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Research Article

A Quaternionic Product of Lines in the Plane $\mathbb{E}^2$

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Abstract: This note introduces a product of plane lines inspired by the product of quaternions. A technical condition is necessary for the existence of this product and some examples (squares, the axes and the bisectrices of the axes) are discussed.

Keywords: Line, quaternion, product, projective geometry.

1. Introduction

The quaternionic algebra $\mathbb{H}$ is a well-known setting of modern mathematics, together with the real algebra $\mathbb{R}$ and the complex algebra $\mathbb{C}$; since the bibliography on quaternions is huge we cite only the very first paper [8] of Sir William Rowan Hamilton. This algebraic structure was designed with the geometric goal of serving as a helpful tool for modelling the rotations of three-dimensional Euclidean space from the very beginning. The projections and involutions are expressed with quaternions in [1]. We note also that recently the applications of quaternions in the differential geometry are surveyed in [7].

The purpose of the present work is to use the product of $\mathbb{H}$ into another framework namely the set of lines of the Euclidean plane. The choice of the identification of a line with a quaternion is based on some previous papers of the author. We point out also that in order to obtain a suitable quaternionic product we introduce a technical condition in our Definition 2.1. These considerations yields a projective way to manage this product and we consider that the potential areas of applications are the incidence geometry [9].

This new product is discussed especially from the point of view of examples. In addition to squares we study some concrete examples by giving also numerical details.

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2. The Quaternionic Product of Two Distinguished Lines

Fix the set of all lines $\mathcal{L} := \{d : ax + by + c = 0; a^2 + b^2 > 0\}$ in the Euclidean plane $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$. The aim of this work is to introduce a product (inspired by quaternions) in $\mathcal{L}$ and hence the starting point of this paper is the identification of the given line

$$d = d(a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, c) : c \in \mathbb{R}\}$$

with the quaternion:

$$q(d) := \bar{k} + a\bar{i} + b\bar{j} + c = (c, a, b, 1) \in \mathbb{R}^4. \tag{1}$$

The quaternion $q(d)$ is pure imaginary if and only if the origin $O(0,0) \in d$. We point out that although there are alternative ways to associate a quaternion to a given line, we choose the expression (1) according to our previous studies, namely (in the chronological order) [2, 4, 5].

From the real algebra structure $\mathbb{H}$ of the quaternions ([6, p. 89]), it follows a product of two lines:

$$d_1 \odot_q d_2 := q^{-1}(q(d_1) \cdot q(d_2)). \tag{2}$$

With the given parameters $(a_i, b_i, c_i), i = 1, 2$, we derive immediately:

$$q(d_1) \cdot q(d_2) = (a_1 b_2 - a_2 b_1 + c_1 + c_2)\bar{k} + (b_1 - b_2 + a_1 c_2 + a_2 c_1)\bar{i} + (a_2 - a_1 + b_1 c_2 + b_2 c_1)\bar{j} +$$

$$+(c_1 c_2 - 1 - a_1 a_2 - b_1 b_2) =: D\bar{k} + A\bar{i} + B\bar{j} + C \tag{3}$$

and due to the expression of the coefficient of $\bar{k}$ in (1), we need a special condition for our approach.

Definition 2.1 *The given pair of lines $d_r = (a_r, b_r, c_r)$, $r = 1, 2$, is called q-distinguished if:*

$$D(d_1, d_2) := a_1 b_2 - a_2 b_1 + c_1 + c_2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + c_1 + c_2 \neq 0. \tag{4}$$

Example 2.2 *i) For a fixed line $d(a, b, c)$ we have the square:*

$$q(d) \cdot q(d) = 2c\bar{k} + 2ac\bar{i} + 2bc\bar{j} + (c^2 - 1 - a^2 - b^2). \tag{5}$$

If $c \neq 0$, then the technical condition (4) is satisfied and then the pair (d, d) is q-distinguished. Also, $q(d) \in \mathbb{R}^4$ is a purely imaginary quaternion only for $c_{\pm} = \pm\sqrt{a^2 + b^2 + 1}$ with $c_- < -1$ and $c_+ > 1$.

ii) Let d_1 and d_2 be concurrent lines in O. Then, (4) means that their normals $N_1 = (a_1, b_1)$, $N_2 = (a_2, b_2)$ are linear independent vectors, i.e., the lines are different.

