0000-0001-6631-8562*
[2] *Department of Mathematics, Eskişehir Technical University, Eskişehir, Türkiye, e.karaman42@gmail.com, 0000-0002-0466-3827*
***Corresponding author**
Received: 02 October 2024, **Accepted:** 31 January 2025, **Available online:** 25 February 2025
How to cite this article: B. Karaman, E. Karaman, *The mathematical dynamics of the Caputo fractional order social media addiction design*, Commun. Adv. Math. Sci., 8(1) (2025), 1-10.

1. Introduction

Nowadays, one of the important real-life problems is the use of social media platforms. They have both advantages and disadvantages. It provides access to information in many parts of education, business, science, and health [1, 2], but intensive usage of social media can negatively affect both family relationships and people's daily lives. Besides using social media extensively can lead to addiction. This situation is one of the significant addictive problems such as alcoholism, smoking, game addiction, etc. There are many studies on this subject [3–9].

Mathematical frameworks for modeling infectious diseases have an essential role in constructing and understanding the dynamics of them. Many researchers have applied to models for alcohol and drug, gambling, smoking, social media, and other addictions [9–13]. Especially after it was noticed that fractional-order models can be better expressed in nonlinear physical problems than integer-order models, they are used to demonstrate the mathematical model of real-life issues such as epidemiology, science, economics, and engineering. There are many studies such as [14–18] etc. in the literature. Demirci proposed a fractional design of Hepatitis B transmission to discuss the analysis of the mathematical model under the effect of vaccination [14]. An approximate solution fractional structure for hepatitis B virus (HBV) infection was obtained in [15]. Then the global stability of the model was constructed in this study. Estimating the disease of COVID-19 in India was designated as a fractional-order SITR by Askar et. al. [16]. They demonstrated the existence and uniqueness of the solution. Also, boundedness

and nonnegativity for the system were established. In [17], they developed a novel fractional-order COVID-19 model clarifying the spread of the disease. The generalized Adams-Bashforth-Moulton algorithm was utilized to get the numerical outcomes. In [18], the authors used the derivatives of type Liouville-Caputo and Conformable to get novel analytic solutions for the electric circuits.

One of the basic goals of the paper is to propose, analyze, and simulate the FSMA design in terms of CF derivative. The proposed design yet has not been solved numerically by using the FBDF. In the Matlab environment, we will utilize the code $flmm2$, which is developed by R. Garrappa (for details see [19, 20]). There are three optional techniques in this code, but we will use the present method to get approximate solutions to the system. Moreover, the stability properties of this present epidemiological design are discussed. Therefore, the global stability of the mentioned structure is examined utilizing the Lyapunov stability theorem. Moreover, the regular state's stability concerning both theoretical and numerical outcomes is not affected by the order of the fractional derivative. Additionally, as the order of the fractional derivative increases, the solutions converge to the regular states faster.

This study consists of five parts. The FSMA design firstly will be introduced in Section 2. The mathematical dynamics of the system will be established in Section 3. We will also search the stability properties in Section 4. In addition, FBDF will be used to obtain numerical solutions of the given structure in Section 5. We will give a summary in the last section 6.

2. The FSMA Design

The part describes the FSMA structure. Its integer-order model is formulated and analyzed by Alemneh and Alemu [7] follow as:

$$\begin{aligned}
\tfrac{dS}{dt} &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
\tfrac{dE}{dt} &= \beta\sigma AS - (\delta+\mu)E,\\
\tfrac{dA}{dt} &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
\tfrac{dR}{dt} &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
\tfrac{dQ}{dt} &= kS + (1-\gamma)\eta R - \mu Q.
\end{aligned} \tag{2.1}$$

There are five categories in the human population according to addiction status in the system (2.1). Individuals in the first category are not addicted but are susceptible to being addicted. They are represented by $S(t)$. Exposed classes are using social media less frequently but do not grow to an addicted level and are denoted by $E(t)$. The third category shows addicted people who spend most of their time on social media and are described by $A(t)$. Recovered populations are defined by $R(t)$. They recovered from their social media addiction. The last category includes the human populations who forever do not use and quit using social media and are specified by $Q(t)$. The total number of members of the population is $N = S+E+A+R+Q$. To depict a deterministic mathematical model, the following assumptions are considered by the authors [7]:

- The epidemic happens in a closed environment,
- The possibility of being addicted to social media is not attached to race, sex, and people's social position, members mix homogeneously, and social media addictive humans will spread to non-addictive when they get in touch with the compression of addiction.

Kongson et. al. [8] used the Atangana-Balenau-Caputo derivative for the fractional-order differentiation. In this research, we will consider an initial value problem of FSMA with a CF derivative.

$$\begin{aligned}
D^{\alpha}S(t) &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
D^{\alpha}E(t) &= \beta\sigma AS - (\delta+\mu)E,\\
D^{\alpha}A(t) &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
D^{\alpha}R(t) &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
D^{\alpha}Q(t) &= kS + (1-\gamma)\eta R - \mu Q,
\end{aligned} \tag{2.2}$$

with initial data $S(0) = S_0 > 0, E(0) = E_0 \geq 0, A(0) = A_0 \geq 0, R(0) = R_0 \geq 0, Q(0) = Q_0 \geq 0$.

