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On Bipolar Soft Topological Spaces

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Abstract — In this present study, some properties of bipolar soft closed sets are introduced and the concept of closure, interior, basis and subspaces which are the building blocks of classical topology are defined on bipolar soft topological spaces. In addition, examples have been presented so that the subject can be better understood.

Keywords — *Bipolar soft set, bipolar soft topology, bipolar soft topological spaces, bipolar soft open(close), bipolar soft interior, bipolar soft basis.*

1 Introduction

Introducing fuzzy sets [11], intiutionistic fuzzy sets [1], soft sets [6] and etc. theories which contribute to solution of problems such as decision making and uncertainity. A lot of researcher has been done on these theories [2, 3, 7, 10].

In the past years, Shabir & Naz [9] and Karaaslan & Karatas [4] differently defined bipolar soft set. Obviously, bipolar soft sets satisfied more sharp results than soft sets. Therefore the concept of bipolar soft topology has a great importance.

In this study, we define a short notation for writing simplicity in the application of bipolar soft sets and investigate the relationship between the soft topological spaces and the bipolar soft topological spaces. Moreover, we define the notion of bipolar soft closure, bipolar soft interior, bipolar soft basis, bipolar soft subspace. The basis theorems of these notations are provided and supported with examples.

2 Preliminary

In this section, we will give some preliminary information about bipolar soft sets and bipolar soft topological spaces. Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A, B, C \subseteq E$.

Definition 2.1. [5] Let $E = \{e_1,\ e_2,\ e_3, ..., e_n\}$ be a set of parameters. The not set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1,\ \neg e_2,\ \neg e_3, ..., \neg e_n\}$ where for all i, $\neg e_i = not\ e_i$.

Definition 2.2. [9] A triplet (F, G, A) is called a bipolar soft set over X, where F and G are mappings, $F : A \to P(X)$ and $G : \neg A \to P(X)$ such that $F(e) \cap G(\neg e) = \emptyset$ for all $e \in A$ and $\neg e \in \neg A$.

Definition 2.3. [9] For two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X, (F_1, G_1, A) is called a bipolar soft subset of (F_2, G_2, B) if

1. $A \subseteq B$ and
2. $F_1(e) \subseteq F_2(e)$ and $G_2(\neg e) \subseteq G_1(\neg e)$ for all $e \in A$.

   This relationship is denoted by $(F_1, G_1, A) \,\widetilde{\subseteq}\, (F_2, G_2, B)$. (F_1, G_1, A) and (F_2, G_2, B) are said to be equal if (F_1, G_1, A) is a bipolar soft subset of (F_2, G_2, B) and (F_2, G_2, A) is a bipolar soft subset of (F_1, G_1, B).

Definition 2.4. [9] Bipolar soft complement of a bipolar soft set (F, G, A) over X is denoted by $(F, G, A)^c$ and is defined by $(F, G, A)^c = (F^c, G^c, A)$ where $F^c : A \to P(X)$ and $G^c : \neg A \to P(X)$ are given by $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in A$ and $\neg e \in \neg A$.

Definition 2.5. [9] Bipolar soft union of two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X is the bipolar soft set (H, I, C) over X where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ F_1(e) \cup F_2(e), & \text{if } e \in A \cap B. \end{cases}$$

$$I(\neg e) = \begin{cases} G_1(\neg e), & \text{if } \neg e \in (\neg A) - (\neg B)\,, \\ G_2(\neg e), & \text{if } \neg e \in (\neg B) - (\neg A)\,, \\ G_1(\neg e) \cap G_2(\neg e), & \text{if } \neg e \in (\neg A) \cap (\neg B)\,. \end{cases}$$

It is denoted by $(F_1, G_1, A) \,\widetilde{\cup}\, (F_2, G_2, B) = (H, I, C)$.

Definition 2.6. [9] Bipolar soft intersection of two bipolar soft sets (F_1, G_1, A) and (F_2, G_2, B) over X is the bipolar soft set (H, I, C) over X where $C = A \cup B$ is non-empty and for all $e \in C$,

$$H(e) = F_1(e) \cap F_2(e) \text{ and } I(\neg e) = G_1(\neg e) \cup G_2(\neg e).$$

It is denoted by $(F_1, G_1, A) \,\widetilde{\cap}\, (F_2, G_2, B) = (H, I, C)$.

Definition 2.7. [9] Let (F_1, G_1, A) and (F_2, G_2, B) be two bipolar soft sets over X. Then,

1. $\left((F_1, G_1, A) \,\widetilde{\cup}\, (F_2, G_2, B)\right)^c = (F_1, G_1, A)^c \,\widetilde{\cap}\, (F_2, G_2, B)^c$,

2. $\left((F_1, G_1, A) \widetilde{\cap} (F_2, G_2, B)\right)^c = (F_1, G_1, A)^c \widetilde{\cup} (F_2, G_2, B)^c$.

Definition 2.8. [9] A bipolar soft set (F, G, A) over X is said to be relative null bipolar soft set, denoted by $\left(\Phi, \widetilde{X}, A\right)$, if for all $e \in A$, $F(e) = \emptyset$ and for all $\neg e \in \neg A$, $G(\neg e) = X$.

