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Research Article

Higher order approximation of functions by modified Goodman-Sharma operators

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ABSTRACT. Here we study the approximation properties of a modified Goodman-Sharma operator recently considered by Acu and Agrawal in [1]. This operator is linear but not positive. It has the advantage of a higher order of approximation of functions compared with the Goodman-Sharma operator. We prove direct and strong converse theorems in terms of a related K-functional.

Keywords: Bernstein-Durrmeyer operator, Goodman-Sharma operator, direct theorem, strong converse theorem, K-functional.

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1. INTRODUCTION

In 1987, W. Chen and independently T. N. T. Goodman and A. Sharma presented at conferences in China and Bulgaria, respectively a new modification of the classical Bernstein operators. For $n \in \mathbb{N}$ and functions $f(x) \in C[0,1]$, they introduce the linear operator (see [5] and [9, 10]):

$$U_n(f,x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1}\Big(\int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt\Big)P_{n,k}(x) + f(1)P_{n,n}(x), \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \qquad k = 0,\dots,n. \tag{1.2}$$

Operators of this kind were investigated by many authors (see [14], [4], [13], [11], [7, 8], [2], etc.) and are generally known as genuine Bernstein-Durrmeyer operators. Note that the operators in (1.1) are actually a limit case of Bernstein type operators with Jacobi weights studied by Berens and Xu [3]. If we set

$$u_{n,k}(f) = \begin{cases} f(0), & k = 0,\\ (n-1)\int_0^1 P_{n-2,k-1}(t)f(t)\,dt, & k = 1,\dots,n-1,\\ f(1), & k = n,\end{cases}$$

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the operators defined in (1.1) take the form

$$U_n(f,x) = \sum_{k=0}^{n} u_{n,k}(f) P_{n,k}(x) \qquad \text{or} \qquad U_n f = \sum_{k=0}^{n} u_{n,k}(f) P_{n,k}.$$

Let us denote, as usual, by

$$\varphi(x) = x(1-x)$$

the weight function which is naturally connected to the second order derivative of the Bernstein operator. Also, we set

$$\widetilde{D} f(x) := \varphi(x) f''(x) \tag{1.3}$$

and

$$\widetilde{D}^2 f := \widetilde{D}\widetilde{D} f, \qquad \widetilde{D}^{\ell+1} f := \widetilde{D}\widetilde{D}^{\ell} f, \qquad \ell = 2, 3 \ldots .$$

Recently, Acu and Agrawal [1] studied a family of Bernstein-Durrmeyer operators, as they modify $U_n f$ by replacing the Bernstein basis polynomials $P_{n,k}$ with linear combinations of Bernstein basis polynomials of lower degree with coefficients which are polynomials of appropriate degree. For special choice of the parameters, these operators lack the positivity but have a higher than $O(n^{-1})$ order of approximation. For example, Acu and Agrawal considered operators with $O(n^{-2})$ and $O(n^{-3})$ rate of approximation, see [1, Section 3].

The results presented in [1] inspired the authors of the current paper to explore in more depth the operators explicitly defined by

$$\widetilde{U}_n(f,x) = \sum_{k=0}^{n} u_{n,k}(f) \widetilde{P}_{n,k}(x), \qquad x \in [0,1], \tag{1.4}$$

where

$$\widetilde{P}_{n,k}(x) = P_{n,k}(x) - \frac{1}{n}\, \widetilde{D} P_{n,k}(x). \tag{1.5}$$

By defining an appropriate K-functional, we prove direct and strong converse inequality of Type B in the terminology of [6].

In order to state our main results, we need some definitions.

Let $L_\infty[0,1]$ be the space of all Lebesgue measurable and essentially bounded functions in $[0,1]$ and $AC_{loc}(0,1)$ consists of the functions absolutely continuous in any subinterval $[a,b] \subset (0,1)$. Let us set

$$W^2(\varphi)[0,1] := \big\{ g \,:\, g, g' \in AC_{loc}(0,1),\ \widetilde{D} g \in L_\infty[0,1] \big\}.$$

By $W_0^2(\varphi)[0,1]$, we denote the subspace of $W^2(\varphi)[0,1]$ of functions g satisfying the additional boundary conditions

$$\lim_{x \to 0^+} \widetilde{D} g = 0, \qquad \lim_{x \to 1^-} \widetilde{D} g = 0.$$

Henceforth, by $\|\cdot\|$ we mean the uniform norm on the interval $[0,1]$. For functions $f \in C[0,1]$ and $t > 0$, we define the K-functional

$$K(f,t) := \inf \big\{ \|f - g\| + t \|\widetilde{D}^2 g\| \,:\, g \in W_0^2(\varphi)[0,1],\ \widetilde{D} g \in W^2(\varphi)[0,1] \big\}. \tag{1.6}$$

Here we investigate the error of approximation of functions $f \in C[0,1]$ by the modified Goodman-Sharma operator (1.4). Our main results read as follows.

Theorem 1.1. *If $n \in \mathbb{N}$, $n \geq 2$, and $f \in C[0,1]$, then*

$$\|\widetilde{U}_n f - f\| \leq (1 + \sqrt{3})\, K\Big(f, \frac{1}{n^2}\Big).$$

Theorem 1.2. *For every function $f \in C[0,1]$ and $n \in \mathbb{N}$, $n \geq 2$, there exist constants $C, L > 0$ such that*

$$K\Big(f, \frac{1}{n^2}\Big) \leq C\,\frac{\ell^2}{n^2}\big(\|\widetilde{U}_n f - f\| + \|\widetilde{U}_\ell f - f)\|\big).$$

for all $\ell \geq Ln$.

Remark 1.1. *Another way to state Theorem 1.1 and Theorem 1.2 is the following: there exists a natural number k such that*

$$K\Big(f, \frac{1}{n^2}\Big) \sim \|\widetilde{U}_n f - f\| + \|\widetilde{U}_{kn} f - f\|.$$

The paper is organized as follows. In Section 1 state of the art is described. Preliminary and auxiliary results are presented in Section 2. Section 3 includes an estimation of the norm of the operator $\widetilde{U}_n$, a Jackson type inequality and a proof of the direct inequality in Theorem 1.1. The last Section 4 is devoted to a converse result for the modified Goodman-Sharma operator (1.4). Inequalities of the Voronovskaya type and Bernstein type for $\widetilde{U}_n$ are proved using the differential operator $\widetilde{D}$, defined in (1.3). Theorem 1.2 represents a strong converse inequality of Type B, according to Ditzian-Ivanov classification in [6]. Complete proof of the converse theorem is given.