3. Theoretical Results of the Structure

A detailed mathematical analysis of the present structure (2.2) is demonstrated in this part. First of all, the existence and uniqueness of the solution of (2.2) are discussed based on the given approach in [21].

Theorem 3.1. *Let the region be defined as* $\Phi \times [0,T_1]$, *where* $\Phi = \{(S,E,A,R,Q) \in \mathbb{R}^5 : \max(|S|,|E|,|A|,|R|,|Q| \leq \xi)\}$ *and* $T_1 < +\infty$. *There is a single solution* $Z(t) \in \Phi$ *of the recommended model (2.2) with an initial condition* $Z_0 = (S_0, E_0, A_0, R_0, Q_0)$, *which is described for all* $t \geq 0$.

Proof. Consider $H(Z) = (H_1(Z), H_2(Z), H_3(Z), H_4(Z), H_5(Z))$ and

$$\begin{aligned}
H_1(Z) &= \pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S,\\
H_2(Z) &= \beta\sigma AS - (\delta+\mu)E,\\
H_3(Z) &= \theta\delta E - (\mu+\varepsilon+\rho)A,\\
H_4(Z) &= (1-\theta)\delta E + \varepsilon A - (\mu+\eta)R,\\
H_5(Z) &= kS + (1-\gamma)\eta R - \mu Q.
\end{aligned}$$

For any $Z, Z^* \in \Phi$ that

$$\begin{aligned}
||H(Z) - H(Z^*)|| &= |H_1(Z) - H_1(Z^*)| + |H_2(Z) - H_2(Z^*)| + |H_3(Z) - H_3(Z^*)|\\
&+|H_4(Z) - H_4(Z^*)| + |H_5(Z) - H_5(Z^*)|\\
&= |\pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S - (\pi + \gamma\eta R^* - \beta\sigma A^*S^* - (k+\mu)S^*)|\\
&+|\beta\sigma AS - (\delta+\mu)E - (\beta\sigma A^*S^* - (\delta+\mu)E^*)|\\
&+|\theta\delta E - (\mu+\varepsilon+\rho)A - (\theta\delta E^* - (\mu+\varepsilon+\rho)A^*)|\\
&+|(1-\theta)\delta E + \varepsilon A - (\mu+\eta)R - ((1-\theta)\delta E^* + \varepsilon A^* - (\mu+\eta)R^*)|\\
&+|kS + (1-\gamma)\eta R - \mu Q - (kS^* + (1-\gamma)\eta R^* - \mu Q^*)|\\
&\leq |\gamma\eta(R-R^*)| + |\beta\sigma(AS - A^*S^*)| + |(k+\mu)(S-S^*)|\\
&+|\beta\sigma(AS - A^*S^*)| + |(\delta+\mu)(E-E^*)| + |\theta\delta(E-E^*)|\\
&+|(\mu+\varepsilon+\rho)(A-A^*)| + |(1-\theta)\delta(E-E^*)| + |\varepsilon(A-A^*)|\\
&+|(\mu+\eta)(R-R^*)| + |k(S-S^*)| + |(1-\gamma)\eta(R-R^*)| + |\mu(Q-Q^*)|\\
&\leq (\mu+2\eta)|R-R^*| + (2\beta\sigma\xi + 2k + \mu)|S-S^*|\\
&+(2\beta\sigma\xi + \mu + 2\varepsilon + \rho)|A-A^*| + (\delta+\mu+1)|E-E^*| + \mu|Q-Q^*|\\
&\leq G||(S,E,A,R,Q) - (S^*,E^*,A^*,R^*,Q^*)||\\
&\leq G||Y - Y^*||,
\end{aligned}$$

where $G = \max\{(\mu+2\eta), (2\beta\sigma\xi + 2k + \mu), (2\beta\sigma\xi + \mu + 2\varepsilon + \rho), (\delta+\mu+1), \mu\}$. Then $H(Z)$ provides the Lipschitz condition. Therefore, we complete the proof. □

3.1 Positivity and boundedness of the solution

This part states the following theorem that ensures the solutions of (2.2) are non-negative and bounded.

Lemma 3.2. *[22] Let us assume that* $u \in C[a,b]$ *and* $D^\alpha u(t) \in C[a,b]$ *for* $\alpha \in (0,1]$, *then we have* $u(t) = u(a) + \frac{1}{\Gamma(\alpha)}(D^\beta u(\xi))(t-\xi)^\alpha$, *in here* $a \leq \xi \leq t$, $\forall t \in [a,b]$.

Lemma 3.3. *Let* $u \in C[a,b]$ *and* $D^\alpha u(t) \in C[a,b]$ *for* $\alpha \in (0,1]$. *If* $D^\alpha u(t) \geq 0$, $\forall t \in (a,b)$, *then* $u(t)$ *is nondecreasing* $\forall t \in [a,b]$. *If* $D^\alpha u(t) \leq 0$, $\forall t \in (a,b)$, *then* $u(t)$ *is nonincreasing on* $[a,b]$.