The relative null bipolar soft set with respect to the universe set of parameters E is called a NULL bipolar soft set over X and is denoted by $\left(\Phi, \widetilde{X}, E\right)$.

Definition 2.9. [9] A bipolar soft set (F, G, A) over X is said to be relative absolute bipolar soft set, denoted by $\left(\widetilde{X}, \Phi, A\right)$, if for all $e \in A$, $F(e) = X$ and for all $\neg e \in \neg A$, $G(\neg e) = \emptyset$.

The relative absolute bipolar soft set with respect to the universe set of parameters E is called a ABSOLUTE bipolar soft set over X and is denoted by $\left(\widetilde{X}, \Phi, E\right)$.

Definition 2.10. [8] Let $\widetilde{\widetilde{\tau}}$ be the collection of bipolar soft sets over X with E as the set of parameters. Then $\widetilde{\widetilde{\tau}}$ is said to be a bipolar soft topology over X if

1. $\left(\Phi, \widetilde{X}, E\right)$ and $\left(\widetilde{X}, \Phi, E\right)$ belong to $\widetilde{\widetilde{\tau}}$
2. the bipolar soft union of any number of bipolar soft sets in $\widetilde{\widetilde{\tau}}$ belongs to $\widetilde{\widetilde{\tau}}$
3. the bipolar soft intersection of finite number of bipolar soft sets in $\widetilde{\widetilde{\tau}}$ belongs to $\widetilde{\widetilde{\tau}}$.

   Then $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is called a bipolar soft topological space over X .

Definition 2.11. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X, then the members of $\widetilde{\widetilde{\tau}}$ are said to be bipolar soft open sets in X.

Definition 2.12. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X. A bipolar soft set (F, G, E) over X is said to be a bipolar soft closed set in X, if its bipolar soft complemet $(F, G, E)^c$ belongs to $\widetilde{\widetilde{\tau}}$.

Definition 2.13. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a bipolar soft topological space over X. A bipolar soft set (F, G, E) over X is said to be a bipolar soft clopen set in X, if it is both a bipolar soft closed set and a bipolar soft open set over X.

3 The Main Results

Definition 3.1. Let (F, G, A) be a bipolar soft set over X. The presentation of

$$(F, G, A) = \{(e,\ F(e),\ G(\neg e)) : e \in A \subseteq E,\ \neg e \in \neg A \subseteq \neg E \text{ and } F(e),\ G(\neg e) \in P(X)\}$$

is said to be a short expansion of bipolar soft set (F, G, A).

From now on, $BSS\,(X)_{E,\neg E}$ denotes the family of all bipolar soft sets over X with E as the set of parameters and $BSTS$ denotes a bipolar soft topolological space.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ be an universe set, $E = \{e_1, e_2, e_3\}$ be the set of parameters and $A = \{e_1, e_3\} \subseteq E$ be a subset of parameters. Then $\neg E = \{\neg e_1, \neg e_2, \neg e_3\}$ and $\neg A = \{\neg e_1, \neg e_3\}$. Suppose that a bipolar soft set (F, G, A) is given as follows.

$$F(e_1) = \{x_1, x_3\}\,,\ F(e_3) = \{x_4\}$$
$$G(\neg e_1) = \{x_2\}\,,\ G(\neg e_3) = \{x_1, x_2\}\,.$$

Then the short expansion of bipolar soft set (F, G, A) is denoted by $(F, G, A) = \{(e_1, \{x_1, x_3\}, \{x_2\}),\ (e_3, \{x_4\}, \{x_1, x_2\})\}$.

Proposition 3.3. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. Then the collection $\tau_e = \{F(e) : (F, G, E) \in \widetilde{\widetilde{\tau}}\}$ for each $e \in E$, defines a topology on X.

Theorem 3.4. [8] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. Then the collection $\widetilde{\widetilde{\tau}}$ consisting of bipolar soft sets (F, G, E) such that $(F, E) \in \widetilde{\tau}$ and $G(\neg e) = F^{'}(e) = U \backslash F(e)$ for all $\neg e \in \neg E$, defines a $BSTS$ over X.

Proposition 3.5. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. Then the collection $\widetilde{\tau} = \left\{(F, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ defines a soft topology and $(X, \widetilde{\tau}, E)$ is a soft topological space over X.

Proof. Suppose that $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is a $BSTS$ over X. Let us show that the collection $\widetilde{\tau} = \left\{(F, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ provides the conditions of soft topological spaces.

1. $\left(\Phi, \widetilde{X}, E\right)$, $\left(\widetilde{X}, \Phi, E\right) \in \widetilde{\widetilde{\tau}}$ implies that (Φ, E), $\left(\widetilde{X}, E\right) \in \widetilde{\tau}$.
2. Let $\{(F_i, E)\}_{i \in I}$ be a collection of sets in $\widetilde{\tau}$. Since $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$, for all $i \in I$ so that $\underset{i \in I}{\widetilde{\bigcup}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$ thus $\underset{i \in I}{\widetilde{\bigcup}} (F_i, E) \in \widetilde{\tau}$.
3. Let $\{(F_i, E)\}_{i = \overline{1,n}}$ be a collection of finite sets in $\widetilde{\tau}$. Then $\overset{n}{\underset{i=1}{\widetilde{\bigcap}}} (F_i, G_i, E) \in \widetilde{\widetilde{\tau}}$ so $\overset{n}{\underset{i=1}{\widetilde{\bigcap}}} (F_i, E) \in \widetilde{\tau}$.