2. Preliminaries and auxiliary results

By $B_n f$, $n \in \mathbb{N}$, we denote the Bernstein operators determined for functions f,

$$B_n(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) P_{n,k}(x), \qquad x \in [0,1],$$

where $P_{n,k}$ are the Bernstein basis polynomials (1.2). The Bernstein operator central moments play important role in many applications and they are defined by

$$\mu_{n,i}(x) = B_n\big((t-x)^i, x\big) = \sum_{k=0}^{n} \Big(\frac{k}{n} - x\Big)^i P_{n,k}(x), \qquad i = 0, 1, \dots .$$

We summarize some well known useful properties of the Bernstein polynomials. Further on we assume $P_{n,k} := 0$ if $k < 0$ or $k > n$.

Proposition 2.1 (see, e.g. [12]). *(a) The following identities are valid:*

$$\sum_{k=0}^{n} k P_{n,k}(x) = nx, \qquad \sum_{k=0}^{n} (n-k) P_{n,k}(x) = n(1-x), \tag{2.7}$$

$$\sum_{k=0}^{n} k(k-1) P_{n,k}(x) = n(n-1)x^2, \tag{2.8}$$

$$\sum_{k=0}^{n} (n-k)(n-k-1) P_{n,k}(x) = n(n-1)(1-x)^2, \tag{2.9}$$

$$P'_{n,k}(x) = n\big[P_{n-1,k-1}(x) - P_{n-1,k}(x)\big], \tag{2.10}$$

$$P''_{n,k}(x) = n(n-1)\big[P_{n-2,k-2}(x) - 2P_{n-2,k-1}(x) + P_{n-2,k}(x)\big]. \tag{2.11}$$

(b) For the low-order moments $\mu_{n,i}(x)$, we have:

$$\mu_{n,0}(x) = B_n\big((t-x)^0, x\big) = 1,$$
$$\mu_{n,1}(x) = B_n\big((t-x), x\big) = 0,$$
$$\mu_{n,2}(x) = B_n\big((t-x)^2, x\big) = \frac{\varphi(x)}{n},$$
$$\mu_{n,3}(x) = B_n\big((t-x)^3, x\big) = \frac{(1-2x)\varphi(x)}{n^2},$$
$$\mu_{n,4}(x) = B_n\big((t-x)^4, x\big) = \frac{3(n-2)\varphi^2(x)}{n^3} + \frac{\varphi(x)}{n^3}.$$

The operators U_n, $\widetilde{U}_n$ and the differential operator $\widetilde{D}$ satisfy interesting properties.

Proposition 2.2. *If the operators U_n, $\widetilde{U}_n$ and the differential operator $\widetilde{D}$ are defined as in (1.1), (1.4) and (1.3), respectively, then*

(a) $\widetilde{D}U_n f = U_n\widetilde{D}f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(b) $\widetilde{U}_n f = U_n\big(f - \frac{1}{n}\widetilde{D}f\big)$ *for* $f \in W_0^2(\varphi)[0,1]$;
(c) $\widetilde{D}\widetilde{U}_n f = \widetilde{U}_n\widetilde{D}f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(d) $U_n\widetilde{U}_n f = \widetilde{U}_n U_n f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(e) $\widetilde{U}_m\widetilde{U}_n f = \widetilde{U}_n\widetilde{U}_m f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(f) $\lim_{n\to\infty} \widetilde{U}_n f = f$ *for* $f \in W^2(\varphi)[0,1]$;
(g) $\|\widetilde{D}U_n f\| \le \|\widetilde{D}f\|$ *for* $f \in W^2(\varphi)[0,1]$.

Proof. For the proof of (a), see [14, Lemma 4.2]. We have

$$\begin{aligned}
\widetilde{U}_n f &= \sum_{k=0}^{n} u_{n,k}(f)\widetilde{P}_{n,k} \\
&= u_{n,0}(f)\Big(P_{n,0} - \frac{1}{n}\widetilde{D}P_{n,0}\Big) + \sum_{k=1}^{n-1} u_{n,k}(f)\Big(P_{n,k} - \frac{1}{n}\widetilde{D}P_{n,k}\Big) + u_{n,n}(f)\Big(P_{n,n} - \frac{1}{n}\widetilde{D}P_{n,n}\Big) \\
&= u_{n,0}(f)P_{n,0} + \sum_{k=1}^{n-1} u_{n,k}(f)P_{n,k} + u_{n,n}(f)P_{n,n} \\
&\quad - \frac{\varphi}{n}\Big(u_{n,0}(f)P''_{n,0} + \sum_{k=1}^{n-1} u_{n,k}(f)P''_{n,k} + u_{n,n}(f)P''_{n,n}\Big) \\
&= U_n f - \frac{1}{n}\varphi\,(U_n f)''.
\end{aligned}$$

Then from (a), we obtain

$$\widetilde{U}_n f = U_n f - \frac{1}{n}\widetilde{D}U_n f = U_n f - \frac{1}{n}U_n\widetilde{D}f = U_n\Big(f - \frac{1}{n}\widetilde{D}f\Big)$$

which proves (b). Now, commutative properties (c) and (d) follow from (b) and (a):

$$\widetilde{D}\widetilde{U}_n f = \widetilde{D}U_n\Big(f - \frac{1}{n}\widetilde{D}f\Big) = U_n\Big(\widetilde{D}f - \frac{1}{n}\widetilde{D}\widetilde{D}f\Big) = \widetilde{U}_n(\widetilde{D}f),$$

and

$$U_n\widetilde{U}_n f = U_nU_n\Big(f - \frac{1}{n}\widetilde{D}f\Big) = U_nU_n f - \frac{1}{n}U_nU_n\widetilde{D}f = U_nU_n f - \frac{1}{n}U_n\widetilde{D}U_n f = \widetilde{U}_nU_n f.$$