We will demonstrate uniform boundedness of the solution thanks to the next lemma.

Lemma 3.4. *[21] Asumme that* $v(t)$ *is a continuous function on* $[t_0,\infty)$. *If* $v(t)$ *provides*

$$D^\alpha v(t) \leq -\beta v(t) + \xi, \quad v(t_0) = v_0 \in \mathbb{R},$$

in here $0 < \alpha \leq 1$, $\alpha, \xi \in \mathbb{R}$ *and* $\beta \neq 0$, *and* $t_0 \geq 0$ *is an initial time. Then*

$$v(t) \leq \left(v_0 - \frac{\xi}{\alpha}\right) E_\alpha\left[-\beta(t-t_0)^\alpha\right] + \frac{\xi}{\beta},$$

where $E_\alpha(t)$ which is called the Mittag-Leffler function (MLF) with one parameter is described as follows

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + 1)},$$

in here $\alpha > 0$.

Mittag-Leffler described the MLF in 1903 [23]. Since it is frequently utilized to calculate the solutions of fractional differential equations, it has great significance in fractional calculus. The advised books to readers are [24]- [26] for a more comprehensive introduction.

Theorem 3.5. *The solutions of our system (2.2) are uniformly bounded and non-negative.*

Proof. Let's describe a function $V(t) = S(t) + E(t) + A(t) + R(t) + Q(t)$. Then

$$\begin{aligned} D^\alpha V(t) &= D^\alpha S(t) + D^\alpha E(t) + D^\alpha A(t) + D^\alpha R(t) + D^\alpha Q(t) \\ &= [\pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S] + [\beta\sigma AS - (\delta+\mu)E] \\ &+ [\theta\delta E - (\mu+\varepsilon+\rho)A] + [(1-\theta)\delta E + \varepsilon A - (\mu+\eta)R] \\ &+ [kS + (1-\gamma)\eta R - \mu Q] \\ &= \pi - \mu S - \mu E - \mu R - \mu Q - (\mu+\rho)A \\ &= \pi - \mu(S+E+A+R+Q) - \rho A \\ &\leq \pi - \mu V(t). \end{aligned}$$

Now, let's apply Lemma (3.4), we obtain

$$\begin{aligned} 0 \leq V(t) &\leq V(0)E_\alpha(-\mu(t)^\alpha) + \pi(t)^\alpha E_{\alpha,\alpha+1}(-\mu(t)^\alpha)) \\ &= V(0)E_\alpha(-\mu(t)^\alpha) + \tfrac{\pi}{\mu}(1 - E_\alpha(-\mu(t)^\alpha)). \end{aligned}$$

Utilizing Lemma 5 and Corollary 6 in [27], we conclude that $0 \leq V(t) \leq \frac{\pi}{\mu}, \quad t \to \infty$. Hence, the solutions of the system (2.2) starting in ${\mathbb{R}_+}^5$ are uniformly bounded within the region $\Omega_1 = \{(S,E,A,R,Q) \in {\mathbb{R}_+}^5 : V(t) \leq \frac{\pi}{\mu} + \varepsilon_0, \varepsilon_0 > 0\}$.

Next, to show that the solutions are non-negative. From the model (2.2), we get

$$\begin{aligned} D^\alpha S(t)|_{S=0} &= \pi + \gamma\eta R \geq 0, \\ D^\alpha E(t)|_{E=0} &= \beta\sigma AS \geq 0, \\ D^\alpha A(t)|_{A=0} &= \theta\delta E \geq 0, \\ D^\alpha R(t)|_{R=0} &= (1-\theta)\delta E + \varepsilon A \geq 0, \\ D^\alpha Q(t)|_{Q=0} &= kS + (1-\gamma)\eta R \geq 0. \end{aligned}$$

The solutions of (2.2) are non-negative according to Lemma 1 and Lemma 2. □

Next, the equilibrium points and reproduction number of (2.2) are introduced. There are two possible equilibrium points in the present design which are called addiction-free and endemic equilibrium points.

The general form of a dynamical system involving the CF derivative is given as

$$D^\alpha z(t) = h(t,z), \quad z(0) = z_0 \tag{3.1}$$

in here $0 < \alpha \leq 1$.

Definition 3.6. *[28] If $h(t,z^*) = 0$ for a point z^*, the point is called an equilibrium point of the system (3.1).*

When $E = A = 0$, then the addiction-free equilibrium point of (2.2) is obtained from Definition (3.6) following as: $P_0 = \left(\frac{\pi}{k+\mu}, 0, 0, 0, \frac{k\pi}{\mu(\mu+k)}\right)$. At this point, there is no addiction in the group.