Hence $\widetilde{\tau}$ defines a soft topology over X. □

Remark 3.6. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X. It can be easily shown that, if the collection $\widetilde{\widetilde{\tau}}$ is finite then $\widetilde{\neg\tau} = \left\{(G, \neg E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ defines a soft topology and $(X, \widetilde{\neg\tau}, \neg E)$ is a soft topological space over X.

Similarly, if the collection $\widetilde{\widetilde{\tau}}$ is finite then $\widetilde{\tau}_{\neg e} = \left\{G\,(\neg e) : (F, G, E) \in \widetilde{\widetilde{\tau}}, \text{ for all } \neg e \in \neg E\right\}$ defines a topology and $(X, \widetilde{\tau}_{\neg e})$ is a topological space over X.

Definition 3.7. Let X be an initial universe set, E be the set of parameters and $\widetilde{\widetilde{\tau}} = \left\{\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right)\right\}$. Then $\widetilde{\widetilde{\tau}}$ is called the bipolar soft indiscrete topology over X and $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is said to be a bipolar soft indiscrete topological space over X.

Definition 3.8. Let X be an initial universe set, E be the set of parameters and $\widetilde{\widetilde{\tau}}$ be the collection of all bipolar soft sets that can be defined over X. Then $\widetilde{\widetilde{\tau}}$ is called the bipolar soft discrete topology over X and $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ is said to be a bipolar soft discrete topological space over X.

Definition 3.9. Let $\left(X, \widetilde{\widetilde{\tau}}_1, E, \neg E\right)$ and $\left(X, \widetilde{\widetilde{\tau}}_2, E, \neg E\right)$be two $BSTS'$s over the same initial universe set X. Then $\widetilde{\widetilde{\tau}}_2$ is said to be bipolar soft finer than $\widetilde{\widetilde{\tau}}_1$, or $\widetilde{\widetilde{\tau}}_1$ is said to be bipolar soft coarser than $\widetilde{\widetilde{\tau}}_2$ if $\widetilde{\widetilde{\tau}}_2 \supseteq \widetilde{\widetilde{\tau}}_1$.

Example 3.10. Let X be an initial universe set and E be the set of parameters. The bipolar soft indiscrete topology is the coarsest bipolar soft topology and the bipolar discrete topology is the finest bipolar soft topology over X.

Proposition 3.11. Let $\left(X, \widetilde{\widetilde{\tau}}_1, E, \neg E\right)$ and $\left(X, \widetilde{\widetilde{\tau}}_2, E, \neg E\right)$be two $BSTS'$s over the same initial universe set X, then $\left(X, \widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2, E, \neg E\right)$ is a $BSTS$ over X.

Proof. 1) $\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right) \in \widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2$.

2) Let $\{(F_i, G_i, E)\}_{i\in I}$ be a family of bipolar soft sets in $\widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2$. Then $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1$ and $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_2$, for all $i \in I$, so $\underset{i\in I}{\widetilde{\cup}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1$ and $\underset{i\in I}{\widetilde{\cup}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_2$. Therefore $\underset{i\in I}{\widetilde{\cup}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2$.

3) Let $\{(F_i, G_i, E)\}_{i=\overline{1,n}}$ be a finite family of bipolar soft sets in $\widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2$. Then $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1$ and $(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_2$ for $i = \overline{1, n}$. Since $\overset{n}{\underset{i=1}{\widetilde{\cap}}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1$ and $\overset{n}{\underset{i=1}{\widetilde{\cap}}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_2$, then $\overset{n}{\underset{i=1}{\widetilde{\cap}}}(F_i, G_i, E) \in \widetilde{\widetilde{\tau}}_1 \cap \widetilde{\widetilde{\tau}}_2$. $\square$

Remark 3.12. The union of two bipolar soft topologies over the same initial universe set X may not be a bipolar soft topology over X.

Example 3.13. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2, e_3\}$. Then $\neg E = \{\neg e_1, \neg e_2, \neg e_3\}$. Suppose that $\widetilde{\widetilde{\tau}}_1 = \left\{\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E)\right\}$, $\widetilde{\widetilde{\tau}}_2 = \left\{\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right), (H_1, K_1, E), (H_2, K_2, E), (H_3, K_3, E)\right\}$ are two bipolar soft topologies defined over X where (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (H_1, K_1, E), (H_2, K_2, E), (H_3, K_3, E) are bipolar soft sets over X, defined as follows:

$$
\begin{aligned}
(F_1, G_1, E) &= \{(e_1, \{x_1, x_3, x_4\}, \{x_2\}), (e_2, \{x_2, x_3\}, \{x_4\}), (e_3, \{x_3, x_4\}, \{x_1\})\}, \\
(F_2, G_2, E) &= \{(e_1, \{x_2, x_4\}, \{x_1\}), (e_2, \{x_1, x_4\}, \{x_2\}), (e_3, \{x_1, x_2\}, \{x_3\})\}, \\
(F_3, G_3, E) &= \{(e_1, \{x_4\}, \{x_1, x_2\}), (e_2, \emptyset, \{x_2, x_4\}), (e_3, \emptyset, \{x_1, x_3\})\},
\end{aligned}
$$