The operators $\widetilde{U}_n$ commute in the sense of (e), since

$$\begin{aligned}\widetilde{U}_m\widetilde{U}_n f &= \widetilde{U}_m U_n\big(f - \frac{1}{n}\,\widetilde{D}f\big)\\ &= U_mU_nf - \frac{1}{n}\,U_mU_n\widetilde{D}f - \frac{1}{m}\,\widetilde{D}U_mU_nf + \frac{1}{mn}\,U_m\widetilde{D}^2U_nf\\ &= U_mU_n\Big(f - \frac{m+n}{mn}\,\widetilde{D}f + \frac{1}{mn}\,\widetilde{D}^2 f\Big).\end{aligned}$$

The same expression on the right-hand side we obtain for $\widetilde{U}_n\widetilde{U}_m f$ because of properties (a), (b) and $U_mU_nf = U_nU_mf$. We recall two more properties of the operator U_n and function $f \in W^2(\varphi)[0,1]$ (see [14, eqs. (4.8), (2.4)]):

$$\|U_nf - f\| \le \frac{1}{n}\,\|\widetilde{D}f\|, \tag{2.12}$$
$$\|U_n\widetilde{D}f\| \le \|\widetilde{D}f\|.$$

Therefore

$$\|\widetilde{U}_nf - f\| = \Big\|U_nf - \frac{1}{n}\,U_n\widetilde{D}f - f\Big\| \le \|U_nf - f\| + \frac{1}{n}\,\|U_n\widetilde{D}f\| \le \frac{2}{n}\,\|\widetilde{D}f\|,$$

hence $\lim_{n\to\infty}\|\widetilde{U}_nf - f\| = 0$, i.e. the limit (f) holds true.

From the proof of Lemma 4.2 in [14] for every $g \in W^2(\varphi)[0,1]$, we have

$$\widetilde{D}U_ng(x) = \sum_{k=1}^{n-1} P_{n,k}(x)\int_0^1 (n-1)P_{n-2,k-1}(t)\widetilde{D}g(t)\,dt,$$

From the last representation, we obtain

$$|\widetilde{D}U_ng(x)| \le \|\widetilde{D}g\|\sum_{k=1}^{n-1} P_{n,k}(x)\int_0^1 (n-1)P_{n-2,k-1}(t)\,dt \le \|\widetilde{D}g\|,$$

which proves (g). □

We now introduce a function that will prove useful in our investigations:

$$\begin{aligned}T_{n,k}(x) &:= k(k-1)\frac{1-x}{x} - 2k(n-k) + (n-k)(n-k-1)\frac{x}{1-x}\\ &= n\Big[-1 - \frac{1-2x}{\varphi(x)}\Big(\frac{k}{n} - x\Big) + \frac{n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2\Big].\end{aligned} \tag{2.13}$$

Observe that

$$T'_{n,k}(x) = -\frac{k(k-1)}{x^2} + \frac{(n-k)(n-k-1)}{(1-x)^2}, \tag{2.14}$$

$$T''_{n,k}(x) = \frac{2k(k-1)}{x^3} + \frac{2(n-k)(n-k-1)}{(1-x)^3} > 0, \qquad x \in (0,1). \tag{2.15}$$

Proposition 2.3.
(a) The following relation concerning $P_{n,k}$, $T_{n,k}$ and differential operator $\widetilde{D}$ holds:

$$\widetilde{D}P_{n,k}(x) = T_{n,k}(x)P_{n,k}(x). \tag{2.16}$$

(b) If α is an arbitrary real number, then

$$\Phi(\alpha) := \sum_{k=0}^{n} \Big(\alpha - \frac{1}{n}\, T_{n,k}(x)\Big)^2 P_{n,k}(x) = \alpha^2 + 2 - \frac{2}{n}.$$

Proof. (a) From (2.10), (2.11) and $\varphi(x)P_{n,k}(x) = \frac{(k+1)(n-k+1)}{(n+1)(n+2)}\, P_{n+2,k+1}(x)$, it follows that

$$\begin{aligned}
\varphi(x)P''_{n,k}(x) &= n(n-1)\big[\varphi(x)P_{n-2,k-2}(x) - 2\varphi(x)P_{n-2,k-1}(x) + \varphi(x)P_{n-2,k}(x)\big] \\
&= n(n-1)\Big[\frac{(k-1)(n-k+1)}{n(n-1)}\, P_{n,k-1}(x) - 2\,\frac{k(n-k)}{n(n-1)}\, P_{n,k}(x) \\
&+ \frac{(k+1)(n-k-1)}{n(n-1)}\, P_{n,k+1}(x)\Big] \\
&= (k-1)(n-k+1)\, P_{n,k-1}(x) - 2k(n-k)\, P_{n,k}(x) \\
&+ (k+1)(n-k-1)\, P_{n,k+1}(x) \\
&= \Big[k(k-1)\frac{1-x}{x} - 2k(n-k) + (n-k)(n-k-1)\frac{x}{1-x}\Big] P_{n,k}(x) \\
&= T_{n,k}(x)P_{n,k}(x),
\end{aligned}$$

i.e. the identity (2.16).

(b) We apply the formulae for the Bernstein operator moments in Proposition 2.1 (b):

$$\begin{aligned}
\Phi(\alpha) &= \sum_{k=0}^{n} \Big[\alpha + 1 + \frac{1-2x}{\varphi(x)}\Big(\frac{k}{n} - x\Big) - \frac{n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2\Big]^2 P_{n,k}(x) \\
&= \sum_{k=0}^{n} \Big[(\alpha+1)^2 + \frac{(1-2x)^2}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^2 + \frac{n^2}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^4 + \frac{2(\alpha+1)(1-2x)}{\varphi(x)}\Big(\frac{k}{n} - x\Big) \\
&- \frac{2(\alpha+1)n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2 - \frac{2n(1-2x)}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^3\Big] P_{n,k}(x) \\
&= (\alpha+1)^2 \mu_{n,0}(x) + \frac{(1-2x)^2}{\varphi^2(x)}\, \mu_{n,2}(x) + \frac{n^2}{\varphi^2(x)}\, \mu_{n,4}(x) + \frac{2(\alpha+1)(1-2x)}{\varphi(x)}\, \mu_{n,1}(x) \\
&- \frac{2(\alpha+1)n}{\varphi(x)}\, \mu_{n,2}(x) - \frac{2n(1-2x)}{\varphi^2(x)}\, \mu_{n,3}(x) \\
&= (\alpha+1)^2 \cdot 1 + \frac{(1-2x)^2}{\varphi^2(x)}\, \frac{\varphi(x)}{n} + \frac{n^2}{\varphi^2(x)}\, \frac{(3n-6)\varphi^2(x) + \varphi(x)}{n^3} \\
&+ \frac{2(\alpha+1)(1-2x)}{\varphi(x)} \cdot 0 - \frac{2(\alpha+1)n}{\varphi(x)}\, \frac{\varphi(x)}{n} - \frac{2n(1-2x)}{\varphi^2(x)}\, \frac{(1-2x)\varphi(x)}{n^2} \\
&= (\alpha+1)^2 + \frac{1-4\varphi(x)}{n\varphi(x)} + \frac{(3n-6)\varphi(x)+1}{n\varphi(x)} - 2(\alpha+1) - \frac{2(1-4\varphi(x))}{n\varphi(x)} \\
&= \alpha^2 + 2\alpha + 1 + \frac{1}{n\varphi(x)} - \frac{4}{n} + 3 - \frac{6}{n} + \frac{1}{n\varphi(x)} - 2\alpha - 2 - \frac{2}{n\varphi(x)} + \frac{8}{n} \\
&= \alpha^2 + 2 - \frac{2}{n}.
\end{aligned}$$