Let now $\mathcal{L}^2(q)$ be the set of q-distinguished pairs of lines. Working in a projective manner it follows a quaternionic product in $\mathcal{L}^2(q)$:

$$d_1 \odot_q d_2 := d\left(\frac{A}{D}, \frac{B}{D}, \frac{C}{D}\right) \tag{6}$$

supposing again that $A^2 + B^2 > 0$. It is worth to point out the combination of Euclidean and projective geometry of our approach; hence, $d_1 \odot_q d_2$ is also the line $d(A, B, C)$.

An important tool of the quaternionic theory is that of conjugate, which for our quaternion (1) means:

$$\overline{q(d)} := -\bar{k} - a\bar{i} - b\bar{j} + c = (c, -a, -b, -1) = -q(a, b, -c) \tag{7}$$

and our projective way of thinking allows the identification: $\overline{q(a,b,c)} = q(a, b, -c)$. The pair of parallel lines $(d(a, b, c), d(a, b, -c))$ is not q-distinguished. The real part C of the quaternion (5) is the Euclidean inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^4$ of the vectors $q(d_1)$ and $\overline{q(d_2)}$.

3. Concrete Examples

In the following we study this new product introduced in (6) through three large examples.

Example 3.1 *Revisiting the Example 2.2 (i) (recall that $c \neq 0$), we have immediately the square of a line d for which $O \notin d$ (recall that $\mathbb{RP}^1$ is the moduli space of lines that contain the origin):*

$$d^2_{\odot_q} : ax + by + \frac{c^2 - a^2 - b^2 - 1}{2c} = 0 \to d^2_{\odot_q} \neq d, \quad d^2_{\odot_q} \parallel d. \tag{8}$$

The expression above suggests as remarkable example the case of right triangle, $\triangle : c^2 = a^2 + b^2$, which gives the associated lines:

$$\begin{cases} d_{(\triangle,+)} : ax + by + \sqrt{a^2 + b^2} = 0, \quad a > 0, \quad b > 0, \\ (d_{(\triangle,+)})^2_{\odot_q} : ax + by - \frac{1}{2\sqrt{a^2+b^2}} = 0. \end{cases} \tag{9}$$

A particular case of the right triangle $\triangle$ is provided by the case when (a, b, c) is a Pythagorean triple and hence we know its parametrization ([3]):

$$a := \beta^2 - \alpha^2, \quad b := 2\alpha\beta, \quad c := \alpha^2 + \beta^2, \quad 0 < \alpha < \beta \in \mathbb{N}^*. \tag{10}$$

It follows the lines:

$$\begin{cases} d(Pythagorean) : (\beta^2 - \alpha^2)x + 2\alpha\beta y + (\alpha^2 + \beta^2) = 0, \\ (d(Pythagorean))^2_{\odot_q} : (\beta^2 - \alpha^2)x + 2\alpha\beta y - \frac{1}{2(\alpha^2+\beta^2)} = 0. \end{cases} \tag{11}$$

For a concrete example we choose the minimal pair $\alpha = 1 < \beta = 2$ *giving the minimal Pythagorean triple* $(a = 3, b = 4, c = 5)$ *with the associated lines:*

$$d(minimal) : 3x + 4y + 5 = 0, \quad (d(minimal))^2_{\odot_q} : 3x + 4y - \frac{1}{10} = 0. \tag{12}$$

This last situation suggests the general case:

$$d_t : (\cos t)x + (\sin t)y + 1 = 0, t \in \mathbb{R} \to (d_t)^2_{\odot_q} : (\cos t)x + (\sin t)y - \frac{1}{2} = 0. \tag{13}$$