and

$$\begin{aligned}(H_1,K_1,E) &= \{(e_1,\{x_2,x_3\},\{x_1\}),(e_2,\{x_1,x_2\},\{x_3,x_4\}),(e_3,\{x_3,x_4\},\{x_2\})\},\\ (H_2,K_2,E) &= \{(e_1,\{x_1,x_4\},\{x_2,x_3\}),(e_2,X,\emptyset),(e_3,\{x_1,x_2,x_3\},\emptyset)\},\\ (H_3,K_3,E) &= \{(e_1,\emptyset,\{x_1,x_2,x_3\}),(e_2,\{x_1,x_2\},\{x_3,x_4\}),(e_3,\{x_3\},\{x_2\})\}.\end{aligned}$$

Then

$$\widetilde{\widetilde{\tau}}_1 \cup \widetilde{\widetilde{\tau}}_2 = \left\{\begin{array}{c}\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right),(F_1,G_1,E),(F_2,G_2,E),\\ (F_3,G_3,E),(H_1,K_1,E),(H_2,K_2,E),(H_3,K_3,E)\end{array}\right\}.$$

For example, we take $(F_1,G_1,E)\widetilde{\cup}(H_1,K_1,E)=(S,T,E)=\{(e_1,X,\emptyset),(e_2,\{x_1,x_2,x_3\},\{x_4\})\ \ (e_3,\{x_3,x_4\},\emptyset)\}$, but $(S,T,E)\notin \widetilde{\widetilde{\tau}}_1 \cup \widetilde{\widetilde{\tau}}_2$. Therefore $\widetilde{\widetilde{\tau}}_1 \cup \widetilde{\widetilde{\tau}}_2$ is not a bipolar soft topology over X.

Theorem 3.14. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X. Then

1. $\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right)$ are bipolar soft closed sets over X,
2. Arbitrary bipolar soft interscetions of the bipolar soft closed sets are bipolar soft closed set over X,
3. Finite bipolar soft unions of the bipolar soft closed sets are bipolar soft closed set over X.

Proof. 1. Since $\left(\Phi,\widetilde{X},E\right)^c=\left(\widetilde{X},\Phi,E\right)\in\widetilde{\widetilde{\tau}}$ and $\left(\widetilde{X},\Phi,E\right)^c=\left(\Phi,\widetilde{X},E\right)\in\widetilde{\widetilde{\tau}}$, then $\left(\Phi,\widetilde{X},E\right),\left(\widetilde{X},\Phi,E\right)$ are bipolar soft closed sets over X.

2. Let $\{(F_i,G_i,E)\}_{i\in I}$ be a family of bipolar soft closed sets over X. Then

$$\left(\widetilde{\bigcap_{i\in I}}(F_i,G_i,E)\right)^c=\widetilde{\bigcup_{i\in I}}(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}.$$

Therefore, $\widetilde{\bigcap}_{i\in I}(F_i,G_i,E)$ is a bipolar soft closed set over X.

3. Let $\{(F_i,G_i,E)\}_{i=\overline{1,n}}$ be a finite family of bipolar soft closed sets over X. Then

$$\left(\widetilde{\bigcup}_{i=1}^{n}(F_i,G_i,E)\right)^c=\widetilde{\bigcap}_{i=1}^{n}(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}.$$

Thus, $\widetilde{\bigcup}_{i=1}^{n}(F_i,G_i,E)$ is a bipolar soft closed set over X. □

Definition 3.15. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X and (F,G,E) be a bipolar soft set over X. Then the bipolar soft closure of (F,G,E), denoted by $\overline{(F,G,E)}$, is the bipolar soft intersection of all bipolar soft closed super sets of (F,G,E).

Obviously, $\overline{(F,G,E)}$ is the smallest bipolar soft closed set over X that containing (F,G,E).

Theorem 3.16. Let $\left(X, \tilde{\tilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X, (F, G, E) and (F_1, G_1, E) be two bipolar soft sets over X. Then

1. $\overline{\left(\Phi, \widetilde{X}, E\right)} = \left(\Phi, \widetilde{X}, E\right)$, $\overline{\left(\widetilde{X}, \Phi, E\right)} = \left(\widetilde{X}, \Phi, E\right)$,
2. $(F, G, E) \widetilde{\subseteq} \overline{(F, G, E)}$,
3. (F, G, E) is a bipolar soft closed set iff $(F, G, E) = \overline{(F, G, E)}$,
4. $\overline{\overline{(F, G, E)}} = \overline{(F, G, E)}$,
5. $(F, G, E) \widetilde{\subseteq} (F_1, G_1, E) \Rightarrow \overline{(F, G, E)} \widetilde{\subseteq} \overline{(F_1, G_1, E)}$
6. $\overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)} = \overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)}$
7. $\overline{(F, G, E) \widetilde{\cap} (F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E)} \widetilde{\cap} \overline{(F_1, G_1, E)}$.

Proof. 1. and 2. are obvious.