The important tool, both mathematically and biologically meaningful, is the main reproduction number R_0 because it displays the spread of the addiction. Also, we will analyze the stability of the equilibrium points by using R_0. The next-generation matrix method [29] is used to obtain R_0. It is found as

$$R_0 = \frac{\beta\pi\sigma\theta\delta}{(k+\mu)(\delta+\mu)(\mu+\varepsilon+\rho)}.$$

If $R_0 > 1$, the endemic equilibrium point of the present model occurs. The point P_1 can be computed by setting the right hand side of the proposed design is equal to zero. $P_1 = (S_1, E_1, A_1, R_1, Q_1)$ is given by

$$\begin{aligned} S_1 &= \tfrac{(\delta+\mu)(\mu+\varepsilon+\rho)}{\theta\delta\beta\sigma}, \\ E_1 &= \tfrac{\kappa_1}{\kappa_2}, \\ A_1 &= \tfrac{\theta\delta E_1}{\mu+\varepsilon+\rho}, \\ R_1 &= \tfrac{\kappa_1+(\delta+\mu)E_1}{\gamma\eta}, \\ Q_1 &= \tfrac{kS_1+(1-\gamma)\eta R_1}{\mu}, \end{aligned}$$

in here $\kappa_1 = \frac{(k+\mu)(\delta+\mu)(\mu+\varepsilon+\rho)}{\beta\sigma\theta\delta} - \pi$ and $\kappa_2 = \frac{\gamma\eta\delta}{\mu+\eta}\left(1-\theta+\frac{\varepsilon\theta}{\mu+\varepsilon+\rho}\right) - \delta - \mu$.

4. Stability Analysis

The stability properties involving local and global asymptotic stability of the mentioned structure are presented.

Theorem 4.1. *The addiction-free equilibrium point, P_0, is locally asymptotically stable if $R_0 < 1$.*

Proof. Let's calculate the Jacobian matrix for the model (2.2) evaluated at P_0. We get

$$\begin{pmatrix} -k-\mu & 0 & -\frac{\beta\pi\sigma}{k+\mu} & \gamma\eta & 0 \\ 0 & -\delta-\mu & \frac{\beta\pi\sigma}{k+\mu} & 0 & 0 \\ 0 & \theta\delta & -\varepsilon-\rho-\mu & 0 & 0 \\ 0 & (1-\theta)\delta & \varepsilon & -\eta-\mu & 0 \\ k & 0 & 0 & (1-\gamma)\eta & -\mu \end{pmatrix} \tag{4.1}$$

Some of the negative eigenvalues of (4.1)

$$\lambda_1 = -\mu, \lambda_2 = -k-\mu, \lambda_3 = -\mu-\eta$$

and the other eigenvalues are derived from the quadratic equation

$$\lambda^2 + \Delta_1\lambda + \Delta_2 = 0$$

in here $\Delta_1 = \varepsilon+\rho+2\mu$ and $\Delta_2 = (\delta+\mu)(\varepsilon+\mu+\rho) - \frac{\beta\pi\sigma\theta\delta}{k+\mu}$. From Routh-Hurwitz criteria, the quadratic equation has strictly negative real root iff $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_1\Delta_2 > 0$. It is seen easily that $\Delta_1 > 0$ and Δ_2 can be rewritten as

$$\Delta_2 = (\delta+\mu)(\varepsilon+\mu+\rho)(1-\frac{\beta\pi\sigma\theta\delta}{k+\mu}) = (\delta+\mu)(\varepsilon+\mu+\rho)(1-R_0).$$

Thus, when $R_0 < 1$, all eigenvalues have the negative real parts. □

Theorem 4.2. *The endemic equilibrium point, P_1, is locally asymptotically stable if $R_0 > 1$.*

Proof. The proof can be done in a similar manner as in [7]. □

The most important concern for the fractional differential equation is about that the global stability of the solution. Now, we will use the Lyapunov functions to construct the stability of fractional systems. We will first give the following Lemma, which introduces the extended Volterra-type Lyapunov function to systems of fractional differential equations through an inequality that was defined by [30] for approximating the CF derivative of the function.

Lemma 4.3. *[30] Let's $v(t) \in \mathbb{R}^+$ be a continuous function. Then for any time $t \geq t_0$,*

$$D^\alpha\left[v(t) - v_* - v_*\ln\frac{v(t)}{v_*}\right] \leq \left(1-\frac{v_*}{v(t)}\right)D^\alpha v(t), v_* \in \mathbb{R}^+, \forall\alpha \in (0,1).$$

The upcoming result displays the solutions of (2.2) are uniformly continuous. The proof is done in the same manner as in [31].

Lemma 4.4. *The solutions S,E,A,R, and Q of system (2.2) are uniformly continuous functions on $[0,\infty)$.*

Theorem 4.5. *The addiction-free equilibrium point P_0 of the present design (2.2) is globally asymptotically stable if $R_0 < 1$ and unstable when $R_0 > 1$.*

Proof. Let's introduce a function $V(S,E,A,R,Q) = E + \frac{\delta+\mu}{\theta\delta}A$, which is called a Lyapunov function. Then,

$$\begin{aligned} D^\alpha V &= D^\alpha E + \tfrac{\delta+\mu}{\theta\delta} D^\alpha A \\ &= \beta\sigma AS - (\delta+\mu)E + \tfrac{\delta+\mu}{\theta\delta}[\theta\delta E - (\mu+\varepsilon+\rho)A] \\ &= \beta\sigma AS - \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}A. \end{aligned} \tag{4.2}$$