□

Auxiliary technical results will be useful for further estimations.

Proposition 2.4. *If* $n \in \mathbb{N}$, $n \ge 2$, *and*

$$\lambda(n) := \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}, \qquad \theta(n) := \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)^2},$$

then

$$\frac{1}{2n^2} \le \lambda(n) \le \frac{1}{n^2}, \tag{2.17}$$

$$\theta(n) \le \frac{4}{9n^3}. \tag{2.18}$$

Proof. Since $\frac{k}{k-1}\frac{n-1}{n} \le 1$ for $k \ge n$, we have for the lower estimate of $\lambda(n)$

$$\lambda(n) \ge \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \cdot \frac{k}{k-1} \cdot \frac{n-1}{n} = \frac{n-1}{n} \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)} = \frac{n-1}{n} \cdot \frac{1}{2(n-1)n} = \frac{1}{2n^2}.$$

For the upper estimates of $\lambda(n)$ and $\theta(n)$, we obtain

$$\lambda(n) < \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)} = \frac{1}{2n(n-1)} \le \frac{1}{n^2},$$

$$\theta(n) < \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)(k+2)} = \frac{1}{3n(n^2-1)} \le \frac{4}{9n^3}.$$

□

3. A DIRECT THEOREM

We will first prove the next upper estimate for the norm of the operator $\widetilde{U}_n$ defined in (1.4).

Lemma 3.1. *If* $n \in \mathbb{N}$ *and* $f \in C[0,1]$, *then*

$$\|\widetilde{U}_n f\| \le \sqrt{3}\,\|f\|, \quad \text{i.e.} \quad \|\widetilde{U}_n\| \le \sqrt{3}. \tag{3.19}$$

Proof. We have

$$\widetilde{P}_{n,k}(x) = P_{n,k}(x) - \frac{1}{n}\,\widetilde{D}P_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P_{n,k}(x).$$

Then for $x \in [0,1]$,

$$\begin{aligned}\big|\widetilde{U}_n(f,x)\big| &= \Big|\sum_{k=0}^{n} u_{n,k}(f)\widetilde{P}_{n,k}(x)\Big| \le \sum_{k=0}^{n} |u_{n,k}(f)|\,\big|\widetilde{P}_{n,k}(x)\big| \\ &\le \|f\| \sum_{k=0}^{n} \big|\widetilde{P}_{n,k}(x)\big| = \|f\| \sum_{k=0}^{n} \Big|1 - \frac{1}{n}\,T_{n,k}(x)\Big| P_{n,k}(x).\end{aligned}$$

Applying Cauchy inequality, we obtain

$$\big|\widetilde{U}_n(f,x)\big| \le \|f\| \sqrt{\sum_{k=0}^{n} \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)^2 P_{n,k}(x)}\,\sqrt{\sum_{k=0}^{n} P_{n,k}(x)}.$$

Since $\sum_{k=0}^{n} P_{n,k}(x) = 1$ identically, by Proposition 2.3 (b) with $\alpha = 1$, we find

$$\big|\widetilde{U}_n(f,x)\big| \le \sqrt{3 - \frac{2}{n}}\,\|f\| < \sqrt{3}\,\|f\|, \qquad x \in [0,1].$$

Hence, inequality (3.19) follows. □

In order to prove a direct theorem for the approximation rate for functions f by the operator $\widetilde{U}_n f$, we need a Jackson type inequality.

Lemma 3.2. *If* $n \in \mathbb{N}$, $f \in W_0^2(\varphi)[0,1]$ *and* $\widetilde{D} f \in W^2(\varphi)[0,1]$, *then*

$$\|\widetilde{U}_n f - f\| \leq \frac{1}{n^2} \|\widetilde{D}^2 f\|. \tag{3.20}$$

Proof. Having in mind the relation

$$U_k f - U_{k+1} f = \frac{1}{k(k+1)} \widetilde{D} U_{k+1} f,$$

(see [14, Lemma 4.1]) and Proposition 2.1 (a) for $f \in W_0^2(\varphi)[0,1]$, we obtain

$$\begin{aligned}
\widetilde{U}_k f - \widetilde{U}_{k+1} f &= U_k f - \frac{1}{k} \widetilde{D} U_k f - U_{k+1} f + \frac{1}{k+1} \widetilde{D} U_{k+1} f \\
&= U_k f - U_{k+1} f + \frac{1}{k+1} \widetilde{D} U_{k+1} f - \frac{1}{k} \widetilde{D} U_k f \\
&= \Big(\frac{1}{k} - \frac{1}{k+1}\Big) \widetilde{D} U_{k+1} f + \frac{1}{k+1} \widetilde{D} U_{k+1} f - \frac{1}{k} \widetilde{D} U_k f \\
&= -\frac{1}{k} \big(\widetilde{D} U_k f - \widetilde{D} U_{k+1} f\big) \\
&= -\frac{1}{k} \big(U_k \widetilde{D} f - U_{k+1} \widetilde{D} f\big) \\
&= -\frac{1}{k} \cdot \frac{1}{k(k+1)} \widetilde{D} U_{k+1} \widetilde{D} f,
\end{aligned}$$

i.e.,

$$\widetilde{U}_k f - \widetilde{U}_{k+1} f = -\frac{1}{k^2(k+1)} \widetilde{D} U_{k+1} \widetilde{D} f. \tag{3.21}$$