3. Suppose that (F, G, E) is a bipolar soft closed. Then $\overline{(F, G, E)}$ is the smallest bipolar soft closed set containing (F, G, E) and $\overline{(F, G, E)} = (F, G, E)$.

Conversely, let $(F, G, E) = \overline{(F, G, E)}$. Since $\overline{(F, G, E)}$ is a bipolar soft closed set, then (F, G, E) is a bipolar soft closed set over X.

4. Since $\overline{(F, G, E)}$ is a bipolar soft closed then we have $\overline{\overline{(F, G, E)}} = \overline{(F, G, E)}$ from the part (3.).

5. Let $(F, G, E) \widetilde{\subseteq} (F_1, G_1, E)$. From the part (2.), $(F, G, E) \widetilde{\subseteq} \overline{(F, G, E)}$ and $(F_1, G_1, E) \widetilde{\subseteq} \overline{(F_1, G_1, E)}$. $\overline{(F, G, E)}$ is the smallest bipolar soft closed set that containing (F, G, E). Then $\overline{(F, G, E)} \widetilde{\subseteq} \overline{(F_1, G_1, E)}$.

6. Since $(F, G, E) \widetilde{\subseteq} (F, G, E) \widetilde{\cup} (F_1, G_1, E)$ and $(F_1, G_1, E) \widetilde{\subseteq} (F, G, E) \widetilde{\cup} (F_1, G_1, E)$ then $\overline{(F, G, E)} \widetilde{\subseteq} \overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)}$ and $\overline{(F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)}$ from the part (5.). Therefore, $\overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)}$.
Conversely, since $(F, G, E) \widetilde{\subseteq} \overline{(F, G, E)}$ and $(F_1, G_1, E) \widetilde{\subseteq} \overline{(F_1, G_1, E)}$. Then $(F, G, E) \widetilde{\cup} (F_1, G_1, E) \widetilde{\subseteq} \overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)}$. Since $\overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)}$ is a bipolar soft closed set and $\overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)}$ is the smallest bipolar soft closed set that containing $(F, G, E) \widetilde{\cup} (F_1, G_1, E)$. Then $\overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)}$.
Hence, $\overline{(F, G, E) \widetilde{\cup} (F_1, G_1, E)} = \overline{(F, G, E)} \widetilde{\cup} \overline{(F_1, G_1, E)}$.

7. Since $(F, G, E) \widetilde{\cap} (F_1, G_1, E) \widetilde{\subseteq} (F, G, E)$ and $(F, G, E) \widetilde{\cap} (F_1, G_1, E) \widetilde{\subseteq} (F_1, G_1, E)$ then $\overline{(F, G, E) \widetilde{\cap} (F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E)}$ and $\overline{(F, G, E) \widetilde{\cap} (F_1, G_1, E)} \widetilde{\subseteq} \overline{(F_1, G_1, E)}$. Therefore, $\overline{(F, G, E) \widetilde{\cap} (F_1, G_1, E)} \widetilde{\subseteq} \overline{(F, G, E)} \widetilde{\cap} \overline{(F_1, G_1, E)}$. □

Example 3.17. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{e_1, e_2, e_3\}$. Then $\neg E = \{\neg e_1, \neg e_2, \neg e_3\}$. Suppose that $\tilde{\tilde{\tau}} = \left\{\left(\Phi, \widetilde{X}, E\right), \left(\widetilde{X}, \Phi, E\right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E)\right\}$, is a bipolar soft topology defined over X where (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E) are bipolar soft sets over X, defined as follows:

$$\begin{aligned}
(F_1, G_1, E) &= \{(e_1, \{x_2, x_3, x_4\}, \{x_5\}), (e_2, \{x_1, x_2, x_3\}, \{x_4, x_5\}), (e_3, \{x_3, x_4, x_5\}, \{x_2\})\}, \\
(F_2, G_2, E) &= \{(e_1, \{x_1, x_2, x_5\}, \{x_3, x_4\}), (e_2, \{x_2, x_4\}, \{x_3, x_5\}), (e_3, \{x_1, x_5\}, \{x_2, x_3\})\}, \\
(F_3, G_3, E) &= \{(e_1, \{x_2\}, \{x_3, x_4, x_5\}), (e_2, \{x_2\}, \{x_3, x_4, x_5\}), (e_3, \{x_5\}, \{x_2\})\}, \\
(F_4, G_4, E) &= \{(e_1, X, \emptyset), (e_2, \{x_1, x_2, x_3, x_4\}, \{x_5\}), (e_3, \{x_1, x_3, x_4, x_5\}, \{x_2\})\}.
\end{aligned}$$