When we use the addiction-free point of (2.2), $S_0 = \frac{\pi}{k+\mu}$, we have from the Equation (4.2),

$$\begin{aligned} D^\alpha V &= \beta\sigma AS_0 - \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}A \\ &= \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}[\tfrac{\beta\sigma\pi\theta\delta}{(k+\mu)(\mu+\varepsilon+\rho)(\delta+\mu)} - 1] \\ &= \tfrac{(\mu+\varepsilon+\rho)(\delta+\mu)}{\theta\delta}[R_0 - 1]. \end{aligned}$$

Thus, $D^\alpha V \le 0$ when $R_0 < 1$. This conclusion gives that P_0 is globally asymptotically stable if $R_0 < 1$. □

Theorem 4.6. *The endemic equilibrium point P_1 of the present model (2.2) is globally asymptotically stable if $R_0 > 1$.*

Proof. We establish a Lyapunov function

$$\begin{aligned} W(S,E,A,R,Q) &= \left(S - S_1 - S_1 \ln \tfrac{S}{S_1}\right) + \left(E - E_1 - E_1 \ln \tfrac{E}{E_1}\right) + \left(A - A_1 - A_1 \ln \tfrac{A}{A_1}\right) \\ &\quad + \left(R - R_1 - R_1 \ln \tfrac{R}{R_1}\right) + \left(Q - Q_1 - Q_1 \ln \tfrac{Q}{Q_1}\right). \end{aligned}$$

By using Lemma 4.3, we have

$$\begin{aligned} D^\alpha W &\le \left(1 - \tfrac{S_1}{S}\right) D^\alpha S + \left(1 - \tfrac{E_1}{E}\right) D^\alpha E + \left(1 - \tfrac{A_1}{A}\right) D^\alpha A \\ &\quad + \left(1 - \tfrac{R1}{R}\right) D^\alpha R + \left(1 - \tfrac{Q_1}{Q}\right) D^\alpha Q \\ &= \left(1 - \tfrac{S_1}{S}\right)[\pi + \gamma\eta R - \beta\sigma AS - (k+\mu)S] + \left(1 - \tfrac{E_1}{E}\right)[\beta\sigma AS - (\delta+\mu)E] \\ &\quad + \left(1 - \tfrac{A_1}{A}\right)[\theta\delta E - (\mu+\varepsilon+\rho)A] + \left(1 - \tfrac{R_1}{R}\right)[(1-\alpha)\delta E + \varepsilon A - (\mu+\eta)R] \\ &\quad + \left(1 - \tfrac{Q_1}{Q}\right)[kS + (1-\gamma)\eta R - \mu Q]. \end{aligned} \tag{4.3}$$

From (2.2), we have that,

$$\begin{aligned} \pi &= \beta\sigma A_1 S_1 + (k+\mu)S_1 - \gamma\eta R_1, \\ (\delta+\mu) &= \beta\sigma \tfrac{A_1 S_1}{E_1}, \\ \theta\delta &= (\mu+\varepsilon+\rho)\tfrac{A_1}{E_1}, \\ \gamma\eta &= 1 - \mu\tfrac{Q_1}{R_1} + k\tfrac{S_1}{R_1}, \\ (\mu+\eta) &= \delta\tfrac{E_1}{R_1} - (\mu+\varepsilon+\rho)\tfrac{A_1}{R_1} + \varepsilon\tfrac{A_1}{R_1}. \end{aligned} \tag{4.4}$$

Next, using the relations (4.4) into (4.3), we get

$$\begin{aligned} D^\alpha W &\le \beta\sigma A_1 S_1 \left[2 - \tfrac{S_1}{S} + \tfrac{A}{A_1} - \tfrac{E}{E_1} - \tfrac{ASE_1}{A_1 S_1 E}\right] + (k+\mu)S_1 \left[2 - \tfrac{S_1}{S} - \tfrac{S}{S_1}\right] \\ &\quad + \gamma\eta R_1 \left[-1 + \tfrac{R}{R_1} + \tfrac{S_1}{S} - \tfrac{RS_1}{R_1 S}\right] \\ &\quad + (\mu+\varepsilon+\rho)\left[-\tfrac{A_1}{A} - \tfrac{A_1 E}{AE_1} + \tfrac{R}{R_1} + \tfrac{R_1 E}{RE_1}\right] \\ &\quad + \varepsilon A_1 \left[1 + \tfrac{A}{A_1} - \tfrac{R}{R_1} - \tfrac{AR_1}{A_1 R}\right] + \mu Q_1 \left[1 - \tfrac{Q}{Q_1} - \tfrac{Q_1 R}{QR_1} + \tfrac{R}{R_1}\right] \\ &\quad + \delta E_1 \left[1 + \tfrac{E}{E_1} - \tfrac{R}{R_1} - \tfrac{ER_1}{E_1 R}\right] + kS_1 \left[\tfrac{S}{S_1} - \tfrac{R}{R_1} - \tfrac{SQ_1}{S_1 Q} + \tfrac{Q_1 R}{QR_1}\right]. \end{aligned}$$