Therefore for every $s > n$, we have

$$\widetilde{U}_n f - \widetilde{U}_s f = \sum_{k=n}^{s-1} \big(\widetilde{U}_k f - \widetilde{U}_{k+1} f\big) = -\sum_{k=n}^{s-1} \frac{1}{k^2(k+1)} \widetilde{D} U_{k+1} \widetilde{D} f.$$

Letting $s \to \infty$ and by Proposition 2.2 (a) and (f), we obtain

$$\widetilde{U}_n f - f = -\sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \widetilde{D} U_{k+1} \widetilde{D} f. \tag{3.22}$$

Then from Proposition 2.1 (g) for $\widetilde{D} f \in W^2(\varphi)[0,1]$

$$\|\widetilde{U}_n f - f\| \leq \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \big\|\widetilde{D} U_{k+1} \widetilde{D} f\big\| \leq \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \big\|\widetilde{D}^2 f\big\|.$$

Proposition 2.4, (2.17), yields

$$\sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \leq \frac{1}{n^2}.$$

Therefore

$$\big\|\widetilde{U}_n f - f\big\| \leq \frac{1}{n^2} \big\|\widetilde{D}^2 f\big\|.$$

□

A direct result on the approximation rate of functions $f \in C[0,1]$ by the operators (1.4) in means of the K-functional (1.6) follows immediately from both lemmas above.

Proof of Theorem 1.1. Let g be arbitrary function, such that $g \in W_0^2(\varphi)[0,1]$ and $\widetilde{D}g \in W^2(\varphi)[0,1]$. Then by Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned}\|\widetilde{U}_n f - f\| &\le \|\widetilde{U}_n f - \widetilde{U}_n g\| + \|\widetilde{U}_n g - g\| + \|g - f\| \\ &\le (1+\sqrt{3})\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\| \\ &\le (1+\sqrt{3})\Big(\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\|\Big).\end{aligned}$$

Taking infimum over all functions g with $g \in W_0^2(\varphi)[0,1]$ and $\widetilde{D}g \in W^2(\varphi)[0,1]$, we obtain

$$\|\widetilde{U}_n f - f\| \le (1+\sqrt{3})\, K\Big(f, \frac{1}{n^2}\Big).$$

□

4. A STRONG CONVERSE RESULT

First, we will prove a Voronovskaya type result for the operator $\widetilde{U}_n$.

Lemma 4.3. *If $\lambda(n) = \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}$, $\theta(n) = \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)^2}$ and $f \in C[0,1]$ is such that $f, \widetilde{D}f \in W_0^2(\varphi)[0,1]$ and $\widetilde{D}^3 f \in L_\infty[0,1]$, then*

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \theta(n)\,\|\widetilde{D}^3 f\|. \tag{4.23}$$

Proof. We have

$$\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f = -\sum_{k=n}^{\infty} \frac{U_{k+1}\widetilde{D}^2 f}{k^2(k+1)} + \sum_{k=n}^{\infty} \frac{\widetilde{D}^2 f}{k^2(k+1)} = \sum_{k=n}^{\infty} \frac{\widetilde{D}^2 f - U_{k+1}\widetilde{D}^2 f}{k^2(k+1)},$$

see the proof of Lemma 3.2, eq. (3.21). Then

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}\,\|\widetilde{D}^2 f - U_{k+1}\widetilde{D}^2 f\|.$$

Using (2.12) with $\widetilde{D}^2 f$ instead of f, we obtain

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \cdot \frac{1}{(k+1)}\,\|\widetilde{D}\widetilde{D}^2 f\| = \theta(n)\,\|\widetilde{D}^3 f\|.$$

□

We need an inequality of Bernstein type.

Lemma 4.4. *Let $n \in \mathbb{N}$, $n \ge 2$ and $f \in C[0,1]$. Then the following inequality holds true*

$$\|\widetilde{D}\widetilde{U}_n f\| \le \widetilde{C}\, n\|f\|, \tag{4.24}$$

where $\widetilde{C} = 6.5 + \sqrt{6}$.

Proof. Since

$$\big|\widetilde{D}\widetilde{U}_n(f,x)\big| \le \sum_{k=0}^{n} |u_{n,k}(f)|\,\big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| \le \|f\| \sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big|,$$

it is sufficient to find an upper estimate for the quantity

$$\sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| = \sum_{k=0}^{n} \big|\varphi(x)\widetilde{P}''_{n,k}(x)\big|.$$

Remind that, according to (2.16), we have the relation

$$\widetilde{D}P_{n,k}(x) = \varphi(x)P''_{n,k}(x) = T_{n,k}(x)P_{n,k}(x).$$

Hence

$$\widetilde{P}_{n,k}(x) = P_{n,k}(x) - \frac{1}{n}\,\widetilde{D}P_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P_{n,k}(x),$$

$$\widetilde{P}''_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)''P_{n,k}(x) + 2\Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)'P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P''_{n,k}(x).$$

Then,

$$\begin{aligned}
\widetilde{D}\widetilde{P}_{n,k}(x) &= \varphi(x)\widetilde{P}''_{n,k}(x)\\
&= -\frac{\varphi(x)}{n}\,T''_{n,k}(x)P_{n,k}(x) - \frac{2\varphi(x)}{n}\,T'_{n,k}(x)P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)\varphi(x)P''_{n,k}(x)\\
&= -\frac{\varphi(x)}{n}\,T''_{n,k}(x)P_{n,k}(x) - \frac{2\varphi(x)}{n}\,T'_{n,k}(x)P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)T_{n,k}(x)P_{n,k}(x).
\end{aligned}$$

Therefore

$$\sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| \le a_n(x) + b_n(x) + c_n(x),$$

where

$$\begin{aligned}
a_n(x) &= \frac{\varphi(x)}{n} \sum_{k=0}^{n} \big|T''_{n,k}(x)\big|P_{n,k}(x),\\
b_n(x) &= \frac{2\varphi(x)}{n} \sum_{k=0}^{n} \big|T'_{n,k}(x)P'_{n,k}(x)\big|,\\
c_n(x) &= \sum_{k=0}^{n} \Big|\Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)T_{n,k}(x)\Big|P_{n,k}(x).
\end{aligned}$$