According to the bipolar soft topological space $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$;

$$\left(\widetilde{\widetilde{\tau}}\right)^c = \left\{\left(\Phi, \widetilde{X}, E\right)^c, \left(\widetilde{X}, \Phi, E\right)^c, (F_1, G_1, E)^c, (F_2, G_2, E)^c, (F_3, G_3, E)^c, (F_4, G_4, E)^c\right\}$$

is the family of all bipolar soft closed sets such that

$$\begin{aligned}
(F_1, G_1, E)^c &= \{(e_1, \{x_5\}, \{x_2, x_3, x_4\}), (e_2, \{x_4, x_5\}, \{x_1, x_2, x_3\}) \ (e_3, \{x_2\}, \{x_3, x_4, x_5\})\}, \\
(F_2, G_2, E)^c &= \{(e_1, \{x_3, x_4\}, \{x_1, x_2, x_5\}), (e_2, \{x_3, x_5\}, \{x_2, x_4\}) \ (e_3, \{x_2, x_3\}, \{x_1, x_5\})\}, \\
(F_3, G_3, E)^c &= \{(e_1, \{x_3, x_4, x_5\}, \{x_2\}), (e_2, \{x_3, x_4, x_5\}, \{x_2\}) \ (e_3, \{x_2\}, \{x_5\})\}, \\
(F_4, G_4, E)^c &= \{(e_1, \emptyset, X), (e_2, \{x_5\}, \{x_1, x_2, x_3, x_4\}), (e_3, \{x_2\}, \{x_1, x_3, x_4, x_5\})\}.
\end{aligned}$$

Let $(K, S, E) = \{(e_1, \{x_3\}, \{x_1, x_2, x_4, x_5\}), (e_2, \{x_5\}, \{x_1, x_2, x_4\}), (e_3, \{x_2\}, \{x_1, x_3, x_5\})\}$ be a bipolar soft set over X. Then the bipolar soft closure of (K, S, E),

$$\overline{(K, S, E)} = (F_2, G_2, E)^c \widetilde{\cap} (F_3, G_3, E)^c \widetilde{\cap} \left(\widetilde{X}, \Phi, E\right) = (F_2, G_2, E)^c.$$

Corollary 3.18. It is clear that whereas only intersection operation on soft closed sets containing (F, E) depending on an appropriate parameter is performed for the soft closure operation of (F, E) in the studies [3, 10], in the bipolar soft set theory, an intersection operation according to an appropriate parameter on the bipolar soft closed sets containing the set and union operation according to not element of parameter on the bipolar soft closed sets containing the set are performed.

Definition 3.19. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and (F, G, E) be a bipolar soft set over X. Then the bipolar soft interior of (F, G, E), denoted by $(F, G, E)^\circ$, is the bipolar soft union of all bipolar soft open subsets of (F, G, E).

Obviously, $(F, G, E)^\circ$ is the biggest bipolar soft open set over X that is contained by (F, G, E).

Theorem 3.20. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X, (F, G, E) and (F_1, G_1, E) be two bipolar soft sets over X. Then

1. $\left(\Phi, \widetilde{X}, E\right)^\circ = \left(\Phi, \widetilde{X}, E\right)$, $\left(\widetilde{X}, \Phi, E\right)^\circ = \left(\widetilde{X}, \Phi, E\right)$,
2. $(F, G, E)^\circ \widetilde{\subseteq} (F, G, E)$,
3. (F, G, E) is a bipolar soft open set iff $(F, G, E) = (F, G, E)^\circ$,
4. $((F, G, E)^\circ)^\circ = (F, G, E)^\circ$,
5. $(F, G, E) \widetilde{\subseteq} (F_1, G_1, E) \Rightarrow (F, G, E)^\circ \widetilde{\subseteq} (F_1, G_1, E)^\circ$,
6. $(F, G, E)^\circ \widetilde{\cap} (F_1, G_1, E)^\circ = \left[(F, G, E) \widetilde{\cap} (F_1, G_1, E)\right]^\circ$,
7. $(F, G, E)^\circ \widetilde{\cup} (F_1, G_1, E)^\circ \widetilde{\subseteq} \left[(F, G, E) \widetilde{\cup} (F_1, G_1, E)\right]^\circ$.

Proof. 1. and 2. are obvious.

3. Suppose that (F,G,E) is a bipolar soft open set. Then (F,G,E) is the biggest bipolar soft open set that is contained by (F,G,E) and $(F,G,E)=(F,G,E)^{\circ}$.

Conversely, let $(F,G,E)=(F,G,E)^{\circ}$. Since $(F,G,E)^{\circ}$ is a bipolar soft open set, then (F,G,E) is a bipolar soft open set over X.

4. Let $(F,G,E)^{\circ}=(K,S,E)$. Then (K,S,E) is a bipolar soft open set iff $(K,S,E)=(K,S,E)^{\circ}$. Therefore, $((F,G,E)^{\circ})^{\circ}=(F,G,E)^{\circ}$.

5. Suppose that $(F,G,E)\widetilde{\subseteq}(F_1,G_1,E)$. From the part (2.), $(F,G,E)^{\circ}\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^{\circ}\widetilde{\subseteq}(F_1,G_1,E)$. $(F_1,G_1,E)^{\circ}$ is the biggest bipolar soft open set that is contained by (F_1,G_1,E). So, $(F,G,E)^{\circ}\widetilde{\subseteq}(F_1,G_1,E)^{\circ}$.

6. Since $(F,G,E)^{\circ}\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^{\circ}\widetilde{\subseteq}(F_1,G_1,E)$, then $(F,G,E)^{\circ}\widetilde{\cap}(F_1,G_1,E)^{\circ}\widetilde{\subseteq}(F,G,E)\widetilde{\cap}(F_1,G_1,E)$. $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^{\circ}$ is the biggest bipolar soft open set that is contained by $(F,G,E)\widetilde{\cap}(F_1,G_1,E)$. Therefore, $(F,G,E)^{\circ}\widetilde{\cap}(F_1,G_1,E)^{\circ}\widetilde{\subseteq}\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^{\circ}$.