Since the arithmetic mean exceeds the geometric mean, then

$$\left(2-\frac{S}{S_1}-\frac{S_1}{S}\right)\leq 0,\quad \left[2-\frac{S_1}{S}+\frac{A}{A_1}-\frac{E}{E_1}-\frac{ASE_1}{A_1S_1E}\right]\leq 0,$$

$$\left[-1+\frac{R}{R_1}+\frac{S_1}{S}-\frac{RS_1}{R_1S}\right]\leq 0,\quad \left[1+\frac{A}{A_1}-\frac{R}{R_1}-\frac{AR_1}{A_1R}\right]\leq 0,$$

$$\left[1-\frac{Q}{Q_1}-\frac{Q_1R}{QR_1}+\frac{R}{R_1}\right]\leq 0,\quad \left[1+\frac{E}{E_1}-\frac{R}{R_1}-\frac{ER_1}{E_1R}\right]\leq 0,$$

and

$$\left[\frac{S}{S_1}-\frac{R}{R_1}-\frac{SQ_1}{S_1Q}+\frac{Q_1R}{QR_1}\right]\leq 0.$$

Therefore, $D^{\alpha}W\leq 0$.

Let N_1 is the largest invariant set in $\{(S,E,A,R,Q);D^{\alpha}W=0\}$. Note that $D^{\alpha}W=0$ if and only if $S=S_1,E=E_1,A=A_1,R=R_1,Q=Q_1$ for any time t. Hence, it can be said that $N_1=\{P_1\}=\{(S_1,E_1,A_1,R_1,Q_1)\}$. When $R_0>1$, we obtain (2.2) is globally asymptotically stable at P_1 thanks to Lyapunov-LaSalle invariance principle. □

5. Numerical Results

Now, we will display the numerical solutions of the FSMA model. The approximate solution is demonstrated by using a fractional backward differentiation formula. For more detailed information, the readers can read the studies [20, 32–34].

Nonnegative parameters are used to get the numerical outcomes. If we choose $\pi=0.5,\mu=0.05,\beta=0.3,\sigma=0.2,\theta=0.7,\rho=0.01,\delta=0.25,\varepsilon=0.7,\kappa=0.01,\gamma=0.35$ and $\eta=0.4$, then we get the reproduction number as $R_0=0.3838<1$ and the result of the numerical solution of the FSMA design as illustrated in Fig. (5.1). In this situation, $P_0=(8.3333,0,0,0,1.6667)$ is obtained. In Fig (5.1), it can be noted that while the number of exposed, addicted, quit-using, and recovered classes quickly increases, the number of susceptible populations decreases the first time. After, the number of exposed, addicted, and recovered populations decreases to zero over time. When exposed and addicted individuals in society heal, the number of recovering populations increases, and after the addict's transmission stops, the number of recovered populations decreases to zero. It should be noted that approximate solutions converge to the point P_0. The number of susceptible and quit-using social media populations is balanced and stable at 8.3333 and 1.6667, respectively.

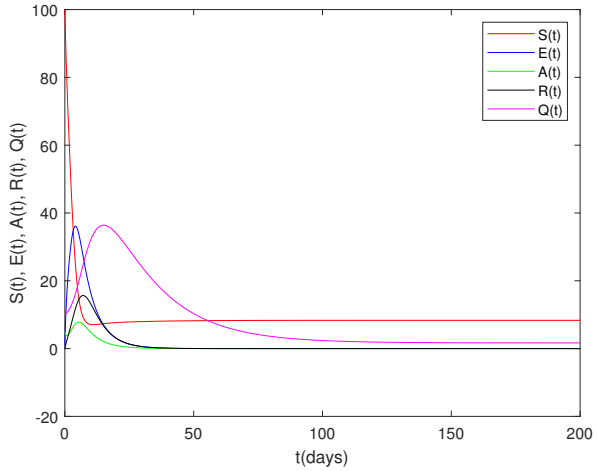

Figure 5.1. The Plot of the model for $\alpha=0.998$ in the case $R_0<1$.

When $\beta=0.8$, $R_0=1.0234>1$ is obtained. Also, the endemic equilibrium point is

$$P_1=(8.1429,0.0505,0.0116,0.0265,1.7663)$$

in this case. Additionally, the plot of the model is displayed in Fig. (5.2). In this step, we would like to say that approximate solutions converge to the endemic equilibrium point P_1. Moreover, each of the numerical solutions of the structure is displayed for various fractional orders $\alpha = 0.88, 0.92, 0.96, 1$ to understand the effect of fractional derivative orders in Fig. (5.3). Remark that the order of the fractional derivative α does not affect on the regular state's stability. All obtained solutions also converge more quickly to the regular states for higher values of α.

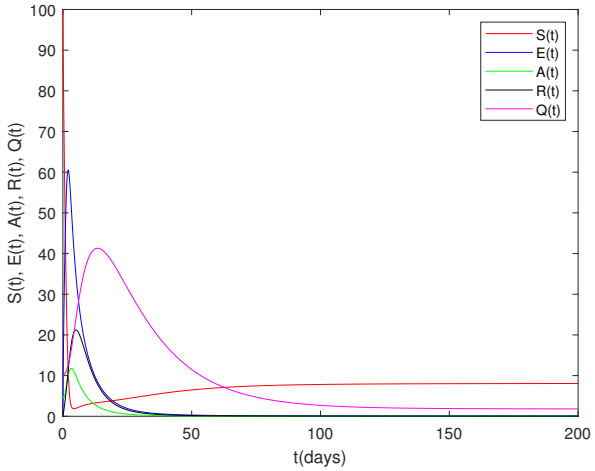

Figure 5.2. Plot of the system for $\alpha = 0.998$ in the case $R_0 > 1$.