1. Estimate for $a_n(x)$. From (2.15) and (2.8)–(2.9),

$$\begin{aligned}
\sum_{k=0}^{n} T''_{n,k}(x)P_{n,k}(x) &= \sum_{k=0}^{n} \Big(\frac{2k(k-1)}{x^3} + \frac{2(n-k)(n-k-1)}{(1-x)^3}\Big)P_{n,k}(x)\\
&= \frac{2}{x^3} \sum_{k=0}^{n} k(k-1)P_{n,k}(x) + \frac{2}{(1-x)^3} \sum_{k=0}^{n} (n-k)(n-k-1)P_{n,k}(x)\\
&= \frac{2}{x^3}\,n(n-1)x^2 + \frac{2}{(1-x)^3}\,n(n-1)(1-x)^2\\
&= \frac{2n(n-1)}{\varphi(x)}.
\end{aligned}$$

Having in mind $T''_{n,k}(x) > 0$ in (2.15), we obtain

$$a_n(x) = \frac{\varphi(x)}{n} \sum_{k=0}^{n} T''_{n,k}(x) P_{n,k}(x) = 2(n-1). \tag{4.25}$$

2. Estimate for $b_n(x)$. Observe that

$$\sum_{k=0}^{n} \left|T'_{n,k}(x)\, P'_{n,k}(x)\right| = \sum_{k=0}^{n} \left|T'_{n,k}(1-x)\, P'_{n,k}(1-x)\right|,$$

hence, there is a symmetry of the function $b_n(x)$ in $x = \frac{1}{2}$. Therefore, it is sufficient to estimate $b_n(x)$ for $x \in \left[0, \frac{1}{2}\right]$.

We will show that in $\left[0, \frac{1}{2}\right]$ the function $b_n(x)$ has exactly $\left\lfloor \frac{n-1}{2} \right\rfloor$ local extrema h_k attained at points in intervals $\left(\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$, respectively. We will estimate all the local maxima h_k and then an estimate for $b_n(x)$ will follow immediately.

(i) First, we prove that

$$S(x) := \frac{-2\varphi(x)}{n} \sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) = 4(n-1).$$

From (2.10) and (2.14),

$$\sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) = n \sum_{k=0}^{n-1} \big(T'_{n,k+1}(x) - T'_{n,k}(x)\big) P_{n-1,k}(x).$$

Since

$$\begin{aligned} T'_{n,k+1}(x) - T'_{n,k}(x) &= \frac{(n-k-1)(n-k-2)}{(1-x)^2} - \frac{(k+1)k}{x^2} + \frac{k(k-1)}{x^2} - \frac{(n-k)(n-k-1)}{(1-x)^2} \\ &= -\frac{2k}{x^2} - \frac{2(n-k-1)}{(1-x)^2}, \end{aligned}$$

using (2.7) we get

$$\begin{aligned} \sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) &= -\frac{2n}{x^2} \sum_{k=0}^{n-1} k P_{n-1,k}(x) - \frac{2n}{(1-x)^2} \sum_{k=0}^{n-1} (n-k-1) P_{n-1,k}(x) \\ &= -\frac{2n}{x^2}\,(n-1)x - \frac{2n}{(1-x)^2}\,(n-1)(1-x) \\ &= -\frac{2n(n-1)}{\varphi(x)}. \end{aligned}$$

Therefore,

$$S(x) = \frac{-2\varphi(x)}{n} \cdot \frac{-2n(n-1)}{\varphi(x)} = 4(n-1). \tag{4.26}$$

(ii) By (2.15), $T''_{n,k}(x) > 0$, hence $-T'_{n,k}(x)$ strictly decreases for $x \in (0,1)$.

For $k = 0, 1$, we have $-T'_{n,k}(0^+) < 0$, then $-T'_{n,k}(x) < 0$, $x \in (0,1)$, and $\varphi(x) T'_{n,1}(x)$ has its only zero in $[0,1)$ at $\xi_1 = 0$.

For $k = 2, \dots, n-2$, we have $-T'_{n,k}(0^+) > 0$, and $T'_{n,k}(x)$ has a unique simple zero at

$$\xi_k = \frac{\sqrt{\binom{k}{2}}}{\sqrt{\binom{k}{2}} + \sqrt{\binom{n-k}{2}}} \in \left(\tfrac{k-1}{n}, \tfrac{k}{n}\right).$$

For $k = n-1, n$, we have $-T'_{n,k}(x) > 0$ for $x \in (0,1)$, and $-\varphi(x)T'_{n,n}(x) = 0$ only for $\xi_n = 1$ in $(0,1]$.

(iii) For the Bernstein basis polynomials on $(0,1)$, we have

$$P'_{n,0}(x) = -n(1-x)^{n-1} < 0,$$
$$P'_{n,k}(x) = n\binom{n}{k}x^{k-1}(1-x)^{n-k-1}\big(\frac{k}{n}-x\big), \text{ and } P'_{n,k}(x) = 0 \text{ only if } x = \frac{k}{n}$$
$$P'_{n,n}(x) = nx^{n-1} > 0.$$

(iv) Now, from (ii) and (iii) for $x \in (0,1)$,

$-\varphi(x)T'_{n,0}(x)\,P'_{n,0}(x) > 0$,
$-\varphi(x)T'_{n,1}(x)\,P'_{n,1}(x) > 0$ for $x \in \big(\xi_1, \frac{1}{n}\big) = \big(0, \frac{1}{n}\big)$,
$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) > 0$ for $x \in \big(\frac{k-1}{n}, \xi_k\big)$, $k = 2, \dots, \lfloor\frac{n-1}{2}\rfloor$,
$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) < 0$ for $x \in \big(\xi_k, \frac{k}{n}\big)$, $k = 2, \dots, \lfloor\frac{n-1}{2}\rfloor$,
$-\varphi(x)T'_{n,n}(x)\,P'_{n,n}(x) > 0$.