Conversely, since $(F,G,E)\widetilde{\cap}(F_1,G_1,E)\widetilde{\subseteq}(F,G,E)$ and $(F,G,E)\widetilde{\cap}(F_1,G_1,E)\widetilde{\subseteq}(F_1,G_1,E)$, then $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^{\circ}\widetilde{\subseteq}(F,G,E)^{\circ}$ and $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^{\circ}\widetilde{\subseteq}(F_1,G_1,E)^{\circ}$. Hence, $\left[(F,G,E)\widetilde{\cap}(F_1,G_1,E)\right]^{\circ}\widetilde{\subseteq}(F,G,E)^{\circ}\widetilde{\cap}(F_1,G_1,E)^{\circ}$.

7. Since $(F,G,E)^{\circ}\widetilde{\subseteq}(F,G,E)$ and $(F_1,G_1,E)^{\circ}\widetilde{\subseteq}(F_1,G_1,E)$, then $(F,G,E)^{\circ}\widetilde{\cup}(F_1,G_1,E)^{\circ}\widetilde{\subseteq}(F,G,E)\widetilde{\cup}(F_1,G_1,E)$. $\left[(F,G,E)\widetilde{\cup}(F_1,G_1,E)\right]^{\circ}$ is the biggest bipolar soft open set that is contained by $(F,G,E)\widetilde{\cup}(F_1,G_1,E)$. So, $(F,G,E)^{\circ}\widetilde{\cup}(F_1,G_1,E)^{\circ}\widetilde{\subseteq}\left[(F,G,E)\widetilde{\cup}(F_1,G_1,E)\right]^{\circ}$. □

Example 3.21. Let us consider the bipolar soft topology over X that is given in Example 3.17. Suppose that

$$(K,S,E)=\{(e_1,\{x_2,x_3,x_4\},\{x_5\}),(e_2,X,\emptyset)\ \ (e_3,X,\emptyset)\}$$

is a bipolar soft set over X. Then the bipolar soft interior of (K,S,E),

$$(K,S,E)^{\circ}=(F_1,G_1,E)\widetilde{\cup}(F_3,G_3,E)\widetilde{\cup}\left(\Phi,\widetilde{X},E\right)=(F_1,G_1,E).$$

Corollary 3.22. It is clear that whereas only union operation on soft open sets contained in (F,E) depending on an appropriate parameter is performed for the soft interior operation of (F,E) in the studies [3, 10], in the bipolar soft set theory, a union operation according to an appropriate parameter on the bipolar soft open sets contained in the set and an intersection operation according to not element of parameter on the bipolar soft open sets contained in the set are performed.

Theorem 3.23. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X, (F,G,E) be a bipolar soft sets over X. Then $\left[\overline{(F,G,E)}\right]^c=[(F,G,E)^c]^{\circ}$.

Proof. From the definitions of a bipolar soft closure and a bipolar soft interior, we have

$$\left[\overline{(F,G,E)}\right]^c=\left(\widetilde{\bigcap_{\substack{(F_i,G_i,E)\widetilde{\supseteq}(F,G,E)\\(F_i,G_i,E)^c\in\widetilde{\widetilde{\tau}}}}}(F_i,G_i,E)\right)^c=\widetilde{\bigcup}(F_i,G_i,E)^c=[(F,G,E)^c]^{\circ}.$$

□

Definition 3.24. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $\widetilde{\widetilde{B}} \subseteq \widetilde{\widetilde{\tau}}$. $\widetilde{\widetilde{B}}$ is said to be a bipolar soft basis for the bipolar soft topology $\widetilde{\widetilde{\tau}}$ if every element of $\widetilde{\widetilde{\tau}}$ can be written as the bipolar soft union of elements of $\widetilde{\widetilde{B}}$.

Theorem 3.25. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $\widetilde{\widetilde{B}}$ be a bipolar soft basis for $\widetilde{\widetilde{\tau}}$. Then, $\widetilde{\widetilde{\tau}}$ equals the collection of all bipolar soft unions of elements of $\widetilde{\widetilde{B}}$.

Proof. This is eaisly seen from the definition of bipolar soft basis. □

Example 3.26. Let us consider the bipolar soft topology over X that is given in Example 3.17. Then $\widetilde{\widetilde{B}} = \left\{\left(\Phi, \widetilde{X}, E\right), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E)\right\}$ is a bipolar soft basis for the bipolar soft topology $\widetilde{\widetilde{\tau}}$.

Definition 3.27. [8] Let (F, G, E) be a bipolar soft set over X and Y be a non-empty subset of X. Then the bipolar sub soft set of (F, G, E) over Y denoted by $({}^Y F,{}^Y G, E)$, is defined as follows

$$ {}^Y F(e) = Y \cap F(e) \text{ and } {}^Y G(\neg e) = Y \cap G(\neg e), \text{ for each } e \in E. $$

Proposition 3.28. [8] Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and Y be a non-empty subset of X. Then $\widetilde{\widetilde{\tau}}_Y = \left\{({}^Y F,{}^Y G, E) : (F, G, E) \in \widetilde{\widetilde{\tau}}\right\}$ is a bipolar soft topology on Y.