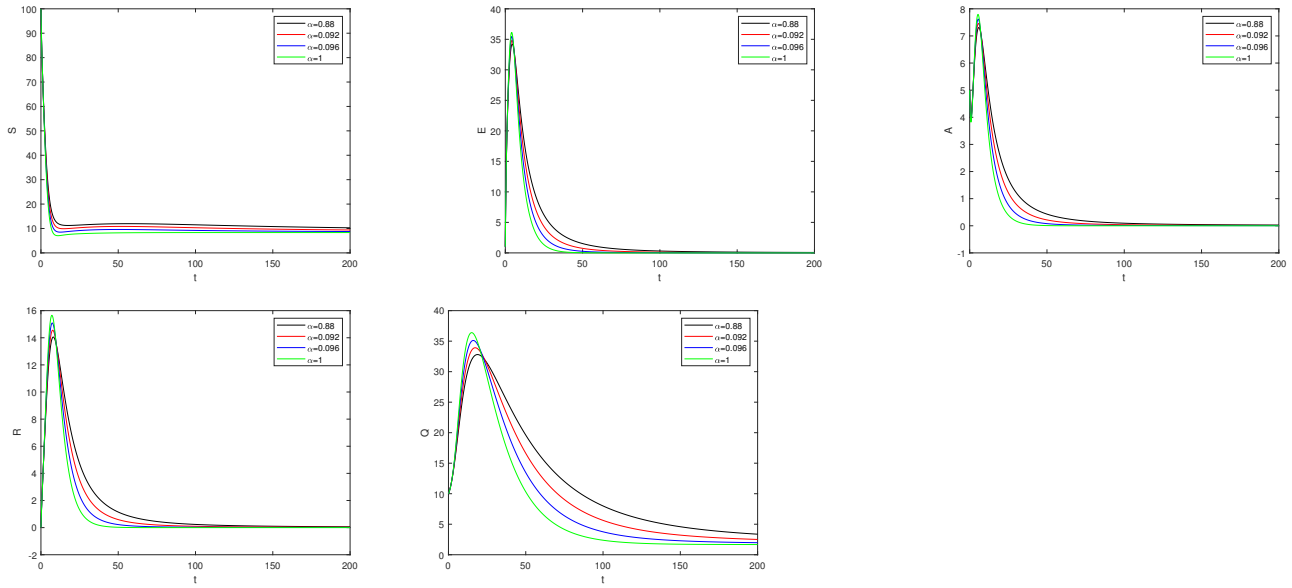

Figure 5.3. Plots of the infection as a function of time (days) with different values of fractional orders

6. Conclusion Remarks

This study focuses on the analyses of the FSMA design with CF derivative. Firstly, the mathematical analysis is examined. We verify the local and global analysis of the equilibria stability. FBDF is used to obtain approximate solutions. Numerical simulations display balance and stability at two equilibrium points P_0 and P_1 when $R_0 < 1$, and $R_0 > 1$, respectively. Furthermore, based on both theoretical and numerical outcomes, we notice that the order of fractional derivatives does not affect on the two equilibria' stability. However, each solution converges more quickly to its stationary state for higher values of the fractional-order derivative. Lastly, we would also like to say that the obtained numerical outcomes are compatible with our theoretical results.

Article Information

Acknowledgements: The authors express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this article.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] M. Drahošóvá, P. Balco, *The analysis of advantages and disadvantages of use of social media in European Union*, Proc. Comput. Sci., **109** (2017), 1005-1009.

[2] R. Faizi, A. E. Afia, R. Chiheb, *Exploring the potential benefits of using social media in education*, IJEP, **3**(4) (2013), 50-53.

[3] A. Simsek, K. Elciyar, T. Kizilhan, *A comparative study on social media addiction of high school and university students*, Contemporary Edu. Tech., **10**(2) (2019), 106-119.

[4] S. Zivnuska, J. R. Carlson, D. S. Carlson, R. B. Harris, K. J. Harris, *Social media addiction and social media reactions: The implications for job performance*, J. Soc. Psychol., **159**(6) (2019), 746-760.

[5] C. S. Andreassen, T. Torsheim, G. S. Burnborg, S. Pallesen, *Development of a Facebook addiction scale*, Psychol. Rep., **110**(2) (2012), 501–517.

[6] J. R. Carlson, S. Zivnuska, D. S. Carlson, R. Harris, K. J. Harris, *Social media use in the workplace: A study of dual effects*, J. Organ. End User Comput., **28**(1) (2016), 15–28.

[7] H. T. Alemneh, N. Y. Alemu, *Mathematical modeling with optimal control analysis of social media addiction*, Infect. Dis. Model., **6** (2021), 405-419.