(v) From the observations in (ii)–(iv), it follows that

$$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) > 0, \qquad k = 0, \dots, n$$

except

$$\begin{aligned} &-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) < 0, && x \in \big(\xi_k, \tfrac{k}{n}\big), && k = 1, \dots, \lfloor\tfrac{n-1}{2}\rfloor, \\ &-\varphi(x)T'_{n,n-k}(x)\,P'_{n,n-k}(x) < 0, && x \in \big(\tfrac{n-k}{n}, \xi_{n-k}\big), && k = 1, \dots, \lfloor\tfrac{n-1}{2}\rfloor. \end{aligned}$$

Hence,

$$\sum_{k=0}^{n}\Big|\frac{-2\varphi(x)T'_{n,k}(x)}{n}\,P'_{n,k}(x)\Big| = S(x) = 4(n-1), \quad x \in \big[0, \tfrac{1}{2}\big] \setminus \bigcup_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\big(\xi_k, \tfrac{k}{n}\big).$$

Therefore, for $k = 1, \dots, \lfloor\frac{n-1}{2}\rfloor$,

$$b_n(x) = \begin{cases} 4(n-1), & x \in \big[\frac{k-1}{n}, \xi_k\big], \\ 4(n-1) + \dfrac{2\varphi(x)}{n}\big|T'_{n,k}(x)P'_{n,k}(x)\big|, & x \in \big[\xi_k, \frac{k}{n}\big]. \end{cases} \tag{4.27}$$

Moreover,

$$b_n(x) = 4(n-1), \qquad x \in \big[\tfrac{n-2}{2n}, \tfrac{n+2}{2n}\big],\ n \text{ even, and } x \in \big[\tfrac{n-1}{2n}, \tfrac{n+1}{2n}\big],\ n \text{ odd.}$$

(vi) This means that we have to estimate the maxima of the functions

$$s_k(x) := \Big|\frac{-2\varphi(x)T'_{n,k}(x)}{n}\,P'_{n,k}(x)\Big|, \qquad x \in \big[\xi_k, \tfrac{k}{n}\big], \quad k = 1, \dots, \lfloor\tfrac{n-1}{2}\rfloor.$$

By (iv) for $k = 1$, we have:

$$s_1(x) = \frac{-2\varphi(x)T'_{n,1}(x)}{n}\,P'_{n,1}(x) = 2n(n-1)(n-2)x\Big(\frac{1}{n}-x\Big)(1-x)^{n-3}.$$

Since

$$\max_{x\in[0,1/n]} x\Big(\frac{1}{n}-x\Big) = \frac{1}{4n^2} \quad \text{and} \quad (1-x)^{n-3} \le 1,$$

we obtain

$$h_1 := \max_{x\in[0,1/n]} s_1(x) \le \frac{2n(n-1)(n-2)}{4n^2} \le \frac{n}{2}. \tag{4.28}$$

Let us fix $k \in \{2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. We estimate the local extremum

$$h_k := \max_{x \in [\xi_k, k/n]} s_k(x).$$

According to (iv), we have

$$s_k(x) = \frac{2\varphi(x)}{n} T'_{n,k}(x) P'_{n,k}(x) = \frac{2\varphi(x)}{n} T'_{n,k}(x) \binom{n}{k} x^{k-1}(1-x)^{n-k-1}\Big(\frac{k}{n} - x\Big),$$

i.e.,

$$s_k(x) = \frac{2}{n} T'_{n,k}(x) P_{n,k}(x) \Big(\frac{k}{n} - x\Big). \tag{4.29}$$

The function $T'_{n,k}(x)$ is strictly increasing in $\big[\frac{k-1}{n}, \frac{k}{n}\big]$ and change sign only at point $\xi_k = \frac{\sqrt{\binom{k}{2}}}{\sqrt{\binom{k}{2}} + \sqrt{\binom{n-k}{2}}}$. Then, for $x \in \big[\xi_k, \frac{k}{n}\big]$,

$$\max_{x \in [\xi_k, k/n]} T'_{n,k}(x) = T'_{n,k}\big(\tfrac{k}{n}\big) = -\frac{k(k-1)n^2}{k^2} + \frac{(n-k)(n-k-1)n^2}{(n-k)^2} = n^2\Big(\frac{1}{k} - \frac{1}{n-k}\Big).$$

The function $h(x) = \frac{1}{x} - \frac{1}{n-x}$ is decreasing in $\big(0, \frac{n}{2}\big)$ since $h'(x) = \big(\frac{1}{x} - \frac{1}{n-x}\big)' < 0$, hence for $k \in \{2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$

$$T'_{n,k}(x) \le n^2\Big(\frac{1}{k} - \frac{1}{n-k}\Big) \le n^2\Big(\frac{1}{2} - \frac{1}{n-2}\Big) \le \frac{n^2}{2}. \tag{4.30}$$

Also, $\frac{k-1}{n} \le \xi_k \le \frac{k}{n}$ and for $x \in \big[\xi_k, \frac{k}{n}\big]$, we have $\frac{k}{n} - x \le \frac{1}{n}$. Since $0 \le P_{n,k}(x) \le 1$ in $[0,1]$, it follows from (4.29) and (4.30) that

$$h_k \le \frac{2}{n} \cdot \frac{n^2}{2} \cdot \frac{1}{n} \le 1.$$

Taking into account (4.28), for $n \ge 2$ we have

$$h_k \le h_1 \le \frac{n}{2}, \qquad k = 1, \dots, \Big\lfloor \frac{n-1}{2} \Big\rfloor. \tag{4.31}$$

Finally, for $b_n(x)$, using (4.27) and (4.31), we obtain the estimate

$$b_n(x) \le 4(n-1) + \max_{1 \le k \le \lfloor \frac{n-1}{2} \rfloor} h_k \le 4(n-1) + \frac{n}{2},$$

or

$$b_n(x) \le 4.5\, n, \qquad x \in [0,1]. \tag{4.32}$$

3. Estimate for $c_n(x)$. We apply Cauchy inequality and Proposition 2.3 (b) with $\alpha = 0$ and $\alpha = 1$:

$$\begin{aligned} c_n(x) &= \sum_{k=0}^{n} \Big| T_{n,k}(x)\Big(1 - \frac{1}{n} T_{n,k}(x)\Big)\Big| P_{n,k}(x) \\ &\le \sqrt{\sum_{k=0}^{n} T^2_{n,k}(x) P_{n,k}(x)} \sqrt{\sum_{k=0}^{n} \Big(1 - \frac{1}{n} T_{n,k}(x)\Big)^2 P_{n,k}(x)} \\ &= \sqrt{\Phi(0) n^2} \cdot \sqrt{\Phi(1)} = n\sqrt{2 - \frac{2}{n}} \cdot \sqrt{3 - \frac{2}{n}}. \end{aligned}$$

Then,

$$c_n(x) \leq \sqrt{6}\, n, \qquad x \in [0,1]. \tag{4.33}$$

From (4.25), (4.32) and (4.33), we obtain

$$\sum_{k=0}^{n} \left|\widetilde{D}\widetilde{P}_{n,k}(x)\right| \leq a_n(x) + b_n(x) + c_n(x) \leq 2(n-1) + 4.5\, n + \sqrt{6}\, n \leq \left(6.5 + \sqrt{6}\right)n.$$

Therefore

$$\left\|\widetilde{D}\widetilde{U}_n f\right\| \leq \widetilde{C} n \|f\|, \qquad \widetilde{C} := 6.5 + \sqrt{6}.$$

□

Now we are ready to prove a strong converse inequality of Type B.