The collection $\widetilde{\widetilde{\tau}}_Y$ is called a bipolar soft subspace topology.

In the above Definition 3.25., Shabir and Bakhtawar have defined bipolar soft subspace according to universal subset $Y \subseteq X$. However, the following definition defines a bipolar soft subspace according to a bipolar soft set.

Theorem 3.29. Let $\left(X, \widetilde{\widetilde{\tau}}, E, \neg E\right)$ be a $BSTS$ over X and $(F, G, E) \widetilde{\subseteq} \left(\widetilde{X}, \Phi, E\right)$. Then the collection

$$ \widetilde{\widetilde{\tau}}_{(F,G,E)} = \left\{(F, G, E) \widetilde{\cap} (F_i, G_i, E) : (F_i, G_i, E) \in \widetilde{\widetilde{\tau}} \text{ for } i \in I\right\} $$

is a bipolar soft topology on (F, G, E) and $\left(X_{(F,G,E)}, \widetilde{\widetilde{\tau}}_{(F,G,E)}, E, \neg E\right)$ is a bipolar soft topological space.

Proof. Since $\left(\Phi, \widetilde{X}, E\right) \widetilde{\cap} (F, G, E) = \left(\Phi, \widetilde{X}, E\right)$ and $\left(\widetilde{X}, \Phi, E\right) \widetilde{\cap} (F, G, E) = (F, G, E)$, then $\left(\Phi, \widetilde{X}, E\right)$, $(F, G, E) \in \widetilde{\widetilde{\tau}}_{(F,G,E)}$.

Moreover,

$$ \widetilde{\bigcap_{i=1}^{n}} \left((F_i, G_i, E) \widetilde{\cap} (F, G, E)\right) = \left(\widetilde{\bigcap_{i=1}^{n}} (F_i, G_i, E)\right) \widetilde{\cap} (F, G, E) $$

and

$$\widetilde{\bigcup_{i\in I}}\left((F_i,G_i,E)\widetilde{\cap}(F,G,E)\right)=\left(\widetilde{\bigcup_{i\in I}}(F_i,G_i,E)\right)\widetilde{\cap}(F,G,E)$$

for $\widetilde{\widetilde{\tau}}=\{(F_i,G_i,E):i\in I\}$. Therefore, the bipolar soft union of any number of bipolar soft sets in $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ belongs to $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ and the finite bipolar soft intersection of bipolar soft sets in $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ belongs to $\widetilde{\widetilde{\tau}}_{(F,G,E)}$. Hence, $\widetilde{\widetilde{\tau}}_{(F,G,E)}$ is a bipolar soft topology on (F,G,E). □

Definition 3.30. Let $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$ be a $BSTS$ over X and $(F,G,E)\widetilde{\subseteq}\left(\widetilde{X},\Phi,E\right)$. Then the collection

$$\widetilde{\widetilde{\tau}}_{(F,G,E)}=\left\{(F,G,E)\widetilde{\cap}(F_i,G_i,E):(F_i,G_i,E)\in\widetilde{\widetilde{\tau}}\text{ for }i\in I\right\}$$

is called a bipolar soft subspace topology on (F,G,E) and $\left(X_{(F,G,E)},\widetilde{\widetilde{\tau}}_{(F,G,E)},E,\neg E\right)$ is called a bipolar soft topological subspace of $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$.

Example 3.31. Let us consider the bipolar soft topology over X that is given in Example 3.17. and $(F,G,E)\widetilde{\subseteq}\left(\widetilde{X},\Phi,E\right)$ such that

$$(F,G,E)=\{(e_1,\{x_1,x_2,x_3\},\{x_4,x_5\}),(e_2,\{x_3,x_5\},\{x_2,x_4\})\ \ (e_3,\{x_2,x_4,x_5\},\{x_1\})\}.$$

Then the collection

$$\widetilde{\widetilde{\tau}}_{(F,G,E)}=\left\{\begin{array}{l}\left(\Phi,\widetilde{X},E\right)\widetilde{\cap}(F,G,E),\left(\widetilde{X},\Phi,E\right)\widetilde{\cap}(F,G,E),(F_1,G_1,E)\widetilde{\cap}(F,G,E),\\(F_2,G_2,E)\widetilde{\cap}(F,G,E),(F_3,G_3,E)\widetilde{\cap}(F,G,E),(F_4,G_4,E)\widetilde{\cap}(F,G,E)\end{array}\right\}$$

is a bipolar soft subspace topology on (F,G,E) and $\left(X_{(F,G,E)},\widetilde{\widetilde{\tau}}_{(F,G,E)},E,\neg E\right)$ is a bipolar soft topological subspace of $\left(X,\widetilde{\widetilde{\tau}},E,\neg E\right)$.

4 Conclusion

In this paper, we introduced some properties of bipolar soft topological spaces and the relationships between soft topological spaces and bipolar soft topological spaces. We hope that, the results of this study may help to next studies for many researchers.

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