Recall that for a given C^2 *periodic and convex function* $p = p(t) = p(t + 2\pi)$ *the convex envelope of the family of lines:*

$$d_p(t) : (\cos t)x + (\sin t)y = p(t)$$

is the oval $\mathcal{C}$ *parametrized by:*

$$\mathcal{C} : (x(t), y(t)) = (p(t)\cos t - p'(t)\sin t, p(t)\sin t + p'(t)\cos t). \tag{14}$$

Therefore, the oval generated by the family $(d_t)^2_{\odot_q}$ *is the Euclidean circle centered in* O *and of radius* $R = \frac{1}{2}$. *Remark also, that the scalar part of the line in the equation* (8) *suggests the real function:*

$$c \in (0, +\infty) \to f(c) := \frac{c^2 - a^2 - b^2 - 1}{2c}$$

which have as oblique asymptotic the line $y = \frac{1}{2}x$.

Example 3.2 *In this example, we will perform the quaternionic product of two different q-distinguished lines. The coordinates lines are so (conform the second Example 2.2) and then we have:*

$$Ox \odot_q Oy : x + y - 1 = 0, \quad Oy \odot_q Ox : x + y + 1 = 0 \tag{15}$$

and hence, generally speaking, the quaternionic product does not preserve the orthogonality nor the concurrency and is not commutative; we have only $Re(q(d_1) \cdot q(d_2)) = Re(q(d_2) \cdot q(d_1))$ *i.e.,* $C(d_1, d_2) = C(d_2, d_1)$. *In fact, if both lines contains the origin, i.e.,* $c_1 = c_2 = 0$, *then* $D(d_1, d_2) = -D(d_2, d_1)$ *and* $d_1 \odot_q d_2 \parallel d_2 \odot_q d_1$. *The bisectrices of the axes are also q-distinguished and their products are vertical lines:*

$$B_1 : x - y = 0, \quad B_2 : x + y = 0 \to B_1 \odot_q B_2 : x - \frac{1}{2} = 0, \quad B_2 \odot_q B_1 : x + \frac{1}{2} = 0. \tag{16}$$

Hence, the quaternionic product does not preserve the concurrence of the given lines.

Example 3.3 *Let three distinct points* $M_i(\alpha_i, \beta_i)$, $i = 1, 2, 3$ *and* d_1 *the line* M_0M_1 *respectively* d_2 *the line* M_0M_2. *Since for* d_1 *the coefficients are:*

$$a_1 = \beta_0 - \beta_1, \quad b_1 = \alpha_1 - \alpha_0, \quad c_1 = \alpha_0\beta_1 - \alpha_1\beta_0 \tag{17}$$

and similar relations hold for d_2 *the condition* (4) *reads:*

$$D(d_1, d_2) = \alpha_1\beta_2 - \alpha_2\beta_1 + 2(\alpha_0\beta_1 - \alpha_1\beta_0) \neq 0. \tag{18}$$

Since the translations are Euclidean isometries let us suppose that M_0 *is* O *and then* $D(d_1, d_2)$ *reduces to the scalar part* c *of the line* M_1M_2; *in the proper triangle* $M_0M_1M_2$ *the vertex* M_0 *does not belong to* M_1M_2 *and therefore* (4) *holds. We study now the possibility to introduce the notion of quaternionic triangle as one in which* $M_0M_1 \odot_q M_0M_2$ *is exactly* M_1M_2. *But, if we write explicitly this equality as the equalities of ratios:*

$$\frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2} = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{-1 - \alpha_1\alpha_2 - \beta_1\beta_2} \tag{19}$$

already the first equality yields the impossible $\alpha_1 - \alpha_2 = 0 = \beta_1 - \beta_2$. *This fact can be probably explained by the complex nature of* $\bar{i}$ *and* $\bar{j}$.

4. Conclusions

In this paper, we study the geometry of 2D Euclidean lines through an algebraic operation inspired by the product of quaternions. Some features of this new product are discussed directly on examples.

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Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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