Proof of Theorem 1.2. We follow the approach of Ditzian and Ivanov [6].

Let $n \in \mathbb{N}$, $n \geq 2$, $f \in C[0,1]$ and $\lambda(n)$, $\theta(n)$ be defined as in Proposition 2.4. From the Voronovskaya type inequality in Lemma 4.3 for the operator $\widetilde{U}_\ell$ and function $\widetilde{U}_n^3 f$ instead of f, we have

$$\begin{aligned}
\lambda(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| &= \left\|\lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f\right\| \\
&= \left\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f + \lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f - \widetilde{U}_\ell\widetilde{U}_n^3 f + \widetilde{U}_n^3 f\right\| \\
&\leq \left\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f + \lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f\right\| + \left\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f\right\| \\
&\leq \theta(\ell)\left\|\widetilde{D}^3\widetilde{U}_n^3 f\right\| + \left\|\widetilde{U}_n^3\left(\widetilde{U}_\ell f - f\right)\right\|.
\end{aligned}$$

Now, using Lemma 4.4 for the function $\widetilde{D}^2\widetilde{U}_n^2 f$ and in addition Lemma 3.1 repeatedly three times, we obtain

$$\begin{aligned}
\lambda(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| &\leq \widetilde{C}\, n\, \theta(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^2 f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell f - f\right\| \\
&= \widetilde{C}\, n\, \theta(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^2(f - \widetilde{U}_n f) + \widetilde{D}^2\widetilde{U}_n^3 f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell f - f\right\| \\
&\leq \widetilde{C}\, n\, \theta(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^2(f - \widetilde{U}_n f)\right\| + \widetilde{C}\, n\, \theta(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell f - f\right\|.
\end{aligned}$$

Applying the Bernstein type inequality Lemma 4.4 twice for $f - \widetilde{U}_n f$ yields

$$\lambda(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| \leq \widetilde{C}^3 n^3 \theta(\ell)\left\|f - \widetilde{U}_n f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell - f\right\| + \widetilde{C}\, n\, \theta(\ell)\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\|.$$

From inequalities (2.17) and (2.18) of Proposition 2.4, we get

$$\frac{1}{2\ell^2}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| \leq \frac{4\widetilde{C}^3 n^3}{9\ell^3}\left\|f - \widetilde{U}_n f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell - f\right\| + \frac{4\widetilde{C}n}{9\ell^3}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\|.$$

Let us choose ℓ sufficiently large such that

$$\frac{4\widetilde{C}n}{9\ell^3} \leq \frac{1}{4\ell^2}, \qquad \text{i.e.} \qquad \ell \geq \frac{16\widetilde{C}}{9}\, n.$$

If we set $L = \frac{16\widetilde{C}}{9}$, for all integers $\ell \geq Ln$ we have

$$\begin{aligned}
\frac{1}{2\ell^2}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| &\leq \frac{4\widetilde{C}^3 n^3}{9\ell^3}\left\|f - \widetilde{U}_n f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell - f\right\| + \frac{1}{4\ell^2}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\|, \\
\frac{1}{4\ell^2}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| &\leq \frac{4\widetilde{C}^3 n^3}{9\ell^3}\left\|f - \widetilde{U}_n f\right\| + 3\sqrt{3}\left\|\widetilde{U}_\ell - f\right\|,
\end{aligned}$$

$$\frac{1}{n^2}\left\|\widetilde{D}^2\widetilde{U}_n^3 f\right\| \leq \widetilde{C}^2\left\|f - \widetilde{U}_n f\right\| + 12\sqrt{3}\,\frac{\ell^2}{n^2}\left\|\widetilde{U}_\ell - f\right\|. \tag{4.34}$$

By using Lemma 3.1,

$$\begin{aligned}\|f-\widetilde{U}_n^3 f\| &\le \|f-\widetilde{U}_n f\| + \|\widetilde{U}_n f-\widetilde{U}_n^2 f\| + \|\widetilde{U}_n^2 f-\widetilde{U}_n^3 f\| \\ &\le (1+\sqrt{3}+(\sqrt{3})^2)\|f-\widetilde{U}_n f\|,\end{aligned}$$

and we obtain the inequality

$$\|f-\widetilde{U}_n^3 f\| \le (4+\sqrt{3})\|f-\widetilde{U}_n f\|. \tag{4.35}$$

It remains to complete the estimation of the K-functional. Since $\widetilde{U}_n^3 f \in W_0^2(\varphi)[0,1]$, from (4.34) and (4.35) it follows

$$\begin{aligned}K\Big(f,\frac{1}{n^2}\Big) &= \inf\Big\{\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\| : \ g\in W_0^2(\varphi)[0,1],\ \widetilde{D}g\in W^2(\varphi)[0,1]\Big\} \\ &\le \|f-\widetilde{U}_n^3 f\| + \frac{1}{n^2}\,\|\widetilde{D}^2\widetilde{U}_n^3 f\| \\ &\le \Big(4+\sqrt{3}+\widetilde{C}^2\Big)\|\widetilde{U}_n f - f\| + 12\sqrt{3}\,\frac{\ell^2}{n^2}\,\|\widetilde{U}_\ell f - f\|.\end{aligned}$$

Therefore,

$$K\Big(f,\frac{1}{n^2}\Big) \le C\,\frac{\ell^2}{n^2}\big(\|\widetilde{U}_n f - f\| + \|\widetilde{U}_\ell f - f)\|\big)$$

for all $\ell \ge Ln$, where $C = 4+\sqrt{3}+\widetilde{C}^2$ and $L=\frac{16\widetilde{C}}{9}$, $\widetilde{C}=6.5+\sqrt{6}$. □

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