[8] J. Kongson, W. Sudsutad, C. Thaiprayoon, J. Alzabut, C. Tearnbucha, *On analysis of a nonlinear fractional system for social media addiction involving Atangana-Balenau-Caputo derivative*, Adv. Differ. Equ., **2021** (2021), 356-385.

[9] Shutaywi, Meshal, et al. *Modeling and analysis of the addiction of social media through fractional calculus*, Front. Appl. Math. Stat., **9** (2023), 1210404.

[10] I. K. Adu, A. L. Mojeeb, C. Yang, *Mathematical model of drinking epidemic*, J. Adv. Math. Comput., **22**(5) (2017), 1-10.

[11] Y. Guo, T. Li, *Optimal control and stability analysis of an online game addiction model with two stages*, Math. Methods Appl. Sci., **43**(7) (2020), 4391-4408.

[12] S. H. Ma, H. F. Huo, X. Y. Meng, *Modelling alcoholism as a contagious disease: A mathematical model with awareness programs and time delay*, Discrete Dyn Nat. Soc., **2015** (2015) Article ID 260195.

[13] S. A. Samad, M. T. Islam,S. T. H. Tomal, M., Biswas, *Mathematical assessment of the dynamical model of smoking tobacco epidemic in Bangladesh*, Int. J. Sci. Manag. Stud., **3**(2) (2020), 36-48.

[14] E. Demirci, *A fractional order model of hepatitis B transmission under the effect of vaccination*, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat., **71**(2) (2022), 566-580. https://doi.org/10.31801/cfsuasmas.1103630

[15] B. Karaman, *The global stability investigation of the mathematical design of a fractional-order HBV infection*, J. Appl. Math. Comput., **68** (2022), 4759–4775. https://doi.org/10.1007/s12190-022-01721-2.

[16] S. S. Askar, G. Dipankar, P. K. Santra, A. A. Elsadany, G. S. Mahapatra, *A fractional order SITR mathematical model for forecasting of transmission of COVID-19 of India with lockdown effect*, Results Phys., **24** (2021), 104067.

[17] I. Owusu-Mensah, L. Akinyemi, B. Oduro, O. S. Iyiola, *A fractional order approach to modeling and simulations of the novel COVID-19*, Adv. Differ. Equ., **2020** (2020), 683.

[18] V. F. Morales-Delgado, J. F. Gomez-Aguilar, M. A. Taneco-Hernandez, *Analytical solutions of electrical circuits described by fractional conformable derivatives in Liouville-Caputo sense*, Int. J. Electron. Commun., **85** (2018), 61-81.

[19] R. Garrappa, *Numerical solution of fractional differential equations: A survey and software Tutorial*, Mathematics **6**(2) (2018), 16.

[20] R. Garrappa, *Trapezoidal methods for fractional differential equations: Theoretical and computational aspects*, Math. Comput. Simul., **110** (2015), 96-112.

[21] H. L. Li, Z. Long, H. Cheng, J. Yao-Lin, T. Zhidong, *Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge*, J. Appl. Math. Comput., **54** (2017), 435-449.

[22] Z. M. Odibat, N. T. Shawaghef, *Generalized Taylor's formula*, Appl. Math. Comput., **186** (2007), 286-293.

[23] G. M. Mittag-Leffler, *Sur l'integrable de Laplace-Abel*, Comptes Rendus de l'Academie des Sciences Series II, **136** (1903), 937–939.

[24] K. Diethelm, *Tha Analysis of Fractional Differential Equations, Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2010.

[25] I. Podlubny, *Fractional Differential Equations*, Academie Press, New York, 1999.

[26] K. B.Oldham, J. Spanier, *The Fractional Calculus*, New York London, Academic Press, 1974.

[27] S. K. Choi, B. Kang, N. Koo, *Stability for Caputo fractional differential systems*, Abstr. Appl. Anal., **2014** (2014), 1-6.

[28] Y. Li, Y. Chen, I. Podlubny, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Comput. Math. Appl., **59** (2010), 1810–1821.

[29] C. Castillo Chavez, Z. Feng, W. Huang, *On the Computation of R_0 and Its Role on Global Stability*, In Mathematical Approaches for Emerging and Remerging Infectious Diseases: An introduction. IMA, Springer, Berlin, 2002.

[30] C. V. Leon, *Volterra Lyapunov functions for fractional-order epidemic systems*, Commun. Nonlinear Sci. Numer. Simul., **24** (2015), 75-85.

[31] L. C. Cardoso, R. F. Camargo, F. L. P. Santos, J. P. C. Santos, *Global stability analysis of a fractional differential system in hepatitis B*, Chaos, Solitons and Fractals, **143** (2021), 110619.

[32] C. Lubich, *Discretized fractional calculus*, SIAM J. Math. Anal., **17** (1986), 704-719.

[33] C. Lubich, *Fractional linear multistep methods for Abel-Volterra integral equations of the second kind*, Math. Comput., **45**(163) (1983), 87-102.

[34] L. Galeone, R. Garrappa, *On multistep methods for differential equations of fractional order*, Mediterr. J. Math., **3** (2006), 565-580.